

# The Location of Extrema of Solutions to a Class of Non-Local Equations in Bounded Domains

József Lőrinczi

Alfréd Rényi Institute of Mathematics  
Budapest, Hungary

## Local vs non-local: a motivation

energy-momentum relation in classical mechanics

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correspondence principle: classical to quantum mechanics

$$p \mapsto -i\hbar\nabla \quad \text{and} \quad E \mapsto i\hbar\partial_t$$

leading to *relativistic Schrödinger equation*

$$i\hbar\frac{\partial\Psi(x, t)}{\partial t} = \left( (-\hbar^2 c^2 \Delta + m^2 c^4)^{1/2} - mc^2 \right) \Psi(x, t) + V(x)\Psi(x, t)$$

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separation of variables gives

$$\Psi(x, t) = \sum_{n=1}^{\infty} e^{-\frac{iE_n t}{\hbar}} \psi_n(x)$$

where

$$\left( (-\hbar^2 c^2 \Delta + m^2 c^4)^{1/2} - mc^2 + V(x) \right) \psi_n(x) = E_n \psi_n(x), \quad n \in \mathbb{N} \quad \text{not a PDE}$$

## Local vs non-local: a motivation

*relativistic Schrödinger operator*

$$H = (-\Delta + m^2)^{1/2} - m + V, \quad m \geq 0$$

Klein-Gordon 1926, Fock '26, Chandrasekhar '31, Weder '74, Herbst '77,  
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NB: since

$$\lim_{c \rightarrow \infty} (\sqrt{p^2 c^2 + m^2 c^4} - mc^2) = \frac{p^2}{2m}$$

we get *classical Schrödinger equation* in non-relativistic limit

$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi(x, t) + V(x) \Psi(x, t) \quad \text{is a PDE}$$

and *classical Schrödinger operator*

$$H = -\frac{1}{2} \Delta + V$$

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for  $m = 0$   $\longrightarrow$  *fractional Laplacian* of order  $0 < \alpha < 2$

$$\left( (-\Delta)^{\alpha/2} f \right) (x) = \frac{2^\alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{d/2} |\Gamma(-\frac{\alpha}{2})|} \lim_{\varepsilon \downarrow 0} \int_{|y-x| > \varepsilon} (f(y) - f(x)) \frac{dy}{|y-x|^{d+\alpha}}, \quad f \in C_c^\infty(\mathbb{R}^d)$$

for  $m > 0$   $\longrightarrow$  *fractional relativistic Laplacian* with rest mass  $m$

$$\begin{aligned} & \left( (-\Delta + m^{2/\alpha})^{\alpha/2} - m \right) f(x) = \\ &= \frac{2^{\frac{\alpha-d}{2}} m^{\frac{d+\alpha}{2\alpha}} \alpha}{\pi^{d/2} \Gamma(1 - \frac{\alpha}{2})} \lim_{\varepsilon \downarrow 0} \int_{|y-x| > \varepsilon} (f(y) - f(x)) \frac{K_{(d+\alpha)/2}(m^{1/\alpha} |y-x|)}{|y-x|^{(d+\alpha)/2}} dy, \quad f \in C_c^\infty(\mathbb{R}^d) \end{aligned}$$

non-local/pseudo-differential operators

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non-local/pseudo-differential operators

NB: A non-local operator if

$$\text{supp } Af \supset \text{supp } f$$

## Further non-local operators and applications

many more non-local operators

- $(-\Delta + m^{2/\alpha})^{\alpha/2} - m$
- $a(-\Delta)^{\alpha/2} - b\Delta$
- $a(-\Delta)^{\alpha/2} + b(-\Delta)^{\beta/2}$
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NB2: recently ongoing studies of equations in full space or bounded domains

- drift-fractional diffusion equations

$$\partial_t u + b \cdot \nabla u + (-\Delta)^{\alpha/2} u = 0$$

- Burgers equation with fractional diffusion

$$\partial_t u + u \partial_x u + (-\Delta)^{\alpha/2} u = 0$$

- non-local porous medium equation

$$\partial_t u + (-\Delta)^{\alpha/2} (u^m) = 0$$

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NB3: applications in

anomalous (non-Gaussian) transport theory, laser cooling technologies, geophysics, foraging models in biology, econophysics, financial maths,

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*Dirichlet exterior value problem*

