On solution of SDE driven by singular Lévy noise

Michał Ryznar
Faculty of Pure and Applied Mathematics
Wrocław University of Science and Technology
Poland

Anomalous transport and anomalous diffusion, Pisa

17.03.2026

- $$\mathcal{K} = -\sum_{i=1}^{d} \left(-\frac{\partial^2}{\partial x_i^2} \right)^{\alpha/2}, \quad \alpha \in (0,2).$$

$$\mathcal{K} f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} (f(x + w e_i) - f(x)) \frac{c_\alpha \, dw}{|w|^{1+\alpha}}.$$

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- $$\mathcal{K}_A f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} (f(x+wa_i(x)) - f(x)) \frac{c_\alpha \, dw}{|w|^{1+\alpha}},$$

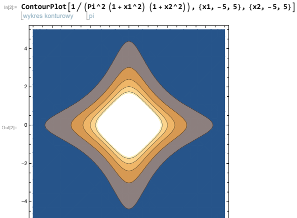
  where $A(x) = (a_{i,j}(x))$ is a $d \times d$ matrix,
  $a_i(x) = (a_{1i}(x), \ldots, a_{di}(x))$.

$$\mathcal{K} = -\sum_{i=1}^{d} \left(-\frac{\partial^2}{\partial x_i^2} \right)^{\alpha/2}, \quad \alpha \in (0,2).$$

$$Z_t = \begin{pmatrix} Z_t^{(1)} \\ \vdots \\ Z_t^{(d)} \end{pmatrix}$$

$Z_t^{(i)}$ one-dimensional standard symmetric α-stable process.
$Z_t^{(1)}, \dots, Z_t^{(d)}$ independent.

For $d = 2$, $\alpha = 1$ the heat kernel corresponding to $\mathcal{K}$ (the transition density of Z) is given by
$p_t(0,x) = p_t(0,(x_1,x_2)) = \frac{t}{\pi(t^2+x_1^2)} \frac{t}{\pi(t^2+x_2^2)}$.



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In[2]:= ContourPlot[1/(Pi^2 (1+x1^2) (1+x2^2)), {x1, -5, 5}, {x2, -5, 5}]
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Under appropriate assumptions on $A(\cdot)$ the operator $\mathcal{K}_A$ is the generator of the process X, which is a solution of the SDE

$$dX_t = A(X_{t-})\,dZ_t.$$

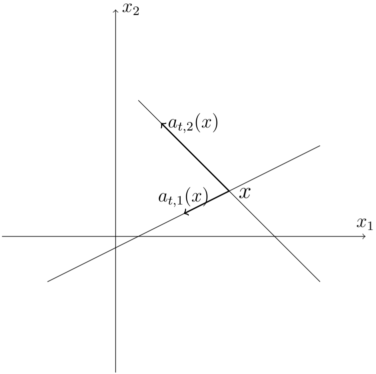
Theorem (R. Bass, Z.-Q. Chen (2006))

If $\sup_x |A(x)| < \infty$, $x \to A(x)$ is continuous and for any $x \in \mathbb{R}^d$, $A(x)$ is invertible, then there is a unique weak solution of the above equation with the initial condition $X_0 = x \in \mathbb{R}^d$. The solution forms a strong Markov process which has a version with cad-lag trajectories.

For $d = 2$ we have

$$\mathcal{K}_A f(x) = \sum_{i=1}^{2} P.V. \int_{\mathbb{R}} \frac{[f(x + a_i(x)w) - f(x)]\, c_\alpha\, dw}{|w|^{1+\alpha}}.$$

where $a_i(x) = (a_{1i}(x), a_{2i}(x))$.



Some results: R. Bass, Z.-Q. Chen (2006), A. Debussche, N. Fournier (2013), J. Chaker (2019), Z.-Q. Chen, Z. Hao, X. M. Zhang (2020) X. M. Friesen, P. Jin, B. Rüdiger (2021), V. Knopova, A. Kulik, R. Schilling (2021).

T. Kulczycki, M. Ryznar, *Semigroup properties of solutions of SDEs driven by Lévy processes with independent coordinates*, Stochastic Process. Appl. (2020).

T. Kulczycki, M. Ryznar, P. Sztonyk, *Strong Feller property for SDEs driven by multiplicative cylindrical stable noise*, Potential Anal. (2021).

