Stationary states for Lévy processes with resetting

Tomasz Grzywny
Wrocław University of Science and Technology

joint project with
K. Budnik, Z. Palmowski, P. Sztonyk, K. Szczypkowski and B. Trojan

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The model

Let

- $\mathbf{Y} = (Y_t)_{t \geq 0}$ be a Lévy process in $\mathbb{R}^d$, $d \in \mathbb{N}$,
- $(T_j : j \in \mathbb{N})$ Poisson arrival moments independent of $\mathbf{Y}$. (We assume $\lambda = 1$)

Definition. Given $c \in [0, 1)$, we define $\mathbf{X} = (X_t)_{t \geq 0}$ by setting

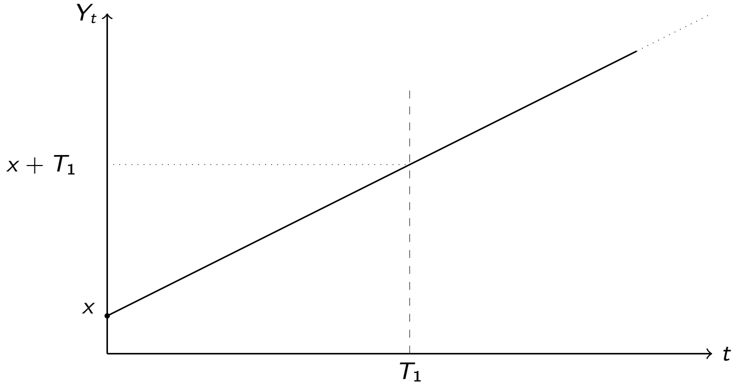
$$X_t = \begin{cases} Y_t, & \text{if } t < T_1, \\ cX_{T_n^-} + Y_t - Y_{T_n}, & \text{for } t \in [T_n, T_{n+1}) \end{cases}$$

where $n \in \mathbb{N}$.

X is obtained from **Y** by resetting

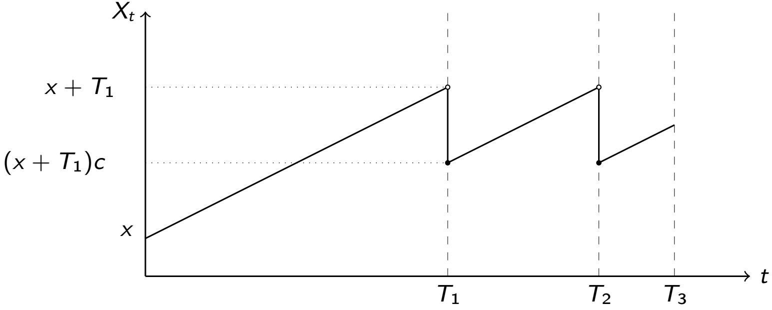
Constant drift

We consider $Y_t = x + t$.



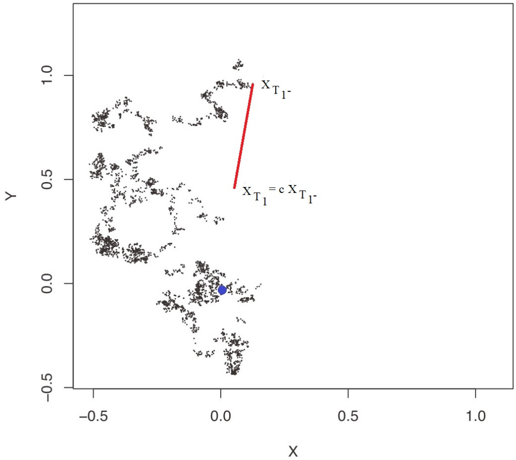
We multiply the position of the process at time T_1 by a factor $c \in [0, 1)$.