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describes quantum particle in a cavity, vibrations of a membrane, plume of contaminated groundwater flow, etc

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*goal*: study  $x^*$  such that

$$|u(x^*)| = \max_{x \in \mathcal{D}} |u(x)|$$

## Problem

*approach*: via *stochastic representation*

$$u(x) = \mathbb{E}^x \left[ e^{-\int_0^t (q(X_s) - \lambda) ds} u(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right], \quad x \in \mathcal{D}, t \geq 0$$

with

$$\tau_{\mathcal{D}} = \inf \{ t > 0 : X_t \notin \mathcal{D} \}$$

*plan of this talk*:

- explain  $\Phi$ ,  $q$ ,  $(X_t)_{t \geq 0}$
- explain main result
- explain consequences

## Brownian motion

*Brownian motion*  $(B_t)_{t \geq 0}$  on  $(\Omega_W, \mathcal{B}(\Omega_W), P_W)$  with values in  $\mathbb{R}^d$

transition probability density

$$p(t, x) \equiv p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}, \quad t > 0, x \in \mathbb{R}^d$$

infinitesimal generator

$$A = \frac{1}{2} \Delta$$

heat semigroup

$$(e^{tA}f)(x) = \int_{\mathbb{R}^d} p_t(x-y)f(y)dy = \mathbb{E}_{P_W}[f(B_t)]$$

## Subordinators

*subordinator*  $(S_t)_{t \geq 0}$  on  $(\Omega_S, \mathcal{B}(\Omega_S), P_S)$

real-valued a.s. non-decreasing Lévy process starting at the origin, i.e.,

$$0 \leq s \leq t \implies P_S(S_t \geq S_s) = 1$$

- *Lévy-Khintchin formula*:  $\mathbb{E}_{P_S}[e^{-uS_t}] = e^{-t\Phi(u)}$ ,  $u \geq 0$

$$\Phi(u) = bu + \int_0^\infty (1 - e^{-uy})\mu(dy)$$

with characteristic pair  $(b, \mu)$  such that

$$b \geq 0, \quad \mu((-\infty, 0)) = 0, \quad \int_{\mathbb{R} \setminus \{0\}} (y \wedge 1)\mu(dy) < \infty$$

- $\Phi$  is a *Bernstein function*

$$\Phi \in \mathcal{B} = \left\{ f \in C^\infty(\mathbb{R}^+) : f \geq 0, (-1)^n \frac{d^n f}{dx^n}(x) \leq 0, \forall n \in \mathbb{N} \right\}$$

- one-to-one correspondence

$$\mathcal{B} \ni \Phi \longleftrightarrow (S_t^\Phi)_{t \geq 0} \text{ subordinator with } (b, \mu)$$

## Subordination

consider

$$A^\Phi = -\Phi(-\Delta), \quad \text{Dom } A^\Phi \subset L^2(\mathbb{R}^d)$$

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$$A^\Phi = -\Phi(-\Delta), \quad \text{Dom } A^\Phi \subset L^2(\mathbb{R}^d)$$

and semigroup  $\{e^{tA^\Phi} : t \geq 0\}$  giving

$$\begin{aligned} (f, e^{-t\Phi(-\Delta)}g) &= \int_0^\infty e^{-t\Phi(\lambda)} d(f, E_\lambda g) && \text{spectral rep } \Phi(-\Delta) \\ &= \int_0^\infty \mathbb{E}_{P_S}^0 [e^{-\lambda S_t^\Phi}] d(f, E_\lambda g) && \text{use } \mathbb{E}_{P_S}^0 [e^{-\lambda S_t^\Phi}] = e^{-t\Phi(\lambda)} \\ &= \mathbb{E}_{P_S}^0 \left[ \int_0^\infty e^{-S_t^\Phi \lambda} d(f, E_\lambda g) \right] && \text{Fubini} \\ &= \mathbb{E}_{P_S}^0 \left[ (f, e^{-S_t^\Phi(-\Delta)}g) \right] && \text{spectral rep } -\Delta \\ &= \mathbb{E}_{P_S}^0 \left[ \int_{\mathbb{R}^d} f(x) \mathbb{E}_{P_W}^x [g(B_{S_t^\Phi})] dx \right] && \text{semigroup for } -\Delta \end{aligned}$$

so that

$$(e^{-t\Phi(-\Delta)}f)(x) = \mathbb{E}_{P_W \times P_S}^{x,0} [f(B_{S_t^\Phi})]$$