Conditions (A) $A(x) = (a_{i,j}(x))$, there are constants $C_1, C_2, C_3 > 0$, such that for any $x, y \in \mathbb{R}^d$, $i, j \in \{1, \ldots, d\}$

$$
\begin{aligned}
&|A(x)| \leqslant C_1, \\
&\det(A(x)) \geqslant C_2, \\
&|A(x) - A(y)| \leqslant C_3 |x - y|.
\end{aligned}
$$

Heat semigroup for $\mathcal{K}_A$.

Theorem (T. Kulczycki, M. R., P. Sztonyk)

Assume (A). There exists a family of operators $P_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$, $t > 0$ such that
(i) $P_{t+s}f(x) = P_t(P_s f)(x)$ for any $s, t > 0$, $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$,
(ii) $\lim_{t\to 0^+} ||P_t f - f||_\infty = 0$ for any $f \in C_0(\mathbb{R}^d)$.
(iii) there exists a nonnegative function $p(t, x, y)$ in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$; for each fixed $t > 0$, $x \in \mathbb{R}^d$ the function $y \to p(t, x, y)$ is Lebesgue measurable, $\int_{\mathbb{R}^d} p(t, x, y)\, dy = 1$ and $P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y)\, dy$ for $f \in C_0(\mathbb{R}^d)$.
(iv) For any $f \in C_0(\mathbb{R}^d)$ we have in some weak sense

$$\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) P_t f = 0.$$

(v) For any $f \in C_0(\mathbb{R}^d)$ we have $P_t f(x) = \mathbb{E}^x f(X_t)$ where X_t is a unique solution of SDE $dX_t = A(X_{t-})\, dZ_t$, $X_0 = x$.

Main results.

Theorem (T. Kulczycki, M. R., P. Sztonyk)

Assume (A). Fix $\tau > 0$. Then for any $\gamma \in (0, \min(\alpha, 1))$, $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and a bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$

$$|P_t f(x) - P_t f(y)| \leqslant c t^{-\gamma/\alpha} |x - y|^\gamma \, ||f||_\infty.$$

Remark (T. Kulczycki, M. R., P. Sztonyk)

There exist a family of matrices A satisfying (A), $t > 0$ and $x \in \mathbb{R}^d$ such that $p(t, x, \cdot) \notin L^\infty(\mathbb{R}^d)$.

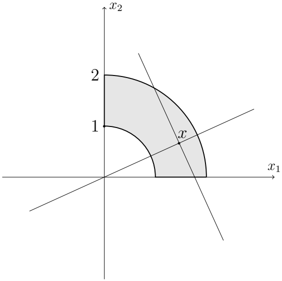
Theorem (T.Kulcycki, M. R., P. Sztonyk)

Assume (A). Fix $\tau > 0$. Then for any $\gamma \in (0, \alpha/d)$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have

$$|P_t f(x)| \leqslant c t^{-\gamma d/\alpha} \, \|f\|_\infty^{1-\gamma} \, \|f\|_1^\gamma.$$

$p(t,x,y)$ is unbounded (for some choices of $\{A(x)\}_{x\in\mathbb{R}^d}$).

Let $d=2$, $\alpha\in(0,1)$ and
$D=\{(x_1,x_2)\in\mathbb{R}^2:\ 1\leqslant\sqrt{x_1^2+x_2^2}\leqslant 2,\ x_1\geqslant 0, x_2\geqslant 0\}$.



- One can choose $A(x)=A(x_1,x_2)$ so that for any $x\in D$

$$A(x)=\begin{bmatrix} a_{11}(x) & a_{12}(x)\\ a_{21}(x) & a_{22}(x)\end{bmatrix}=|x|^{-1}\begin{bmatrix} x_1 & -x_2\\ x_2 & x_1\end{bmatrix}.$$

$p(t,x,y)$ is unbounded (for some choices of $\{A(x)\}_{x\in\mathbb{R}^d}$).

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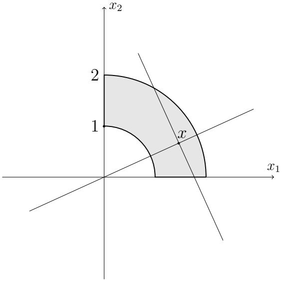
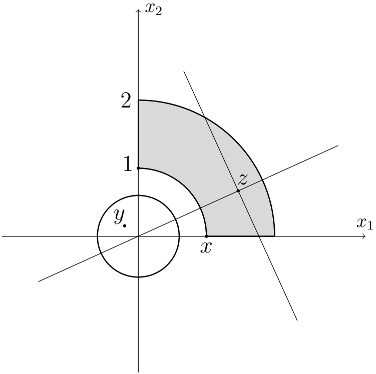


