Stable processes with reflections

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Fractional Laplacian

Let $d \in \mathbb{N} := \{1, 2, \dots\}$, $\alpha \in (0, 2)$, and

$$\nu(x) := c_{d,\alpha} |x|^{-d-\alpha}, \quad x \in \mathbb{R}^d.$$

The constant $c_{d,\alpha}$ is such that

$$\int_{\mathbb{R}^d} (1 - \cos \xi \cdot x) \nu(x) \mathrm{d}x = |\xi|^{\alpha}, \quad \xi \in \mathbb{R}^d.$$

Let $\nu(x, y) := \nu(y - x) = c_{d,\alpha} |y - x|^{-d-\alpha}$. We interpret $\nu(x, y)\mathrm{d}y$ as intensity of *jumps*. For $u \in C_c^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, we let

$$\Delta^{\alpha/2} u(x) := \lim_{\epsilon \to 0^+} \int_{\{|y-x|>\epsilon\}} \big[u(y) - u(x)\big] \nu(x, y) \mathrm{d}y.$$

This is the fractional Laplacian, often denoted by $-(-\Delta)^{\alpha/2}$.

Transition semigroup

By the Lévy–Khinchine formula, there are smooth probability densities $(p_t, t > 0)$ with $p_t * p_s = p_{t+s}$ and

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\xi\cdot x} p_t(x)\mathrm{d}x = \mathrm{e}^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d.$$

We denote $p_t(x, y) := p_t(y - x)$, for $t > 0$, $x, y \in \mathbb{R}^d$. Then,

$$p_t(x, y) = t^{-d/\alpha} p_1(t^{-1/\alpha}(x - y)) \approx t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}.$$

We get a Feller semigroup of operators on $C_0(\mathbb{R}^d)$,

$$P_t f(x) := \int_{\mathbb{R}^d} f(y) p_t(x, y)\mathrm{d}y, \quad x \in \mathbb{R}^d,\ t \geq 0,$$

with $\Delta^{\alpha/2}$ as generator. Of course, $P_t P_s = P_{t+s}$, $s, t > 0$.

The isotropic α-stable Lévy process in $\mathbb{R}^d$

Consider the space $\mathcal{D}[0,\infty)$ of càdlàg functions $\omega : [0,\infty) \to \mathbb{R}^d$.

We denote $X_t(\omega) := \omega_t$, $t \geq 0$, and $X_{t-} := \lim_{s\uparrow t} X_s$.

For $x \in \mathbb{R}^d$, $0 < t_1 < t_2 < \ldots < t_n$, and $A_1, A_2, \ldots, A_n \subset \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{P}^x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) \\
&:= \int_{A_1} \mathrm{d}x_1 \int_{A_2} \mathrm{d}x_2 \cdots \int_{A_n} \mathrm{d}x_n \\
&\quad \times\ p_{t_1}(x, x_1)\, p_{t_2-t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n).
\end{aligned}$$

This defines the measures $\mathbb{P}^x$ and the corresponding expectations $\mathbb{E}^x$.

We call $(X_t, \mathbb{P}^x)$ the isotropic α-stable Lévy process. It is strong Markov.

Trajectory of the process

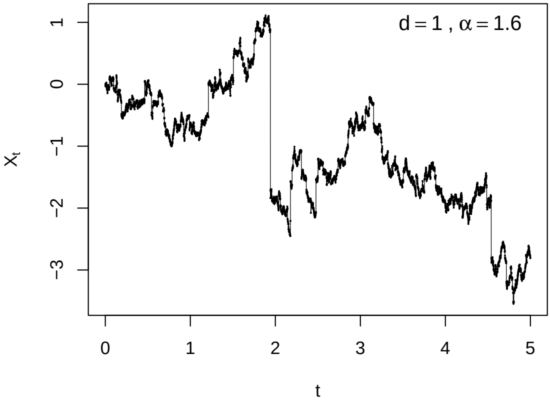


Figure: A trajectory of the symmetric α-stable Lévy process in $\mathbb{R}^1$ with $\alpha = 1.6$

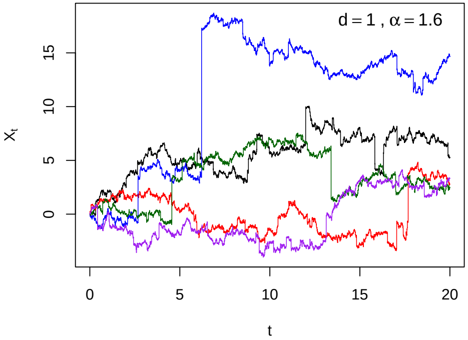


Figure: Trajectories of the symmetric α-stable Lévy process in $\mathbb{R}^1$ with $\alpha = 1.6$

The law of $(X_t, t \geq 0)$ under $\mathbb{P}^x$ is as $(x + X_t, t \geq 0)$ under $\mathbb{P}^0$.