Additive-increase and multiplicative-decrease process

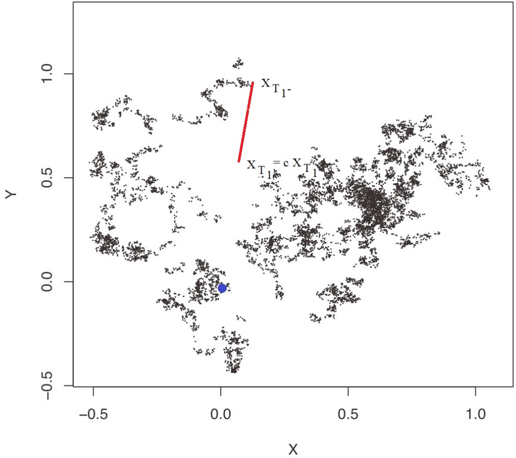


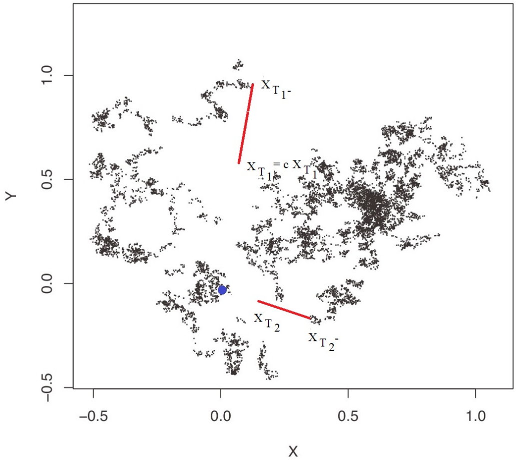
In the ideal TCP **congestion avoidance algorithm**: when a congestion signal is received (e.g. missing packets are detected), then the **window transferring size** is proportionally decreased and retransmission starts again. Otherwise, it grows with constant speed.

Search process



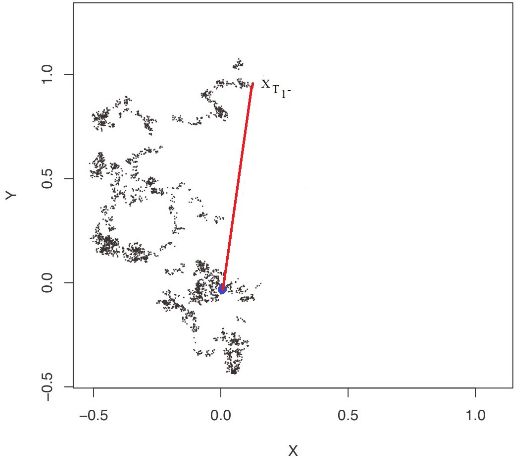
Search process





X_{T_1-}
$X_{T_1} = c\,X_{T_1-}$
X_{T_2}
X_{T_2-}
Y
X
1.0
0.5
0.0
−0.5
−0.5
0.0
0.5
1.0

Total resetting vs partial resetting



– Total resetting $c = 0$
– **Partial resetting** $c \in (0, 1)$

Goals

We aim at:

- convergence of X_t to X_∞ as $t \to \infty$,
- moments (and convergence) of X_t and X_∞,
- distribution/density of $p(t; x, y) \sim X_t$, $\rho(y) \sim X_\infty$,
- asymptotics of $p(t; x, y)$ as $t \to \infty$ and $y = y(t)$,
- estimates of $p(t; x, y)$.

The semigroup

For $\mathbf{Y} = (Y_t)_{t\geq 0}$ that is absolutely continuous we let

$$T_t f(x) = \mathbb{E}^x f(Y_t) = \int_{\mathbb{R}^d} p_0(t; x, y) f(y)\, \mathrm{d}y.$$

For $\mathbf{X} = (X_t)_{t\geq 0}$ we let

$$P_t f(x) = \mathbb{E}^x f(X_t) = \int_{\mathbb{R}^d} p(t; x, y) f(y) \mathrm{d}y.$$

The generator

Let $(L, D(L))$ be the generator of $\{T_t\}$ on $C_0(\mathbb{R}^d)$.

Fact. Let $c \in (0,1)$. The generator $(G, D(G))$ of $\{P_t\}$ is

$$Gf(x) = Lf(x) + f(cx) - f(x)$$

and $D(G) = D(L)$.

