Large deviations and dynamical phase transitions

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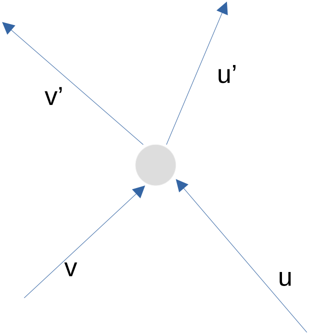
LD for binary collision models

Binary collision



Elastic collision $\rightarrow$ conservation laws

Space homogeneous binary collision models

N velocities $\{v_1, ..., v_N\}$, $v_i \in \mathbb{R}^d$.

Each pair $\{v_i, v_j\}$ has an exponential clock. As the clock rings

$$(v_i, v_j) \to (v_i', v_j')$$

according to a certain probability $p(v_i, v_j, dv', dv_*')$.

Conservation laws

$$v_i + v_j = v_i' + v_j', \qquad |v_i|^2 + |v_j|^2 = |v_i'|^2 + |v_j'|^2$$

The Kac's walk

Due to the conservation laws, there exist $\omega \in S^{d-1}$

$$v_i' = v_i + (\omega \cdot (v_j - v_i))\omega, \qquad v_j' = v_j - (\omega \cdot (v_j - v_i))\omega,$$

Continuous time Markov chain on $\left(\mathbb{R}^d\right)^N$ with generator

$$L_N F(\underline{v}) = \frac{1}{N} \sum_{\{i,j\}} \int_{\mathbb{S}_{d-1}} \mathrm{d}\omega \, B\big(v_i - v_j, \omega\big)\big(F\big(T^{\omega}_{i,j}\underline{v}\big) - F(\underline{v})\big),$$

where $T^{\omega}_{i,j}\underline{v} = (v_1, .., v_i', .., v_j', .., v_N)$. Absorbing state $v = (0, .., 0)$.

Hard sphere kernel $B = [\omega \cdot (v - v_*)]_-$

Kinetic limit

Initial distribution $F_0^N = f_0^{\otimes N}$ on configuration space $\mathbb{R}^{Nd}$.
As $N \to \infty$

$$F_t^{N,j}(v_1, .., v_j) \to \prod_{i=1}^{j} f_t(v_i) \quad \text{(propagation of chaos)}$$

with f_t solution to the homogeneous Boltzmann equation (Kac '56).

Empirical measure $\pi_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{v_i(t)}$

$$\mathrm{d}\pi_t^N \to f_t \, \mathrm{d}v \quad \text{(LLN)}$$

(Sznitman '84) hard-sphere collision kernel.

The homogeneous Boltzmann equation

$$\partial_t f_t(v) = \int_{\mathbb{R}^d} \mathrm{d}v_* \int_{S^{d-1}} \mathrm{d}\omega \, B(v - v_*, \omega) \big[f_t(v') f_t(v'_*) - f_t(v) f_t(v_*) \big]$$

- Existence (Arkeryd '72); uniqueness if B bounded.
- Hard sphere kernel: uniqueness in the class of energy conserving solutions (Mischler, Wennberg '99).
- Existence of weak solutions with increasing energy (Lu, Wennberg '02).
- Stationary solution: any Maxwellian $M_{u,\beta}$.

Discrete energy model

$\{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i \in \mathbb{N}$. Collision $(\epsilon_i, \epsilon_j) \to (\epsilon_i', \epsilon_j')$, with

$$\epsilon_i + \epsilon_j = \epsilon_i' + \epsilon_j',$$

according to the uniform collision kernel

$$B(\epsilon, \epsilon_*, \epsilon', \epsilon_*') = \frac{1}{\epsilon + \epsilon_* + 1} \mathbb{1}_{\{\epsilon+\epsilon_* = \epsilon' + \epsilon_*'\}} \mathbb{1}_{\left\{\{\epsilon, \epsilon_*\} \neq \{\epsilon', \epsilon_*'\}\right\}}$$

Absorbing state $\{0, .., 0\}$.