Under $\mathbb{P}^0$, $(cX_t, t \geq 0)$ equals in law to $(X_{c^\alpha t}, t \geq 0)$.

Exit time and reflection

For any open set $U \subset \mathbb{R}^d$, the *time of the first exit* is

$$\tau_U := \inf\{t > 0 : X_t \notin U\}.$$

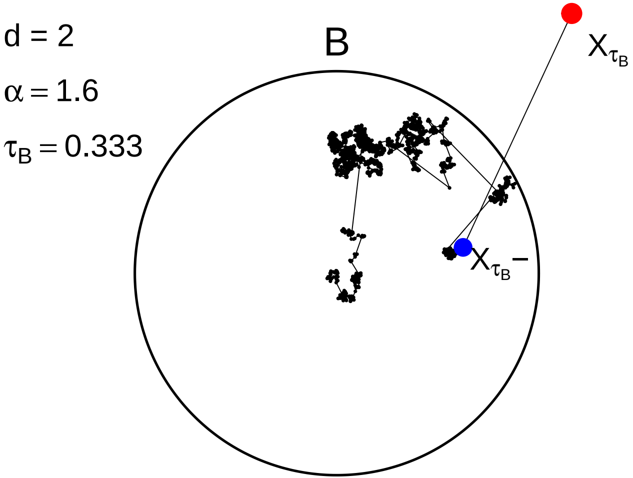
Fix an open bounded Lipschitz set $D \subset \mathbb{R}^d$.

We will consider τ_D, $X_{\tau_D -}$ and X_{τ_D}.

We have $\mathbb{P}^x(X_{\tau_D -} \in \partial D) = \mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$ for $x \in D$.

Goal: We want to *reflect* X_t at $t = \tau_D$ *back* to D.

Trajectory $t \mapsto X_t$, X_{τ_B}, and $X_{\tau_B -}$ for the unit ball B in $\mathbb{R}^2$



The killed stable semigroup and Ikeda–Watanabe formula

For $t > 0$, $x \in D$, and bounded or nonnegative functions f, we let

$$P_t^D f(x) := \mathbb{E}^x\big[t < \tau_D;\ f(X_t)\big] = \int_D f(y)\, p_t^D(x,y)\mathrm{d}y.$$

The *Dirichlet heat kernel* $p_t^D(x,y)$ is given *pointwise* by the Hunt formula:

$$p_t^D(x,y) = p_t(x,y) - \mathbb{E}^x\big[t < \tau_D; p_{t-\tau_D}(X_{\tau_D}, y)\big].$$

This *killed semigroup* P_t^D is (strong) Feller: $P_t^D B_b(D) \subset C_0(D)$.

The law of $(\tau_D, X_{\tau_D -}, X_{\tau_D})$, for $x \in D$, is given by the I-W formula:

$$\mathbb{P}^x[\tau_D \in J, X_{\tau_D -} \in A, X_{\tau_D} \in B] = \int_J \int_A \int_B p_u^D(x,y)\nu(y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}u.$$

Here $J \subset [0,\infty)$, $A \subset D$, $B \subset D^c$.

The reflections

We want a Markov process $(Y_t, t \geq 0)$ that agrees with X_t for $t < \tau_D$.

At $t = \tau_D$, we perform a *reflection*:

For $z := X_{\tau_D} \in D^c$, we let $Y_{\tau_D} := y \in D$ with distribution $\mu(z, \mathrm{d}y)$, etc.

This yields the (total) jump intensity kernel on D

$$\gamma(x, \mathrm{d}y) := \nu(x, \mathrm{d}y) + \int_{D^c} \nu(x, dz)\, \mu(z, \mathrm{d}y).$$

In short, $\gamma = \nu + \nu \mathbf{1}_{D^c} \mu$ on D.