Formally,

$$G^* f(y) = L^* f(y) + \frac{1}{c} f(y/c) - f(y)$$

as well as (heat equation, **Fokker-Planck** equation)

$$\partial_t\, p(t; x, y) = G_x\, p(t; x, y), \qquad \partial_t\, p(t; x, y) = G_y^*\, p(t; x, y)$$

and (**harmonicity**)

$$G^* \rho(y) = 0\,.$$

Additive-increase and multiplicative-decrease process

Literature:

- Ott, Kemperman, Mathis (1996)
- Dumas, Guillemin, Robert (2002)
- Guillemin, Robert, Zwart (2004)
- Kemperman, Ott (2008), Leeuwaarden, Löpker (2008)
- Leeuwaarden, Löpker, Ott (2009)
- Baccelli, Carofiglio, Foss (2008)

Question: can we replace drift with other stable subordinator or multi-dimensional process?

Comment: one-dimensional Brownian motion + resetting + other mechanisms

- Bello, Chechkin, Hartmann, Palmowski, Metzler (2023)
- Kolb, Wübker (2024)
- Dybiec, Żbik (2024) (Lévy flights)
- Boxma, Kella, Perry (2025)
- Colantoni, D'Ovidio, Pagnini (2025)

One-dimensional Gaussian case

Inverting (Bello, Chechkin, Hartmann, Palmowski, Metzler (2023)):

$$p(t;x,y) = e^{-t}\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-x)^2}{2t}\right) + e^{-t}\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!\Gamma\left(n+\frac{1}{2}\right)}\times$$

$$\sum_{m=0}^{n}\frac{1}{(c^{-2};c^{-2})_m\,(c^2;c^2)_{n-m}}c^{-2m}\Bigg[(n-1)!c^m\sqrt{\frac{1}{2}t}\,{}_1F_1\left(-n+\frac{1}{2};\frac{1}{2};-\frac{(y-c^nx)^2}{2c^{2m}t}\right)$$

$$-\Gamma\left(n+\frac{1}{2}\right)|y-c^nx|\;{}_1F_1\left(1-n;\frac{3}{2};-\frac{(y-c^nx)^2}{2c^{2m}t}\right)\Bigg]$$

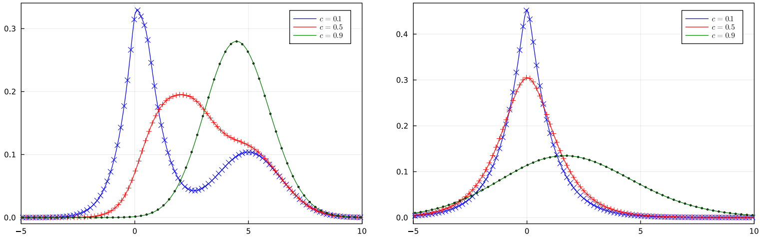
and

$$\rho(x) = \frac{1}{\sqrt{2}(c^2;c^2)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{c^n}{(c^2;c^2)_n}e^{-\sqrt{2}c^{-n}|x|}.$$

where ${}_1F_1$ denotes the Kummer confluent hypergeometric function, and $(a;q)_n$ is the q-Pochhammer symbol

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

One-dimensional Gaussian case



Transition density $p(t; x, y)$ for $x = 5$ and $t = 1$ on the left panel and $t = 10$ on the right panel.

The perturbation formula gives

$$p(t;x,y) = e^{-t}p_0(t;x,y) + \int_0^t \int_{\mathbb{R}^d} e^{-s}p_0(s;x,z)p(t-s;cz,y)\,\mathrm{d}z\,\mathrm{d}s.$$

Fact. The unique probabilistic solution can be written in the form

$$p(t;x,y) = e^{-t}\sum_{j=0}^{\infty} p_j(t;x,y)$$

where

$$p_{j+1}(t;x,y) = \int_0^t \int_{\mathbb{R}^d} p_0(s;x,z)p_j(t-s;cz,y)\,\mathrm{d}z\,\mathrm{d}s.$$

If $c = 0$

Fact

$$p(t; x, y) = e^{-t} p_0(t; x, y) + \int_0^t e^{-s} p_0(s; 0, y) \, \mathrm{d}s.$$

Strictly α-stable processes

We assume that $\mathbf{Y}$ is a strictly α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2]$ that admits a transition density.