## Source term

heat source

$q$  is  $\Phi$ -Kato class      $q = q^+ - q^-$  with  $q^- \in \mathcal{K}^\Phi$ ,  $q^+ \in \mathcal{K}_{loc}^\Phi$

where

$$h \in \mathcal{K}^\Phi \iff \lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ \int_0^t |h(X_s)| ds \right] = 0$$

$$h \in \mathcal{K}_{loc}^\Phi \iff h \mathbf{1}_K \in \mathcal{K}^\Phi, \forall \text{ compact } K \subset \mathbb{R}^d$$

NB:

- 1:  $L_{loc}^\infty(\mathbb{R}^d) \subset \mathcal{K}_{loc}^\Phi \subset L_{loc}^1(\mathbb{R}^d)$ ,  $\mathcal{K}^\Phi$  allows local singularities
- 2: Khasminskii's Lemma holds:

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ \int_0^t |q(X_s)| ds \right] = \beta < 1 \implies \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ e^{\int_0^t |q(X_s)| ds} \right] = \frac{1}{1 - \beta}$$

--> exponential integrability, allows to obtain Feynman-Kac representation

## Non-local heat operator

*non-local heat operator*

$$L = \Phi(-\Delta) + q, \quad \text{Dom } L = \text{Dom } \Phi(-\Delta) \cap \text{Dom } q$$

with

$\Phi \in \mathcal{B}_0 := \{\text{Bernstein function with parameter } b = 0\}$

$q$  in  $\Phi$ -Kato class

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### *Feynman-Kac formula*

with  $\Phi \in \mathcal{B}_0$ ,  $L = \Phi(-\Delta) + q$ ,  $(X_t)_{t \geq 0}$  generated by  $-\Phi(-\Delta)$ ,  $\Phi$ -Kato class  $q$

$$(e^{-tL}f)(x) = \mathbb{E}^x \left[ e^{-\int_0^t q(X_s) ds} f(X_t) \right], \quad t > 0, x \in \mathbb{R}^d$$

NB:  $e^{-tL} \mathbf{1}_{\mathbb{R}^d} \neq \mathbf{1}_{\mathbb{R}^d}$  but by unitary map image is Markov semigroup

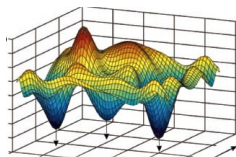
## Perturbed process

source term gives rise to drift and bias in jump rates – associated **Feller process**

$$\begin{aligned} X_t &= X_0 + B_t + \int_0^t \nabla \log u_1(X_s) ds \\ &\quad + \int_0^t \int_{|z| \leq 1} \frac{u_1(X_s + z) - u_1(X_s)}{u_1(X_s)} z \nu(z) dz ds \\ &\quad + \int_0^t \int_{|z| \leq 1} \int_0^\infty \mathbf{1}_{\left\{ w \leq \frac{u_1(X_{s-} + z)}{u_1(X_{s-})} \right\}} z \tilde{N}(ds, dz, dw) \\ &\quad + \int_0^t \int_{|z| > 1} \int_0^\infty \mathbf{1}_{\left\{ w \leq \frac{u_1(X_{s-} + z)}{u_1(X_{s-})} \right\}} z N(ds, dz, dw) \end{aligned}$$

with stationary distribution  $u_1^2 dx$ ,  $u_1$  eigenfunction to  $\lambda_1 = \inf \text{Spec } H$

random motion in penalty/reward landscape



## The role of the source term/potential

if *local time*  $(\ell_t^x)_{t \geq 0}$  exists (else use occupation measure), then

$$\int_0^t q(X_s) ds = \int_{\mathbb{R}^d} q(y) \ell_t^y dy$$

thus

$$e^{-tL} f(x) = \mathbb{E}^x [e^{-\int_{\mathbb{R}^d} q(y) \ell_t^y dy} f(X_t)]$$

- so  $q$  is a (soft) killing/mass generation rate
- hard killing corresponds to “infinite potential well”

$$q(x) = \begin{cases} v(x) & \text{if } x \in \mathcal{U} \\ \infty & \text{if } x \in \mathcal{U}^c \end{cases}$$