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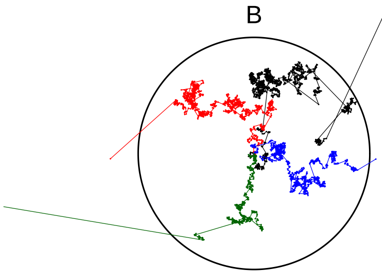
In short, $\gamma = \nu + \nu \mathbf{1}_{D^c} \mu$ on D. E.g., $\mu(z, \cdot) = \delta_0(\cdot)$ is "resetting to 0":



The reflection kernel μ

Assume μ is a Markov kernel from D^c to D, that is:

- for every $z \in D^c$, $\mu(z, \cdot)$ is a probability measure on D;
- for every Borel set $A \subset D$, the map

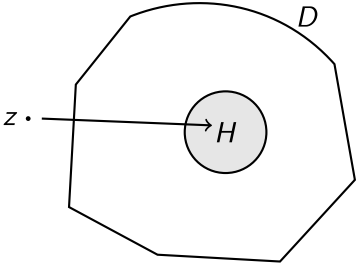
$$D^c \ni z \mapsto \mu(z, A)$$

  is Borel measurable.

Uniform return assumption (URA):

There is a compact set $H \Subset D$ such that

$$\inf_{z \in D^c} \mu(z, H) > 0.$$



Specific goals and questions

1. Construct the corresponding semigroup $(K_t, t > 0)$ and describe its long-time behavior.
2. Describe the generator and boundary conditions.
3. Construct the reflected process Y.
4. Can we calculate the stationary measure?
5. Can we use the strong Markov property at τ_D?

Comments on the literature

Diffusions and early reflections.

Similar “reflections” appeared first in Feller [12] for one-dimensional diffusions, as *instantaneous return processes* with non-local boundary conditions; see also Watanabe [20].

Multidimensional developments.

Galakhov and Skubachevskiĭ [14], Ben-Ari and Pinski [3], Arendt, Kunkel, and Kunze [1], Kunze [16].

General Markov process framework.

Ikeda, Nagasawa, and Watanabe [15], Meyer [17], Sharpe [18], Werner [21].

Specific jump processes

Basic idea.

For jump processes, one may define Y_{τ_D} depending on $(X_{\tau_D-}, X_{\tau_D})$.

Selected constructions.

Bogdan, Burdzy and Chen [5]: censored processes (return to X_{τ_D-}).

Barles, Chasseigne, Georgelin and Jakobsen [2]: geometric reflections (half-space).

See also Bobrowski [4].

Our contribution.

Bogdan and Kunze [8] and [9] and Bogdan, Fafuła, and Sztonyk [6].

The Servadei-Valdinoci process

Dipierro, Ros-Oton and Valdinoci [11] essentially postulate $\mu(z, \mathrm{d}y) = \nu(z, \mathrm{d}y)/\nu(z, D)$, $z \in D^c$. The kernel violates URA.

Vondraček [19] proposes a variant of [11] with exponential waiting time in D^c. See also Kassmann [13].

The important papers [2] and [11] address Neumann-type problems but do not construct the semigroup or the Markov process.

Our contribution.

The semigroup and the Servadei-Valdinoci process are defined by Bogdan, Fafuła, and Sztonyk in [6] but only for $D = (0, \infty) \subset \mathbb{R}$.

The Dirichlet heat kernel of the stable Lévy process X

The *survival probability*:

$$\mathbb{P}^x(\tau_D > t) = \int_D p_t^D(x,y)\,\mathrm{d}y, \quad t > 0,\, x \in D.$$

The *Green function*:

$$G_D(x,y) := \int_0^\infty p_t^D(x,y)\,\mathrm{d}t, \quad x, y \in D.$$

The *expected exit time*:

$$\mathbb{E}^x \tau_D = \int_D G_D(x,y)\,\mathrm{d}y, \quad x \in D.$$

Exercise: For all $t > 0$, $x \in D$,

$$\int_D p_t^D(x,y)\mathrm{d}y + \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_s^D(x,v)\nu(v,z) = 1.$$

Construction of the semigroup $(K_t, t > 0)$

For $t > 0$, $x, y \in D$, $n \in \mathbb{N}$,

$$\begin{aligned} p_0(t,x,y) &:= p_t^D(x,y), \\ p_n(t,x,y) &:= \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_D p_{n-1}(s,x,v)\nu \mathbf{1}_{D^c}\mu(v,\mathrm{d}w)p_0(t-s,w,y), \\ k_t(x,y) &:= \sum_{n=0}^{\infty} p_n(t,x,y). \end{aligned}$$