Then

$$p_0(t; x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y-x, z\rangle} e^{-t\Psi(z)} \, \mathrm{d}z$$

and

$$t\Psi(z) = \Psi(t^{1/\alpha} z), \qquad p_0(tu; x, y) = t^{-d/\alpha} p_0(u; t^{-1/\alpha} x, t^{-1/\alpha} y).$$

Main examples:

– Brownian motion in $\mathbb{R}^d$

$$\Psi(z) = |z|^2 .$$

– isotropic α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$

$$\Psi(z) = |z|^\alpha .$$

– cylidrical α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$

$$\Psi(z_1, \ldots, z_d) = |z_1|^\alpha + \ldots + |z_d|^\alpha .$$

– α-stable subordinator with $\alpha \in (0, 1)$

Recall that

$$p(t;x,y) = e^{-t} \sum_{j=0}^{\infty} p_j(t;x,y).$$

Let $m = c^{\alpha}$.

Theorem

For all $n \in \mathbb{N}$, $t > 0$, $x, y \in \mathbb{R}^d$,

$$p_n(t;x,y) = t^n \int_0^1 p_0(tu; c^n x, y) P_n(u) \, \mathrm{d}u$$

where $(P_n : n \in \mathbb{N})$ are defined by

$$P_1(u) = \frac{1}{1-m} \mathbb{1}_{(m,1]}(u),$$

$$P_{n+1}(u) = (u - m^{n+1})_+^n \int_u^1 \frac{P_n(v)}{(v - m^{n+1})^{n+1}} \, \mathrm{d}v \,.$$

For all $t > 0$, $x, y \in \mathbb{R}^d$,

$$p(t; x, y) = e^{-t} p_0(t; x, y) + e^{-t} \sum_{j=1}^{\infty} t^j \int_0^1 p_0(tu; c^j x, y) P_j(u) \, \mathrm{d}u \,.$$

Corollary

For all $n \in \mathbb{N}$, $t > 0$, $y \in \mathbb{R}^d$,

$$p(t; 0, y) = \int_0^\infty p_0(u; 0, y) \mu_t(\mathrm{d}u)$$

where

$$\mu_t(\mathrm{d}u) = e^{-t} \delta_t(\mathrm{d}u) + e^{-t} \sum_{j=1}^{\infty} t^j P_j\left(\frac{u}{t}\right) \frac{\mathrm{d}u}{t}.$$

Remark: For $c = 0$ we have $P_{j+1}(v) = (1 - v)^j / j!$.

Strictly α-stable processes with partial resetting

$$\mu_t(\mathrm{d}u) = e^{-t}\delta_t(\mathrm{d}u) + e^{-t}\sum_{j=1}^{\infty} t^j P_j\left(\frac{u}{t}\right)\frac{\mathrm{d}u}{t}$$

Theorem

The family of probability measures $(\mu_t : t > 0)$ *converges weakly to a probability measure* μ *which is uniquely characterized by its moments, that is*

$$\int_0^\infty u^k \mu(\mathrm{d}u) = \frac{k!}{(m;m)_k}, \qquad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

In fact, the measure μ *has moments of all orders* $\gamma \in \mathbb{R}$, *and*

$$\lim_{t\to\infty}\int_0^\infty u^\gamma \mu_t(\mathrm{d}u) = \int_0^\infty u^\gamma \mu(\mathrm{d}u) = \frac{\Gamma(\gamma+1)}{\Gamma_m(\gamma+1)}(1-m)^{-\gamma}$$

where

$$\Gamma_m(x) = (1-m)^{1-x}\frac{(m;m)_\infty}{(m^x;m)_\infty}.$$

Theorem

For every $x, y \in \mathbb{R}^d$,

$$\lim_{t\to+\infty} p(t; x, y) = \int_0^\infty p_0(u; 0, y)\, \mu(\mathrm{d}u) = \rho(y).$$

Moreover, for every $x \in \mathbb{R}^d$,

$$\lim_{t\to+\infty} \sup_{y\in\mathbb{R}^d} |p(t; x, y) - \rho(y)| = 0.$$

Furthermore, for every $x \in \mathbb{R}^d$,

$$\lim_{t\to+\infty} \int_{\mathbb{R}^d} |p(t; x, y) - \rho(y)|\, \mathrm{d}y = 0\,.$$