## Non-local domain heat operator

### *non-local domain heat operator*

define on  $L^2(\mathbb{R}^d)$  non-local heat operator  $L = \Phi(-\Delta) + q$  as form sum

take convex bounded domain  $\mathcal{D} \subset \mathbb{R}^d$

define domain operator  $L^{\mathcal{D}}$  as Friedrichs extension of  $L|_{C_c^\infty(\mathcal{D})}$

form-domain of  $L^{\mathcal{D}} =$  functions in form-domain of  $L$  and a.s. zero in  $\mathcal{D}^c$

## Feynman-Kac semigroup for domain operator

### Feynman-Kac semigroup

$$T_t^{\mathcal{D}} f(x) = \mathbb{E}^x \left[ e^{-\int_0^t q(X_s) ds} f(X_t) \mathbf{1}_{\{t < \tau_{\mathcal{D}}\}} \right], \quad x \in \mathcal{D}, t > 0, f \in L^2(\mathcal{D})$$

then for all  $t > 0$

- 1:  $T_t^{\mathcal{D}}$  integral operator with kernel  $T^{\mathcal{D}}(t, x, y) = T^{\mathcal{D}}(t, y, x)$ ,  $x, y \in \mathbb{R}^d$
- 2:  $\mathbb{R}^+ \times \mathcal{D} \times \mathcal{D} \ni (t, x, y) \mapsto T^{\mathcal{D}}(t, x, y) \in \mathbb{R}$  continuous
- 3:  $\{T_t^{\mathcal{D}} : t \geq 0\}$  strongly continuous on  $L^p(\mathcal{D})$ ,  $p \geq 1$ , and  $T_t^{\mathcal{D}} = e^{-tH^{\mathcal{D}}}$
- 4:  $\mathcal{D}$  outer cone property,  $f \in L^\infty(\mathcal{D}) \implies T_t^{\mathcal{D}} f$  continuous in  $\bar{\mathcal{D}}$  and 0 on  $\partial\mathcal{D}$
- 5: Hartman-Wintner:  $\lim_{|u| \rightarrow \infty} \frac{\Phi(|u|^2)}{\log |u|} = \infty \implies T_t^{\mathcal{D}}$  Hilbert-Schmidt on  $L^2(\mathcal{D})$

## Feynman-Kac formula for domain operator

recall Feynman-Kac formula

$$(e^{-tL}f)(x) = \mathbb{E}^x[e^{-\int_0^t q(X_s)ds} f(X_t)], \quad t > 0, x \in \mathbb{R}^d, f \in L^2(\mathbb{R}^d)$$

and eigenvalue equation

$$Lu = \lambda u, \quad u \in \text{Dom } L \setminus \{0\}$$

by e.v. equation

$$e^{-tL}u = e^{-\lambda t}u, \quad t \geq 0$$

- FK-formula gives

$$u(x) = e^{\lambda t}(e^{-tL}u)(x) = \mathbb{E}^x[e^{-\int_0^t (q(X_s) - \lambda)ds} u(X_t)], \quad t \geq 0, x \in \mathbb{R}^d$$

- Dirichlet exterior condition gives

$$\begin{aligned} u(x) &= \mathbb{E}^x[e^{-\int_0^t (q(X_s) - \lambda)ds} u(X_t)(\mathbf{1}_{\{t < \tau_D\}} + \mathbf{1}_{\{t \geq \tau_D\}})] \\ &= \mathbb{E}^x[e^{-\int_0^t (q(X_s) - \lambda)ds} u(X_t)\mathbf{1}_{\{t < \tau_D\}}] + \underbrace{\mathbb{E}^x[e^{-\int_0^t (q(X_s) - \lambda)ds} u(X_t)\mathbf{1}_{\{t \geq \tau_D\}}]}_{=0} \end{aligned}$$

## Non-local heat-Dirichlet eigenvalue problem

non-local heat-Dirichlet eigenvalue problem then

$$\begin{cases} Lu = \lambda u & \text{in } \mathcal{D} \\ u = 0 & \text{in } \mathcal{D}^c \end{cases}, \quad u \in \text{Dom } L \setminus \{0\} \iff L^{\mathcal{D}} u = \lambda u, \quad u \in \text{Dom } L^{\mathcal{D}} \setminus \{0\}$$

a countable set of eigenvalues/eigenfunctions exists

for  $q \not\equiv 0$

$$\lambda_1^{q, \mathcal{D}} < \lambda_2^{q, \mathcal{D}} \leq \dots \rightarrow \infty$$