This is a perturbation series; see [7] and [10]. In short,

$$\begin{aligned} p_0(t) &:= p_t^D, \\ p_n(t) &:= \int_0^t p_{n-1}(s)\, \nu \mathbf{1}_{D^c}\mu \, p_0(t-s)\, ds, \qquad n \geq 1, \\ k_t &:= \sum_{n=0}^{\infty} p_n(t). \end{aligned}$$

Chapman-Kolmogorov equations

Corollary

$$\int_D k_t(x,y)k_s(y,z)\mathrm{d}y = k_{t+s}(x,z), \quad t>0, x,y\in D.$$

In short, $k_t k_s = k_{t+s}$. This follows from

Lemma

$$\sum_{m=0}^{n} \int_D k_m(s,x,z)\, k_{n-m}(t,z,y)\, dz = k_n(s+t,x,y),\ t>0, x,y\in D, n\in\mathbb{N}_0.$$

In short, $\sum_{m=0}^{n} k_m(s)k_{n-m}(t) = k_n(s+t)$.

Compare with $(s+t)^n = \sum_{m=0}^{n} \binom{n}{m} s^m t^{n-m}$ and $e^{s+t} = e^s e^t$.

Main results of [8]

Theorem

$$\int_D k_t(x,y)\,dy = 1, \qquad t>0,\ x\in D.$$

Proof. We first show that

$$k_t(x,D) := \int_D k_t(x,y)\,dy \le 1.$$

Indeed, $p_t^D(x,D) \le p(t,x,D) = 1$. Moreover,

$$\begin{aligned} p_1(t,x,D) &:= \int_0^t ds \int_D dv \int_D p_s^D(x,v)\,\nu \mathbf{1}_{D^c}\mu(v,dw)\,p_{t-s}^D(w,D) \\ &\le \int_0^t ds \int_D dv \int_D p_s^D(x,v)\,\nu(v,D^c). \end{aligned}$$

Hence, by our Exercise,

$$p_0(t,x,D) + p_1(t,x,D) \le 1, \quad \text{etc.}$$

Main results of [8]

For the equality $k_t(x, D) = 1$, we get and use the lower bound: For $t > 0$,

$$p_0(t, x, D) + p_1(t, x, D) \geq c > 0, \quad x \in D.$$

This follows from some deep estimates of p^D. Extensions??

Also, $k_t(x, y) \geq \delta > 0$, $x \in D$, $y \in F$, for some $F \Subset D$ with $|F| > 0$.

Theorem

(K_t) has a unique stationary distribution κ, i.e.,

$$\int_D K_t f(x)\, \kappa(dx) = \int_D f(x)\, \kappa(dx), \qquad t \geq 0,$$

and for each probability measure ρ on D,

$$\|\rho K_t - \kappa\|_{TV} \leq M e^{-\omega t}, \quad t > 0.$$

This is similar, but less explicit than [3]. (We give κ in [9].) Extensions??

The space $C_\mu(D)$ and the resolvent

Given $f \in C_b(D)$, define

$$f_\mu(x) := \begin{cases} f(x), & x \in D, \\ \mu(x, f), & x \in D^c. \end{cases}$$

Assume $z \mapsto \mu(z, \cdot)$ is weakly continuous at ∂D^c. Let

$$C_\mu(D) := \{ f \in C_b(D) : f_\mu \text{ is continuous on } \overline{D} \}.$$

For $\lambda > 0$, $R_\lambda f(x) := \int_0^\infty e^{-\lambda t} K_t f(x)\, dt$. Then $\overline{R_\lambda(C_b(D))} = C_\mu(D)$.
Informally: The boundary condition on D^c is:

$$u(z) = \int_D u(x) \mu(z, \mathrm{d}x), \quad z \in D^c.$$

Remarks and issues

1. (K_t) is a C_b-semigroup and has the strong Feller property, but it is not Feller (on $C_0(D)$) nor symmetric nor bounded on $L^2(D)$ in general.
2. The existence of (Y_t) requires a separate approach. In [9], we use the potential theory of Bliedtner and Hansen.
3. Concatenation of Markov processes should also apply—also called piecing-out, resetting, resurrection, instantaneous return, Neumann-type conditions.
4. Test functions $C_c^\infty(D)$ are not in the domain of the generator.
5. The range of the resolvent is a specific function space with boundary condition expressed via μ.
6. It is convenient to use the Dynkin operator to describe the generator.
7. This is about constructing new semigroups by positive nonlocal perturbations of P_t^D.
8. Reflected trajectories in models without URA can accumulate at the boundary.