Strictly α-stable processes with partial resetting

Corollary (Moments of the ergodic measure)

For every $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} |y|^\gamma \rho(y) \mathrm{d}y = \frac{\Gamma(\gamma/\alpha + 1)}{\Gamma_m(\gamma/\alpha + 1)} (1-m)^{-\gamma/\alpha} \, \mathbb{E}|Y_1|^\gamma.$$

Corollary (Ergodic measure by β-potentials of $\mathbf{Y}$)

For every $y \in \mathbb{R}^d$,

$$\rho(y) = \frac{1}{(m,m)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{m^{\frac{1}{2}k(k-1)}}{(m,m)_k} \, U^{(m^{-k})}(y)$$

where, for $\beta > 0$,

$$U^{(\beta)}(y) = \int_0^\infty e^{-\beta u} \, p_0(u; 0, y) \, \mathrm{d}u.$$

Strictly α-stable processes with partial resetting

We have

- $\rho \in C_0^\infty(\mathbb{R}^d)$, $p \in C^\infty((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$,
- $\partial_t \, p(t; x, y) = G_x \, p(t; x, y) = G_y^* \, p(t; x, y)$ and $G^* \rho(y) = 0$.

Theorem

The process $(X_t)_{t \geq 0}$ gives a strongly continuous semigroup on $L^2(\mathbb{R}^d, \rho(y)\mathrm{d}y)$.

- *there is $g \in C_0^\infty(\mathbb{R}^d)$ such that $Gg \neq G^{\circledast} g$ in $L^2(\mathbb{R}^d, \rho(y)\mathrm{d}y)$, where*

$$\langle Gf, g \rangle_{d\rho} = \left\langle f, G^{\circledast} g \right\rangle_{d\rho}.$$

The process $(X_t)_{t \geq 0}$ has NESS (non-equilibrium stationary states).

- Brownian motion in $\mathbb{R}^d$, $d \in \mathbb{N}$. Then

$$\rho(y) = \frac{1}{(m,m)_\infty} \frac{|y|^{1-d/2}}{(2\pi)^{d/2}} \sum_{k=0}^{\infty} (-1)^k \frac{m^{\frac{1}{2}k(k-d/2)}}{(m,m)_k} K_{d/2-1}(m^{-\frac{1}{2}k}|y|)$$

and

$$\lim_{|y|\to+\infty} \frac{\rho(y)}{|y|^{-\frac{d-1}{2}} e^{-|y|}} = \frac{1}{2} \frac{1}{(m,m)_\infty} (2\pi)^{-\frac{d-1}{2}} .$$

- Brownian motion in $\mathbb{R}$. Then

$$\rho(y) = \frac{1}{2} \frac{1}{(m,m)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{m^{\frac{k^2}{2}}}{(m,m)_k} e^{-m^{-\frac{k}{2}}|y|}$$

and

$$\lim_{|y|\to+\infty} \rho(y)\, e^{|y|} = \frac{1}{2} \frac{1}{(m,m)_\infty} .$$

Not really possible to repeat for the isotropic α-stable process with $\alpha \in (0,2)$.

Isotropic α-stable process with resetting and $\alpha \in (0,2)$

Recall $m = c^\alpha$. Here $\nu(y) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}|y|^{-d-\alpha}$.

Theorem

For any fixed $\kappa_1, \kappa_2 \in (0,1)$,

$$\lim_{\substack{|y|\to+\infty \\ t\to+\infty}} \sup_{\substack{m\in(0,\kappa_1] \\ |x|\leq\kappa_2|y|}} \left|(1-m)\frac{p^{(m)}(t;x,y)}{\nu(y)} - 1\right| = 0\,.$$

Corollary

We have

$$\lim_{|y|\to+\infty} \frac{\rho(y)}{\nu(y)} = \frac{1}{1-m} \qquad \textit{and} \qquad \rho(y) \approx 1 \wedge \nu(y)\,.$$

Furthermore, for any fixed $\delta, r > 0$, *we get on* $[\delta,\infty) \times B_r \times \mathbb{R}^d$ *that*

$$p(t;x,y) \approx \rho(y)\,.$$

Other processes with jumps

We obtain similar results for

– cylidrical α-stable process in $\mathbb{R}^d$ with $\alpha \in (0, 2)$,

– α-stable subordinator with $\alpha \in (0, 1)$.