Discrete energy model

$\{\epsilon_1, .., \epsilon_N\}$, $\epsilon_i \in \mathbb{N}$. Collision $(\epsilon_i, \epsilon_j) \to (\epsilon'_i, \epsilon'_j)$, with

$$\epsilon_i + \epsilon_j = \epsilon'_i + \epsilon'_j,$$

according to the uniform collision kernel

$$B(\epsilon, \epsilon_*, \epsilon', \epsilon'_*) = \frac{1}{\epsilon + \epsilon_* + 1} \mathbb{1}_{\{\epsilon+\epsilon_*=\epsilon'+\epsilon'_*\}} \mathbb{1}_{\left\{\{\epsilon,\epsilon_*\}\neq\{\epsilon',\epsilon'_*\}\right\}}$$

Absorbing state $\{0, .., 0\}$.

LLN for the empirical measure: discrete HBE

$$\partial_t f_t(\epsilon) = \sum_{\epsilon_*,\epsilon',\epsilon'_*} B(\epsilon, \epsilon_*, \epsilon', \epsilon'_*) \left[f_t(\epsilon') f_t(\epsilon'_*) - f_t(\epsilon) f_t(\epsilon_*) \right].$$

Large Deviation Principle

Probability of "atypical paths" $\pi_t(\mathrm{d}v) = \tilde{f}_t(v)\,\mathrm{d}v,\ t \in [0, T]$

$$P(\pi^N \sim \pi) \sim e^{-N\mathcal{I}(\pi)}$$

Rate function $\mathcal{I}(\pi) = \mathcal{H}(\pi_0) + \mathcal{J}(\pi)$

- $\mathcal{H}$ "static contribution " (LDP of the initial distribution).
- $\mathcal{J}$ "dynamical contribution" ($\tilde{f}_t$ does not solves HBE).

Kac's walk, empirical observable

Kac's walk on

$$\Sigma_e^N := \{\mathbf{v} \in \mathbb{R}^{Nd} \colon \sum_i v_i = 0, \ \frac{1}{N} \sum_i |v_i|^2 = e\}$$

$\mathcal{T}_{i,j}$ set of collision times of the pair (v_i, v_j)

The *empirical flux* is the measure on $[0, T] \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ defined by

$$Q^N(\mathrm{d}t, \mathrm{d}v, \mathrm{d}v_*, \mathrm{d}v', \mathrm{d}v'_*) \\ = \frac{1}{N} \sum_{\{i,j\}} \sum_{\tau \in \mathcal{T}_{i,j}} \delta_\tau(\mathrm{d}t)\, \delta_{v_i(\tau^-)}(\mathrm{d}v)\, \delta_{v_j(\tau^-)}(\mathrm{d}v_*)\, \delta_{v_i(\tau)}(\mathrm{d}v')\, \delta_{v_j(\tau)}(\mathrm{d}v'_*)$$

Q^N records the collision times and incoming/outgoing velocities

Balance equation

π^N empirical measure, Q^N emprirical flow.

For any $\phi \in C_b(\mathbb{R}^d)$

$$\pi_T^N(\phi_T) - \pi_0^N(\phi_0) = \int Q^N(\bar{\nabla}\phi),$$

where

$$\bar{\nabla}\phi := \phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*).$$

We call (π^N, Q^N) the empirical **measure-flux pair**

LLN for the measure-flux pair

Given $m \in \mathcal{P}_e(\mathbb{R}^d)$, choose $v_1, .., v_N \in \Sigma_e$ with probability

$$\nu^N = m^{\otimes N}\Big(\cdot \mid \sum_{i=1}^N v_i = 0, \quad \frac{1}{N}\sum_{i=1}^N |v_i|^2 = e\Big)$$

As $N \uparrow +\infty$ (π^N, Q^N) converge to $(f_\cdot(v)\,\mathrm{d}v, Q^{f\otimes f})$, where

$$\mathrm{d}Q^{f\otimes f} = \frac{1}{2} f_t(v) f_t(v_*) B(v - v_*, \omega)\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$f_0\,\mathrm{d}v = \mathrm{d}m$ and $\forall \phi \in C_b(\mathbb{R}^d)$

$$\int \mathrm{d}v\, f_T(v)\phi(v) - \int \mathrm{d}v\, f_0(v)\phi(v) = \int Q^{f\otimes f}(\bar{\nabla}\phi) \quad (HBE)$$

LDP for microcanonical initial data

Assume that m has some exponential moments (+ technical assumptions).Choose $v_1, .., v_N \in \Sigma_e$ with probability $\nu^N = m^{\otimes N}(\cdot|\Sigma_e)$.