The ladder space

We want to keep track of the number of (attempted) jumps to D^c.

To this end, we define the ladder space

$$\mathbb{D} := \mathbb{N}_0 \times D.$$

An element of $\mathbb{D}$ is of the form

$$x = (m, x), \qquad m \in \mathbb{N}_0,\ x \in D.$$

We endow $\mathbb{D}$ with the product topology and σ-algebra.

The (lifted) kernel on the ladder space

We define $\mathbb{k} : (0,\infty) \times \mathbb{D} \times \mathbb{D} \to [0,\infty)$ by

$$\mathbb{k}(t,(m,x),(n,y)) := \begin{cases} k_{n-m}(t,x,y), & n \geq m, \\ 0, & n < m. \end{cases}$$

The function $\mathbb{k}$ is a transition density on $\mathbb{D}$ with respect to

$$d\mathbb{y} := \text{counting measure on } \mathbb{N}_0 \; \times \; \text{Lebesgue measure on } D;$$

For $\mathbb{x} = (m,x)$, $\mathbb{y} = (n,y)$, $\mathbb{z} = (\ell,z)$,

$$\int_{\mathbb{D}} \mathbb{k}(t,\mathbb{x},\mathbb{y})\,\mathbb{k}(s,\mathbb{y},\mathbb{z})\,d\mathbb{y} = \mathbb{k}(t+s,\mathbb{x},\mathbb{z}).$$

This follows from the Lemma. In particular, for $(m,x) \in \mathbb{D}$,

$$\sum_{n=m}^{\infty} \int_D \mathbb{k}(t,(m,x),(n,y))dy = 1.$$

The ladder process

The kernel $\mathbb{k}$ defines a Markov semigroup on $\mathbb{D}$.

We obtain a Markov process

$$(\mathbb{Y}_t, t \geq 0), \qquad \mathbb{Y}_t \in \mathbb{D}.$$

We write

$$\mathbb{Y}_t = (N_t, Y_t),$$

where

- $N_t \in \mathbb{N}_0$ is the level,
- $Y_t \in D$ is the spatial component.

Interpretation

The process $(\mathbb{Y}_t)$ evolves as follows:

- Between jumps of N_t, the spatial component behaves like the killed process in D.
- When a jump occurs, the level increases:

$$N_t \to N_t + 1.$$

- The spatial component is redistributed according to $\nu \mathbf{1}_{D^c} \mu$.

Thus, N_t counts the number of returns to D.

Projection onto D

The reflected stable process can be obtained as the projection:

$$Y_t := \pi(\mathbb{Y}_t), \qquad \pi(m, x) = x.$$

Note that the $\mathbb{P}^{(x,m)}$ distribution of (Y_t) depends only on x, not on m. It follows that the process $\pi(\mathbb{Y}_t)$ indeed has the transition kernel

$$k_t(x, y) = \sum_{n=0}^{\infty} k_n(t, x, y).$$

The counting process (N_t) is not Markovian.

Filtration and stopping times

Let $(\mathbb{Y}_t, t \geq 0)$ be the ladder process on $\mathbb{D}$.

We consider its natural filtration

$$\mathcal{F}_t := \sigma(\mathbb{Y}_s : 0 \leq s \leq t).$$

Then,

$$\tau_D = \inf\{t > 0 : N_t > N_0\}$$

is a stopping time with respect to $(\mathcal{F}_t)$.

Strong Markov property at τ_D

The process $(\mathbb{Y}_t)$ is Hunt, hence strong Markov with respect to $(\mathcal{F}_t)$.

Thus, for bounded measurable F and $t \geq 0$,

$$\mathbb{E}^{\mathbb{x}}\left[F(\mathbb{Y}_{\tau_D+t}) \,\middle|\, \mathcal{F}_{\tau_D}\right] = \mathbb{E}^{\mathbb{Y}_{\tau_D}}[F(\mathbb{Y}_t)]\,.$$

This yields a formula for the stationary measure of K_t, as in [3].