for  $q \equiv 0$

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## Location of extrema

choose  $q \in L^\infty$

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key step    using  $u(x) = \mathbb{E}^x [e^{-\int_0^t (q(X_s) - \lambda) ds} u(X_t) \mathbf{1}_{\{t < \tau_D\}}]$

for global maximiser  $x = x^*$

$$\begin{aligned} |u(x^*)| &\leq e^{\lambda t} \mathbb{E}^{x^*} [e^{-\int_0^t q(X_s) ds} |u(X_t)| \mathbf{1}_{\{\tau_D > t\}}] \\ &\leq |u(x^*)| e^{\lambda t} e^{(\|q^-\|_{\infty, D} - \inf_D q^+) t} \mathbb{P}^{x^*}(\tau_D > t) \end{aligned}$$

giving

$$e^{t(\|q^-\|_{\infty, D} - \inf_D q^+ + \lambda)} \mathbb{P}^{x^*}(\tau_D > t) \geq 1, \quad \forall t \geq 0$$

## Location of extrema

for some  $t > 0$  derive uniform bound (independent of  $x^*$  or  $\mathcal{D}$ )

$$\mathbb{P}^{x^*}(\tau_{\mathcal{D}} > t) < \delta < 1$$

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- *by convexity*

with  $z \in \partial\mathcal{D}$  s.t.  $\text{dist}(x^*, z) = r$ , and half-space  $\mathcal{H} \subset \mathcal{D}^c$  s.t.  $\mathcal{H} \cap \mathcal{D} = \{z\}$

$$\mathbb{P}^{x^*}(\tau_{\mathcal{D}} \leq t) \geq \mathbb{P}^{x^*}(X_t \in \mathcal{H})$$

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$$\mathbb{P}^{x^*}(\tau_{\mathcal{D}} \leq t) \geq \mathbb{P}^{x^*}(X_t \in \mathcal{H})$$

then for  $s \geq r^2$

$$\begin{aligned} P_W^{x^*}(B_s \in \mathcal{H}) &= P_W^0(B_s^1 \geq r) = \frac{1}{\sqrt{4\pi}} \int_{\frac{r}{\sqrt{s}}}^{\infty} e^{-\frac{y^2}{4}} dy \\ &\geq \frac{1}{\sqrt{4\pi}} \int_1^{\infty} e^{-\frac{y^2}{4}} dy = F(-1) \end{aligned}$$

## Location of extrema

- *by subordination*

$$\begin{aligned}\mathbb{P}^{x^*}(X_t \in \mathcal{H}) &= \int_0^\infty P_W^{x^*}(B_s \in \mathcal{H})P_S(S_t \in ds) \\ &\geq \int_{r^2}^\infty P_W^{x^*}(B_s \in \mathcal{H})P_S(S_t \in ds) \geq F(-1)P_S(S_t \geq r^2)\end{aligned}$$

## Location of extrema

- *by subordination*

$$\begin{aligned}\mathbb{P}^{x^*}(X_t \in \mathcal{H}) &= \int_0^\infty P_W^{x^*}(B_s \in \mathcal{H})P_S(S_t \in ds) \\ &\geq \int_{r^2}^\infty P_W^{x^*}(B_s \in \mathcal{H})P_S(S_t \in ds) \geq F(-1)P_S(S_t \geq r^2)\end{aligned}$$

- *estimate*

$$P_S(S_t \leq r^2) = P_S(e^{-r^{-2}S_t} \geq e^{-1}) \leq e \mathbb{E}_{P_S}[e^{-r^{-2}S_t}] = e^{1-t\Phi(r^{-2})}$$

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- *choosing*

$$t = \frac{\varkappa}{\Phi(1/r^2)}, \quad \varkappa > 1$$

gives

$$\mathbb{P}^{x^*}(\tau_{\mathcal{D}} > t) < \delta \quad \text{with} \quad \delta = 1 - F(-1)(1 - e^{1-\varkappa})$$

## Location of extrema

optimising over  $x$  in

$$e^{t(\|q^-\|_{\infty, \mathcal{D}} - \inf_{\mathcal{D}} q^+ + \lambda)} \mathbb{P}^{x^*}(\tau_{\mathcal{D}} > t) \geq 1$$