For $r \in [0,1)$, we set

$$\varphi(r) = \sum_{j=0}^{\infty} \frac{r^j}{(m;m)_{j+1}}.$$

Theorem (A)

Let $\delta > 0$. Then

$$p(t;0,y) = e^{-t}(4\pi t)^{-\frac{d}{2}} e^{-\frac{|y|^2}{4t}} \left(1 + \left(\frac{4t^2}{|y|^2}\right)\varphi\left(\frac{4t^2}{|y|^2}\right) + \mathcal{O}\left(\frac{t}{|y|^2}\right)\right)$$

as $t \to +\infty$ in the region

$$\frac{|y|^2}{4t^2} \geq 1 + \delta.$$

Brownian motion with partial resetting

Corollary

Let $\delta > 0$. *Then*

$$\frac{p(t;0,y)}{|y|^{-\frac{d-1}{2}} e^{-|y|}} \longrightarrow 0.$$

as $t \to +\infty$ *in the region*

$$\frac{|y|^2}{4t^2} \geq 1 + \delta.$$

Theorem (B)

Let $\delta > 0$. *Then*

$$p(t;0,y) = \frac{1}{2} \frac{1}{(m;m)_\infty} (2\pi)^{-\frac{d-1}{2}} |y|^{-\frac{d-1}{2}} e^{-|y|} \left(1 + \mathcal{O}\left(t^{-1}\right)\right)$$

as $t \to +\infty$ *in the region*

$$m^2 + \delta \leq \frac{|y|^2}{4t^2} \leq 1 - \delta.$$

Remark: For $c = 0$, and $\frac{|y|^2}{4t^2} = 1$ we have additional $\frac{1}{2}$.

Relativistic processes with total resetting

Let $\alpha \in (0,2), M > 0$ and Y_t be the relativistic stable process, i.e.,

$$\Psi(z) = \left(M^{2/\alpha} + |z|^2\right)^{\alpha/2} - M, \quad z \in \mathbb{R}^d.$$

Theorem

- *If $M < 1$ then*

$$p(t,x,y) \sim \frac{\alpha M^{\frac{d+\alpha-1}{2\alpha}}}{2^{\frac{d-\alpha+1}{2}} \pi^{\frac{d-1}{2}} \Gamma(1-\frac{\alpha}{2})(1-M)^2} \frac{e^{-M^{1/\alpha}|y|}}{|y|^{\frac{d+\alpha+1}{2}}}.$$

- *If $M = 1$ then*

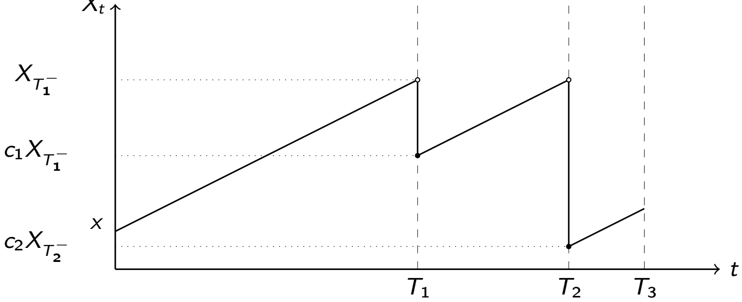
$$p(t,x,y) \sim \frac{e^{-|y|}}{2^{\frac{d+\alpha-1}{2}} \pi^{\frac{d-1}{2}} |y|^{\frac{d-\alpha+1}{2}}} \int_0^{2^{\alpha/2} \frac{t}{|y|^{\alpha/2}}} \eta(s,1)\, \mathrm{ds},$$

where η is the density of the $\alpha/2$–stable subordinator.

We consider $Y_t = x + t$ and a sequence

$$(c_j : j \in \mathbb{N})$$

of independent random variables with values in $[0, 1]$.