Theorem (B., Benedetto, Bertini, Caglioti (2024))

$LDP^{(*)}$ for the measure-flux pair (μ^N, Q^N) with rate function

$$I_e(\pi, Q) := H_e(\pi_0) + J_e(\pi, Q),$$

(*): LD upper bound, matching lower bound on a restricted class of paths.

Relative entropy functional

Given two probability measures μ, ν

$$\mathrm{Ent}(\mu|\nu) = \begin{cases} \int \mathrm{d}\mu \log \frac{\mathrm{d}\mu}{\mathrm{d}\nu} & \text{if } \mu \ll \nu \\ +\infty & \text{otherwise} \end{cases}$$

Given two finite positive measure Q, $\tilde{Q}$

$$\mathrm{Ent}(Q|\tilde{Q}) = \begin{cases} \int \mathrm{d}Q \log \frac{\mathrm{d}Q}{\mathrm{d}\tilde{Q}} - \mathrm{d}Q + \mathrm{d}\tilde{Q} & \text{if } Q \ll \tilde{Q} \\ +\infty & \text{otherwise} \end{cases}$$

LD rate function

Static part (see also (Kim, Ramanan '18), (Nam '20))

$$H_e(\pi_0) = \begin{cases} \mathrm{Ent}(\pi_0|m) + \gamma_0^*\big[e - \pi_0\big(|v|^2/2\big)\big] \text{ if } \pi_0 \in \mathcal{P}_{\le e} \\ +\infty \text{ otherwise} \end{cases}$$

Dynamical part

$$J_e(\pi, Q) = \begin{cases} \mathrm{Ent}(Q|Q^{\pi\otimes\pi}) & \text{if } \pi_t \in \mathcal{P}_{\le e}, \forall t \\ +\infty & \text{otherwise.} \end{cases}$$

where $\mathrm{d}Q^{\pi\otimes\pi} := \frac{1}{2}\,\mathrm{d}\pi_t \otimes \mathrm{d}\pi_t\, B\,\mathrm{d}\omega\,\mathrm{d}t$ "typical flow"

Comments

- the zero level set of I_e is the **unique energy conserving solution** to the HBE with initial value $\pi_0 = m$.
- LB lower bounds for pairs (π, Q) such that $Q(|v|^2 + |v_*|^2 + |v'|^2 + |v'_*|^2) < +\infty$ (conserving energy paths)
- LDP$^{(*)}$ for the discrete energy model.

Large deviation results

- C.Léonard (1995): LD upper bound for Kac's walk
- F.Rezakhanlou (1998): LDP for non-homogeneous case, finite set of velocities (conservation of momentum, not of energy)
- B.B., D.Benedetto, L.Bertini, C.Orrieri (2021): LDP$^{(*)}$ for a Kac's like walk (conservation of momentum, not of energy)
- D.Heydecker (2022): LDP$^{(*)}$ for Kac's walk; **emergence of a contradiction for Lu and Wennberg solutions**

Atypical paths

Discrete energy model

LDP$^{(*)}$ (B., Benedetto, Bertini, Caglioti '23)

Proposition (B., Benedetto, Bertini, Caglioti '23)

There exist atypical paths which condensate to the zero energy state in finite time. Their probability is exponentially small in the number of particle.

Discrete energy model

LDP$^{(*)}$ (B., Benedetto, Bertini, Caglioti '23)

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There exist atypical paths which condensate to the zero energy state in finite time. Their probability is exponentially small in the number of particle.

Construction.
Consider a trajectory with modified collision kernel, such that in each collision the whole energy is transferred to a single particle).