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gives

**theorem**

if  $\Phi \in \mathcal{B}_0$ ,  $q \in L^\infty(\mathbb{R}^d)$ , then

$$\text{dist}(x^*, \partial\mathcal{D}) \geq \frac{1}{\sqrt{\Phi^{-1}\left(\frac{\|q^-\|_{\infty, \mathcal{D}} - \inf_{\mathcal{D}} q^+ + \lambda q, \mathcal{D}}{\theta}\right)}}$$

with universal constant

$$\theta = -\min_{x > 1} \frac{1}{x} \log(1 - F(-1)(1 - e^{1-x})) \approx 0.0833 \left(\approx \frac{1}{12}\right)$$

## Location of extrema

*weak lower scaling condition (WLSC):*  $\exists \gamma > 0, \underline{c} \in (0, 1]$

$$\Phi(au) \geq \underline{c} a^\gamma \Phi(u), \quad u > 0, a \geq 1$$

### theorem

if

$\Phi \in \mathcal{B}_0$ , WLSC with  $\gamma > 0$  and  $\underline{c} \in (0, 1]$

$q$  is  $\Phi$ -Kato class with  $q^- \in L^p(\mathbb{R}^d)$ ,  $p > \frac{d}{2\gamma}$  (possibly  $q \equiv 0$ )

$$\eta = 1 - \frac{d}{2\gamma p}$$

then

$\exists \Theta_1 = \Theta_1(d, \gamma, \underline{c}, \eta, \text{inrad } \mathcal{D}) > 0$  and  $\Theta_2 = \Theta_2(\eta) > 0$

$$\Theta_1 \|q^-\|_p^{1/\eta} - \inf_{\mathcal{D}} q^+ + \lambda^{q, \mathcal{D}} \geq \Theta_2 \Phi \left( \frac{1}{\text{dist}(x^*, \partial \mathcal{D})^2} \right)$$

## Consequences

- location  $x^*$  relates with a balance of survival times

mean survival time in  $\mathcal{D}$  under  $q$  using  $\mathbb{E}^{x^*} [e^{-\int_0^t q(X_s) ds} \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] \geq e^{-\lambda^{q, \mathcal{D}} t}$

$$\int_0^\infty \mathbb{E}^{x^*} [e^{-\int_0^t q(X_s) ds} \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] dt \geq \frac{1}{\lambda^{q, \mathcal{D}}} \quad (\text{whenever } \lambda^{q, \mathcal{D}} > 0)$$

mean survival time in  $\mathcal{B}_r(x)$  in absence of  $q$

$$c_1(d) \mathbb{E}^x [\tau_{\mathcal{B}_r(x)}] \leq \frac{1}{\Phi(1/r^2)} \leq c_2(d) \mathbb{E}^x [\tau_{\mathcal{B}_r(x)}]$$

e.g. if  $\|q^-\|_{\infty, \mathcal{D}} > 0$  and  $q^+ \equiv 0$ , then

$$\frac{\int_0^\infty \mathbb{E}^{x^*} [e^{-\int_0^t q(X_s) ds} \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] dt}{\mathbb{E}^{x^*} [\tau_{\mathcal{B}_{\text{inrad } \mathcal{D}}(x^*)}]} \geq \text{const}$$

- *Faber-Krahn isoperimetric inequality*

since  $B_r(x^*) \subset \mathcal{D}$  whenever  $r \leq \text{dist}(x^*, \partial\mathcal{D})$  and

$$|\mathcal{D}| \geq |B_r(x^*)| \geq \omega_d \left( \Phi^{-1} \left( \frac{\|q^-\|_{\infty, \mathcal{D}} - \inf_{\mathcal{D}} q^+ + \lambda^{q, \mathcal{D}}}{\theta} \right) \right)^{-d/2}$$

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gives **Faber-Krahn inequality**

$$|\mathcal{D}| \left( \Phi^{-1} \left( \frac{\|q^-\|_{\infty, \mathcal{D}} - \inf_{\mathcal{D}} q^+ + \lambda^{q, \mathcal{D}}}{\theta} \right) \right)^{d/2} \geq \omega_d$$

compare case with Laplacian

$$|\mathcal{D}| \|q + \lambda\|_{\infty}^{d/2} \geq j_{\frac{d}{2}-1}^d \omega_d \quad \text{De Carli, Hudson (2012)}$$