It also follows that (Y_t) is a Hunt process with infinite lifetime.

Stationary measure: explicit formula

Assume that there exists a probability measure m on D such that

$$\mu(w, dz) = m(dz), \qquad w \in D^c.$$

Then the stationary density κ of (K_t) is given by

$$\kappa(y) = c \int_D G_D(x, y)\, m(dx),$$

i.e.

$$\kappa = c\, mG_D,$$

where G_D is the Green function of X and D and c is normalization.

Hunt process

The process $(Y_t, t \geq 0)$ associated with (K_t) is a Hunt process on D with infinite lifetime.

In particular:

- it is strong Markov,
- it has right-continuous paths with left limits,
- it is quasi-left continuous.

This follows from the construction of two excessive functions with different asymptotics at the boundary of $\mathbb{D}$ for the ladder process $(\mathbb{Y}_t)$, in fact for the semigroup K_t; a construction based on the properties of the killed process X on D.

Existence of an excessive function

Let $\delta_D(x) = \mathrm{dist}(x, \partial D)$, $x \in \mathbb{R}^d$.

As usual, $v \geq 0$ is called λ-supermedian for (K_t) if

$$e^{-\lambda t} K_t v \leq v, \qquad t > 0.$$

It is called λ-excessive if, additionally,

$$e^{-\lambda t} K_t v(x) \to v(x) \quad \text{as } t \downarrow 0.$$

Theorem (3.1)

Let $\lambda > 0$. There exists a continuous function

$$v : D \to (0, \infty)$$

which is λ-excessive for K_t and satisfies

$$v(x) \to \infty \quad \text{as } \delta_D(x) \to 0.$$

Lemma (3.5)

Let $\lambda \geq 0$ *and* $\alpha \in (0,1]$. *If* h *is* λ*-supermedian (excessive) for* (K_t) *on* D*, then*

$$\mathbb{h}(m,x) := \alpha^m h(x), \qquad (m,x) \in \mathbb{D},$$

is λ*-supermedian (excessive) for* $(\mathbb{k}_t)$.

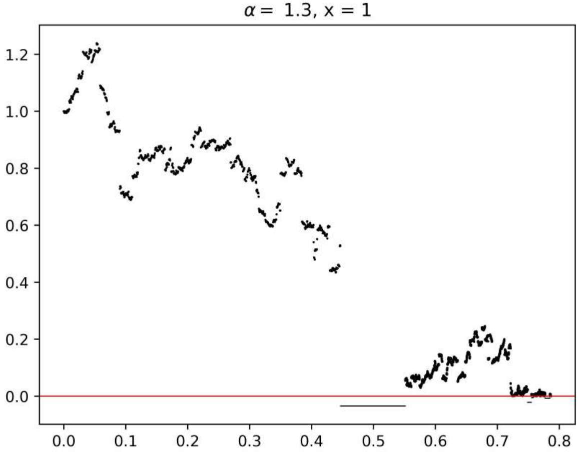
Thus, excessive functions lift multiplicatively along the levels.

The Servadei-Valdinoci process

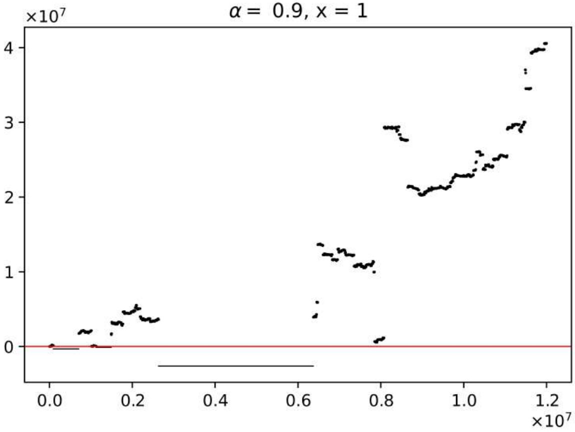
These are different reflections from $(-\infty)$ to $(0, \infty)$, corresponding to

$$\nu(x, y)\mathbf{1}_{x>0 \text{ or } y>0}.$$

Here are a few pictures showing the behavior near the interface 0.