$f_t \to \delta_0 \quad$ weakly as $t \to \infty$
Condensation to the zero energy state when $t \to +\infty$

Condensation in finite time

Time reparametrization: $t^* \in (0, T)$, $\alpha(t) = \frac{t}{1-t/t^*}$,

$$\bar{f}_t(\epsilon) = \begin{cases} f_{\alpha(t)}(\epsilon) & t \in [0, t^*) \\ \delta_{\epsilon,0} & t \in [t^*, T], \end{cases}$$

Probability of the atypical path

$$P(\pi^N \sim \bar{f}_t(\epsilon)) \sim e^{-Nc},$$

with $c > 0$.

Kac's walk

Theorem (B., Benedetto, Bertini, Caglioti,(2024))

Given a non decreasing energy profile $\mathcal{E}(t)$ $t \in [0, T]$ piece-wise constant, with $\mathcal{E}(T) \leq e$, there exists a weak solution to the homogeneous BE with an energy profile $\mathcal{E}$ and its asymptotic probability is

$$e^{-NI_e(f\,\mathrm{d}v, Q^{f\otimes f})} = e^{-NH_e f\,\mathrm{d}v)}$$

Remark: the cost is due only to the initial distribution

Increasing energy solutions

Construction of Lu-Wennberg solutions

Sequence of one-particle initial distribution f_0^n:

- $f_0^n \to \bar{f}_0$ weakly
- $\lim_{n\uparrow+\infty} \int f_0^n(v)|v|^2 \, \mathrm{d}v = e > e_0 := \int \bar{f}_0(v)|v|^2 \, \mathrm{d}v$

For $t > 0$ f_t evolves following the Boltzmann equation, and $\mathcal{E}(0) = e_0 < e = \mathcal{E}(0^+)$, i.e. energy has a jump at $t = 0$.

The complete statistics of the collisions

Particle system with energy Ne, zero momentum

$q_{N,T}$ number of collisions per particle and per unit time.

Typical behavior

$$\lim_{T\to+\infty}\lim_{N\to\infty} q_{N,T} = \frac{1}{2}\int B(v-v_*,\omega)M_{e,0}(dv)M_{e,0}(dv_*)d\omega$$

Asymptotic probability of atypical flow

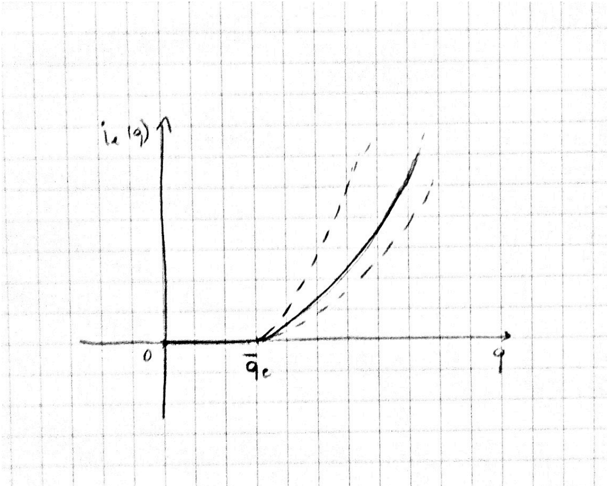
(B., Benedetto, Bertini, Heydecker,2025)

$$P^N\big(q_{N,T} \sim q\big) \sim e^{-NT\, i_e(q)}$$

with $i_e(q) = 0$ in $[0, \bar{q}_e]$ where

$$\bar{q}_e := \frac{1}{2}\int B(v - v_*, \omega) M_{e,0}(dv) M_{e,0}(dv_*) d\omega$$

The rate function



Anomalous diffusion

A linear Boltzmann equation for phonons

(B.,Olla, Spohn 2010) kinetic limit of an harmonic chain of oscillator perturbed by a stochastic conservative noise

$$\partial_t f_t(x,k) + \frac{1}{2\pi}\nabla\omega(k)\cdot\nabla_x f_t(x,k) = \int_{\mathbb{T}^d} \mathrm{d}k'\, \sigma(k,k')\big[f_t(x,k') - f_t(x,k)\big],$$