Assumptions (semi-group & scaling):

Chapman–Kolmogorov equations: for all $s, t > 0$, $x \in M$ and $A \in \mathscr{M}$,

$$p_0(s+t; x, A) = \int_M p_0(s; x, \mathrm{d}z) p_0(t; z, A).$$

The following scaling condition: for all $t > 0$, $x \in M$ and $\mathscr{M}$-measurable functions $f : M \to [0, \infty]$,

$$\int_M p_0(t; x, \mathrm{d}y) f(y) = \int_M p_0(1; \mathfrak{D}_{1/t}(x), \mathrm{d}y) f(\mathfrak{D}_t(y)).$$

Dilations: $\{\mathfrak{D}_t : t > 0\}$, that is, $\mathfrak{D}_s \mathfrak{D}_t = \mathfrak{D}_{st}$, $s, t > 0$.

Let $(m_j \in [0,1] : j \in \mathbb{N})^*$ be **a sequence of independent random variables**.

We study

$$p(t;x,A) = e^{-t} \sum_{n=0}^{\infty} \mathbb{E}[p_{n,0}(t;x,A)]$$

where, for $k \in \mathbb{N}_0$,

$$p_{0,k}(t;x,A) = p_0(t,x,A)$$

$$p_{n+1,k}(t;x,A) = \int_{\mathbb{R}} \mathrm{d}s \left(\int_M p_0(s;x,\mathrm{d}z) p_{n,k+1}(t-s;\mathfrak{D}_{m_{k+1}}(z),A) \right), \qquad n \in \mathbb{N}_0.$$

* $m_j = 0$ only if $\mathfrak{D}_t(x_0) = x_0$ for all $t > 0$, and $p_0(t;x,M) = 1$

The main object

$$\mu_t(\mathrm{d}u) = e^{-t} \sum_{j=0}^{\infty} t^j \mathbb{E}\Big[P_j\Big(\frac{\mathrm{d}u}{t}\Big)\Big].$$

Moments

$$\mathbb{M}_\gamma(t) = \int_0^\infty u^\gamma \mu_t(\mathrm{d}u) = e^{-t} t^\gamma \sum_{j=0}^{\infty} t^j \mathbb{A}(\gamma, j).$$

Assume that $(m_j \in [0,1] : j \in \mathbb{N})$ is **asymptotically K-periodic** (in distribution)*, and satisfies

$$\mathbb{E}[m_{j_0+jK}] < 1 \qquad \text{for all } j \in \mathbb{N}_0.$$

Theorem

We have $\mu_t \Rightarrow \mu$ *as* $t \to \infty$.

*otherwise there exist non-trivial counter-examples

Properties of μ

- μ is determined by its natural moments,

$$\lim_{t\to\infty}\int_0^\infty u^k \mu_t(\mathrm{d}u) = \int_0^\infty u^k \mu(\mathrm{d}u).$$

- μ can be expressed by

$$I = \int_0^\infty e^{-\xi_t}\mathrm{d}t$$

where ξ_t is a certain *Markov additive process* (MAP). More precisely,

$$\mu(\mathrm{d}u) = \frac{1}{K}\sum_{i=1}^{K}\mathbb{P}[I \in \mathrm{d}u \,|\Theta(0) = i].$$

Other properties under additional constraints on $(m_j : j \in \mathbb{N}_0)$.

As before in $\mathbb{R}^d$ or $\mathbb{R}$ take the distribution of:

– constant drift (understanding splines),

– Brownian motion,

– isotropic α-stable process, $\alpha \in (0, 2)$,

– cylidrical α-stable process, $\alpha \in (0, 2)$,

– α-stable subordinator, $\alpha \in (0, 1)$.

Examples

In $\mathbb{R}^3$ for $x = (x_1, x_2, x_3)$ and $t \geq 0$ consider $\delta_t(x) = (tx_1, t^2x_2, t^3x_3)$ and

$$\|x\| = |x_1| + \sqrt[2]{|x_2|} + \sqrt[3]{|x_3|}.$$

Then,

$$\|\delta_t(x)\| = t\|x\|.$$

For $Q = 1 + 2 + 3 = 6$ and $\alpha \in (0, 2)$, let

$$\nu(A) = \int_A \frac{\Omega(x)}{\|x\|^{Q+\alpha}} \mathrm{d}x.$$

There is an α **strictly stable semigroup of symmetric measures** on $L^2(\mathbb{R}^3)^*$:

$$\int_{\mathbb{R}^3} p_0(t; x, \mathrm{d}y) f(y) = \int_{\mathbb{R}^3} p_0(1; \mathfrak{D}_{1/t}(x), \mathrm{d}y) f(\mathfrak{D}_t(y)).$$

where $\mathfrak{D}_t = \delta_{t^{1/\alpha}}$. Then apply Dziubański (1991).