- *exit times*

$$\frac{C_1(d)}{\Phi(1/(\text{inrad } \mathcal{D})^2)} \leq \sup_{x \in \mathcal{D}} \mathbb{E}^x[\tau_{\mathcal{D}}] \leq \frac{C_2(d)}{\Phi(1/(\text{inrad } \mathcal{D})^2)}$$

$$\sup_{x \in \mathcal{D}} \mathbb{E}^x[\tau_{\mathcal{D}}^p] \geq \frac{\Gamma(p+1)}{(\lambda_1^{\mathcal{D}})^p}, \quad p \geq 1 \quad \text{and} \quad \mathbb{E}^x[e^{p \tau_{\mathcal{D}}}] \geq \frac{p}{\lambda_1^{\mathcal{D}} - p}, \quad p < \lambda_1^{\mathcal{D}}$$

- *safe distance and entropic repulsion*

NB: for given process  $x^*$  can get near to  $\partial\mathcal{D}$  if  $q$  has deep enough hole close by

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how close? for fixed  $\mathcal{D}, q$  def for  $u = u_1$

$$L^\Phi = \frac{1}{\sqrt{\Phi^{-1} \left( \frac{\|q^-\|_{\infty, \mathcal{D}} - \inf_{\mathcal{D}} q^+ + \lambda^{q, \mathcal{D}}}{\theta} \right)}} \leq \text{dist}(x^*, \partial\mathcal{D})$$

since

$$\underbrace{(-\Delta)^{\alpha/2}}_{\rightarrow \Phi_1} < \underbrace{(-\Delta + m^{2/\alpha})^{\alpha/2} - m}_{\rightarrow \Phi_2} < \underbrace{-\Delta}_{\rightarrow \Phi_3}$$

also

$$\lambda_{\Phi_1}^{q, \mathcal{D}} \leq \lambda_{\Phi_2}^{q, \mathcal{D}} \leq \lambda_{\Phi_3}^{q, \mathcal{D}}$$

and

$$L^{\Phi_3} \leq L^{\Phi_2} \leq L^{\Phi_1}$$

- *bounds on the spectrum*

probabilistic:  $\lambda^{q, \mathcal{D}} \geq \frac{1}{\int_0^\infty \mathbb{E}^{x^*} [e^{-\int_0^t q(X_s) ds} \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] dt}$

geometric:  $\lambda_1^{q, \mathcal{D}} \geq \theta \Phi \left( \frac{1}{(\text{inrad } \mathcal{D})^2} \right) - \|q^-\|_{\infty, \mathcal{D}} + \inf_{\mathcal{D}} q^+$

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- *Dirichlet eigenvalues*

$$\lambda_1^{\mathcal{D}} \geq \left( \frac{\Gamma(p+1)}{\sup_{x \in \mathcal{D}} \mathbb{E}^x[\tau_{\mathcal{D}}^p]} \right)^{1/p}, \quad \forall p \geq 1$$

$$\frac{1}{\sup_{x \in \mathcal{D}} \mathbb{E}^x[\tau_{\mathcal{D}}]} \leq \lambda_1^{\mathcal{D}} \leq \frac{C(d)}{\sup_{x \in \mathcal{D}} \mathbb{E}^x[\tau_{\mathcal{D}}]}$$

## Connection with other problems: hot-spots

- *hot-spots*

$$\begin{cases} \partial_t u = \Delta u & (t, x) \in (0, \infty) \times \mathcal{D} \\ u(t, x) = 0 & (t, x) \in (0, \infty) \times \partial\mathcal{D} \\ u(0, x) = \phi(x) & x \in \mathcal{D} \end{cases}$$

define

$$\mathcal{M}(t) = \left\{ x \in \mathcal{D} : u(t, x) = \max_{y \in \mathcal{D}} u(t, y) \right\}$$

then:  $\mathcal{M}(t) \subset \{x \in \mathcal{D} : \nabla u(t, x) = 0\} = \{\hat{x}(t)\}$ , and

$$\hat{x}(t) \rightarrow x^*, \quad \text{as } t \rightarrow \infty$$

Brascamp, Lieb (1976), Kawohl (1985), Sakaguchi (1992), Klamkin (1994),  
Magnanini, Sakaguchi (2004), Ishige, Kabeya (2012), ...