- $f_t(x,k)$ energy density of a phonon with wave number $k \in \mathbb{T}^d$
- $\nabla\omega$ velocity; ω dispersion relation of the harmonic lattice
- $\sigma(k,k')$ scattering kernel; $\sigma(k,k') \sim |k|^2|k'|^2$ for k, k' small (momentum conserving noise)

Kinetic description

The LBE is the Fokker-Planck equation of the random flight $(K_t, X_t)_{t\geq 0}$, where

- the wave number $(K_t)_{t\geq 0}$ is continuous time Markov chain on $\mathbb{T}^d$ with rate $r(k, \mathrm{d}k') = \sigma(k, k')\,\mathrm{d}k'$
- the position $X_t = \int_0^t \mathrm{d}s \nabla\omega(K_s)$ is an additive functional

Microscopic traslational invariance: $\nabla\omega(k) \sim k/|k|$, $|k| \ll 1$

$$d = 1, \quad \frac{1}{n^{2/3}} X_{nt} \to Z_t \quad \text{3/2-stable symmetric Lévy process}$$

(Jara, Komorowski, Olla 2009), (B., Bovier 2010); (B., 2014) diffusion with an anomalous scaling in $d = 2$, diffusion for $d \geq 3$

LLN of empirical measure and flow

Empirical measure and flow (averaged on time) of the continuous time Markov chain K,

$$\bar{\mu}_T(f)(K) = \frac{1}{T}\int_0^T f(K_t), \quad \bar{Q}_T(F) = \frac{1}{T}\sum_{\tau_{jump}\in[0,T]} F(K_{\tau^-}, K_\tau)$$

for any $f \in C(\mathbb{T}^d)$, $F \in C(\mathbb{T}^d \times \mathbb{T}^d)$

[LLN]. As $T \to +\infty$

$$\bar{\mu}_T(f) \to \int_{\mathbb{T}^d} \mathrm{d}k\, f(k), \qquad \bar{Q}_T(F) \to \iint_{\mathbb{T}^d\times\mathbb{T}^d} \mathrm{d}k\, \mathrm{d}k'\, \sigma(k,k')F(k,k')$$

(B., Bertini 2015)

LDP

Theorem (B., Bertini 2015)

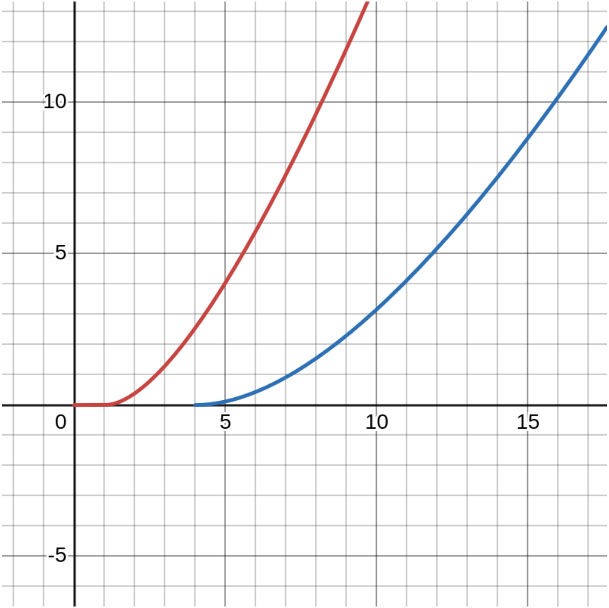
LDP for the measure-flow pair (μ, Q). The rate function vanishes on the set of measure-flow pairs of the form
$(\mu, Q) = \big(\alpha\lambda + (1-\alpha)\delta_0, \alpha\bar{Q}\big)$, *with* $\alpha \in [0,1]$.

Corollary

LDP for the observable $q_T := \frac{1}{T}\#$ of jumps in $[0, T]$, with rate function i which vanishes for any $q \in [0, \bar{q}]$, where $\bar{q}$ is the typical number of collisions.

See also (Lepri, 2024).

The rate function



Thank you.