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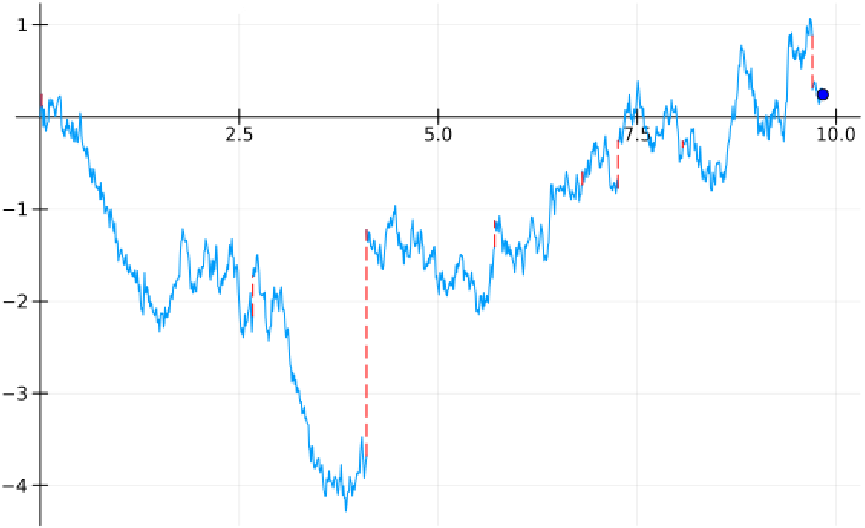
*stability in the sense that $\pi_t = \delta_{t^{1/\alpha}} \pi_1$

Brownian motion with partial resetting (with sequence of c_j)



Moment analysis of μ_t

Theorem

For all $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}$,

$$\int_0^1 u^\gamma P_n(u)\,\mathrm{d}u = \frac{1}{(m;m)_n}\left(\prod_{k=1}^{n} \frac{1-m^{k+\gamma}}{k+\gamma}\right) = \frac{\Gamma(\gamma+1)}{\Gamma_m(\gamma+1)}\frac{\Gamma_m(n+\gamma+1)}{\Gamma(n+\gamma+1)}\frac{1}{\Gamma_m(n+1)}.$$

Corollary

For all $t > 0$ and $\gamma \in \mathbb{R}$,

$$\int_0^\infty u^\gamma \mu_t(\mathrm{d}u) = e^{-t}\frac{\Gamma(\gamma+1)}{\Gamma_m(\gamma+1)}\sum_{j=0}^{\infty}\frac{t^{j+\gamma}}{\Gamma(j+\gamma+1)}\frac{\Gamma_m(j+\gamma+1)}{\Gamma_m(j+1)}.$$

Corollary

For all $t > 0$ and $k \in \mathbb{N}$,

$$\int_0^\infty u^k \mu_t(\mathrm{d}u) = e^{-t} \frac{k!}{(m;m)_k} \int_{mt}^{t} \int_{mu_{k-1}}^{u_{k-1}} \dots \int_{mu_1}^{u_1} e^{u_0} \, \mathrm{d}u_0 \dots \mathrm{d}u_{k-1}$$
$$= k! \sum_{j=0}^{k} \left(\prod_{i=0,\, i\neq j}^{k} \frac{1}{m^j - m^i} \right) e^{-(1-m^j)t} .$$

The idea of the proof of the **asymptotics** is based on the representation

$$p(t; 0, y) = \int_0^\infty p_0(u; 0, y)\, \mu_t(\mathrm{d}u) = \nu(y) \int_0^\infty \frac{p_0(u; 0, y)}{u\nu(y)} u\, \mu_t(\mathrm{d}u)$$

and the following asymptotic result

$$\lim_{u|y|^{-\alpha} \to 0^+} \frac{p_0(u; 0, y)}{u\nu(y)} = 1\,.$$