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NB: Rauch-conjecture for Neumann b.c. in general false

Rauch (1974), Bañuelos, Burdzy (1999), Burdzy, Werner (1999) ...

we get:  $\text{dist}(x^*, \partial\mathcal{D}) \geq \sqrt{\frac{\theta}{\lambda_1^{\mathcal{B}}}} \text{inrad } \mathcal{D}$

to compare with (for classical case)

$$\text{dist}(x^*, \partial\mathcal{D}) \geq c \text{inrad } \mathcal{D} \left( \frac{\text{inrad } \mathcal{D}}{\text{diam } \mathcal{D}} \right)^{d^2-1}$$

Brasco, Magnanini, Salani (2011)

## Connection with other problems: torsion

- *torsion*

$$\left\{ \begin{array}{ll} -\Delta v = 1 & \text{in } \mathcal{D} \\ v = 0 & \text{on } \partial D \end{array} \right. \quad \text{vs} \quad \left\{ \begin{array}{ll} -\Delta u = \lambda u & \text{in } \mathcal{D} \\ u = 0 & \text{on } \partial D \end{array} \right.$$

Makai (1962)

Cima, Derrick (2011): triangle

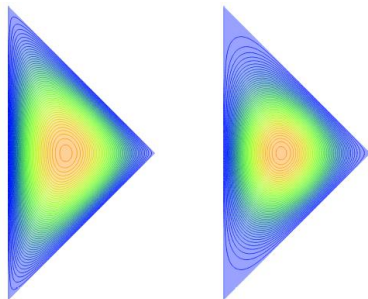
Benson, Laugesen, Siudeja (2016)

▲ right isosceles triangle:

maximiser of torsion:  $x^T = (0.39168, 0)$

maximiser of g.s.  $u$ :  $x^* = (0.39183, 0)$

▲ half-disk:  $(0.48022, 0)$  vs  $(0.48051, 0)$



torsional rigidity:  $T_{\mathcal{D}} = \|v\|_{1,\mathcal{D}} = \max \left\{ \frac{\|u\|_1^2}{\|\nabla u\|_2^2} : u \in W_0^{1,2}(\mathcal{D}), u \neq 0 \right\}$

$$= \max \{ 2\|u\|_1 - \|\nabla u\|_2^2 : u \in W_0^{1,2}(\mathcal{D}) \} = \|\nabla v\|_{2,\mathcal{D}}^2$$

satisfies

*Saint-Venant inequality*  $|\mathcal{B}|^{-\frac{d+2}{d}} T_{\mathcal{B}} \geq |\mathcal{D}|^{-\frac{d+2}{d}} T_{\mathcal{D}}$

*Kohler-Jobin inequality*  $\lambda_1^{\mathcal{D}} T_{\mathcal{D}}^{\frac{2}{d+2}} \geq \lambda_1^{\mathcal{B}} T_{\mathcal{B}}^{\frac{2}{d+2}}$

NB: jointly imply Faber-Krahn inequality

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NB: jointly imply Faber-Krahn inequality

non-local torsion equation

$$\begin{cases} \Phi(-\Delta)v = 1 & \text{in } \mathcal{D} \\ v = 0 & \text{in } \mathcal{D}^c \end{cases}$$

is solved by

$$v(x) = \mathbb{E}^x[\tau_{\mathcal{D}}]$$

we get:  $\sup_{\mathcal{D}} v(x) = v(x^{\top}) \leq C_d v(x^*)$

compares location of “safest” point  $x^{\top}$  with point  $x^*$  of highest clustering of paths

# THANK YOU !

with Anup Biswas (IISER, Pune)

- (1) Universal constraints on the location of extrema of eigenfunctions of non-local heat operators, *J. Diff. Equations* **267**, 267–306, 2019
- (2) Maximum principles and Aleksandrov-Bakelman-Pucci type estimates for non-local Schrödinger equations with exterior conditions, *SIAM J. Math. Anal.* **51**, 1543–1581, 2019
- (3) Maximum principles for time-fractional Cauchy problems with spatially non-local components, *Fract. Calc. Appl. Anal.* **21**, 1335–1359, 2018
- (4) Ambrosetti-Prodi type results for Dirichlet problems of fractional-like Laplace operators, *Integr. Equat. Oper. Th.* **92**, 2020
- (5) Hopf's lemma for viscosity solutions to a class of non-local equations with applications, *Nonlinear Anal. Theory Methods Appl.* **204**, 112194, 2021