$\times 10^7$ $\alpha = 0.9$, x = 1



α = 0.9, x = 1
×10^7
4
3
2
1
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
×10^7

Summary

In [8], we propose a framework for constructing semigroups with a specific reflection mechanism from the killed semigroup. The restriction to $\Delta^{\alpha/2}$ can easily be relaxed, but the assumption URA is restrictive. In [9], we construct the corresponding Hunt by constructing a barrier and using the potential theory of Bliedtner and Hansen.

This area of research is motivated by the Neumann-type boundary-value problems [2, 11] and by the problem of piecing-out or concatenation of Markov processes in the sense of Ikeda, Nagasawa and Watanabe [15], Sharpe [18] and Werner [21].

Besides construction, questions arise on large-time and boundary behavior of the semigroup (process) and on applications to nonlocal differential equations with those boundary conditions.

[1] W. Arendt, S. Kunkel, and M. Kunze.
Diffusion with nonlocal boundary conditions.
J. Funct. Anal., 270(7):2483–2507, 2016.

[2] G. Barles, E. Chasseigne, C. Georgelin, and E. R. Jakobsen.
On Neumann type problems for nonlocal equations set in a half space.
Trans. Amer. Math. Soc., 366(9):4873–4917, 2014.

[3] I. Ben-Ari and R. G. Pinsky.
Spectral analysis of a family of second-order elliptic operators with nonlocal boundary condition indexed by a probability measure.
J. Funct. Anal., 251(1):122–140, 2007.

[4] A. Bobrowski.
Concatenation of dishonest Feller processes, exit laws, and limit theorems on graphs, 2022.

[5] K. Bogdan, K. Burdzy, and Z.-Q. Chen.
Censored stable processes.
Probab. Theory Related Fields, 127(1):89–152, 2003.

[6] K. Bogdan, D. Fafuł a, and P. Sztonyk.
Nonlocal operators with Neumann conditions.
Dissertationes Math., 604:91, 2025.

[7] K. Bogdan, W. Hansen, and T. Jakubowski.
Time-dependent Schrödinger perturbations of transition densities.
Studia Math., 189(3):235–254, 2008.

[8] K. Bogdan and M. Kunze.
The fractional Laplacian with reflections.
Potential Anal., 61(2):317–345, 2024.

[9] K. Bogdan and M. Kunze.
Stable processes with reflections.
Stochastic Process. Appl., 187:Paper No. 104654, 14, 2025.

[10] K. Bogdan and S. Sydor.
On nonlocal perturbations of integral kernels.
In *Semigroups of operators—theory and applications*, volume 113 of *Springer Proc. Math. Stat.*, pages 27–42. Springer, Cham, 2015.

[11] S. Dipierro, X. Ros-Oton, and E. Valdinoci.
Nonlocal problems with Neumann boundary conditions.
Rev. Mat. Iberoam., 33(2):377–416, 2017.

[12] W. Feller.
Diffusion processes in one dimension.
Trans. Amer. Math. Soc., 77:1–31, 1954.

[13] M. Felsinger, M. Kassmann, and P. Voigt.
The Dirichlet problem for nonlocal operators.
Math. Z., 279(3-4):779–809, 2015.

[14] E. I. Galakhov and A. L. Skubachevskiĭ.
On Feller semigroups generated by elliptic operators with integro-differential boundary conditions.
J. Differential Equations, 176(2):315–355, 2001.

[15] N. Ikeda, M. Nagasawa, and S. Watanabe.
A construction of Markov processes by piecing out.
Proc. Japan Acad., 42:370–375, 1966.

[16] M. Kunze.
Diffusion with nonlocal Dirichlet boundary conditions on unbounded domains.
Studia Math., 253(1):1–38, 2020.

[17] P.-A. Meyer.
Renaissance, recollements, mélanges, ralentissement de processus de markov.
Annales de l'institut Fourier, 25(3-4):465–497, 1975.

[18] M. Sharpe.
General theory of Markov processes, volume 133 of *Pure and Applied Mathematics*.
Academic Press, Inc., Boston, MA, 1988.

[19] Z. Vondraček.
A probabilistic approach to a non-local quadratic form and its connection to the Neumann boundary condition problem.
Math. Nachr., 294(1):177–194, 2021.

[20] S. Watanabe.

On stable processes with boundary conditions.
J. Math. Soc. Japan, 14(2):170 – 198, 1962.

[21] F. Werner.
Concatenation and pasting of right processes.
Electron. J. Probab., 26:Paper No. 50, 21, 2021.