- One can choose $A(x)=A(x_1,x_2)$ so that for any $x\in D$
$$A(x)=\begin{bmatrix} a_{11}(x) & a_{12}(x)\\ a_{21}(x) & a_{22}(x)\end{bmatrix}=|x|^{-1}\begin{bmatrix} x_1 & -x_2\\ x_2 & x_1\end{bmatrix}.$$
- We have $a_1(x)=(a_{11}(x),a_{21}(x))=|x|^{-1}(x_1,x_2)=x/|x|$,
$a_2(x)=(a_{21}(x),a_{22}(x))=|x|^{-1}(-x_2,x_1)$.

$p(t, x, y)$ is unbounded (for some choices of $\{A(x)\}_{x \in \mathbb{R}^d}$).

Let $d = 2$, $\alpha \in (0, 1)$, $x = (1, 0)$.



One can show that there exists

$t > 0$, $\delta > 0$ such that for almost all $y \in B(0, \delta)$ we have $p(t, x, y) \geqslant c|y|^{\alpha-1}$.

Levi's method.

- E. Levi, *Sulle equazioni lineari totalmente ellittiche alle derivate parziali*, Rend. Circ. Mat. Palermo 24 (1907), 275-317
  O. Ladyzenskaja, V. Solonnikov, N. Uralceva (1968)
  A. Friedman (1975)
  For nonlocal operators:
  A. Kochubei (1988)
  K. Bogdan, T. Jakubowski, Z.-Q. Chen, P. Kim, T. Kumagai, T. Grzywny, K. Szczypkowski, V. Knopova, A. Kulik, T. Grzywny, R. Song, Z. Vondracek

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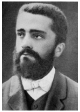

Eugenio Elia Levi (18 October 1883 - 28 October 1917) was an Italian mathematician, known for his fundamental contributions in group theory, in the theory of partial differential operators and in the theory of functions of several complex variables.

- $$\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) p(t,\cdot,y)(x) = 0. \qquad \lim_{t\to 0^+} p(t,x,y)\,dy = \delta_x.$$

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- $$p(t,x,y) = \tilde{p}(t,x,y) + r(t,x,y). \qquad \lim_{t\to 0^+} \tilde{p}(t,x,y)\,dy = \delta_x.$$

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- $$\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) r(t,\cdot,y)(x) = \underbrace{-\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) \tilde{p}(t,\cdot,y)(x)}_{q_0(t,x,y)}.$$

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- $$r(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z)q_0(s,z,y)\,dz\,ds.$$

- $$\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) p(t,\cdot,y)(x) = 0. \qquad \lim_{t\to 0^+} p(t,x,y)\,dy = \delta_x.$$

- $$p(t,x,y) = \tilde{p}(t,x,y) + r(t,x,y). \qquad \lim_{t\to 0^+} \tilde{p}(t,x,y)\,dy = \delta_x.$$

- $$\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) r(t,\cdot,y)(x) = \underbrace{-\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right)\tilde{p}(t,\cdot,y)(x)}_{q_0(t,x,y)}.$$

- $$r(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z) q_0(s,z,y)\,dz\,ds.$$

- $$p(t,x,y) = \tilde{p}(t,x,y) + \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z) q_0(s,z,y)\,dz\,ds.$$

$$q_{n+1}(t,x,y) = \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z) q_n(s,z,y)\, dz\, ds.$$

$$p(t,x,y) = \tilde{p}(t,x,y) + \sum_{n=0}^{\infty} \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t-s,x,z) q_n(s,z,y)\, dz\, ds.$$

$$\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) p(t,\cdot,y)(x) = 0. \qquad \lim_{t\to 0^+} p(t,x,y)\, dy = \delta_x.$$

Idea of the proof.

$$\mathcal{K}_A f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} \frac{[f(x + a_i(x)w) - f(x)]\, c_\alpha \, dw}{|w|^{1+\alpha}}.$$

"freezing operator":

$$\mathcal{K}_A^y f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} \frac{[f(x + a_i(y)w) - f(x)]\, c_\alpha \, dw}{|w|^{1+\alpha}}.$$

$p_y(t,x) \sim \mathcal{K}_A^y$, $\left(\frac{\partial}{\partial t} - \mathcal{K}_A^y\right) p_y(t,\cdot)(x) = 0$.
$\tilde{p}(t,x,y) = p_y(t, y-x)$?
Problem: $\int_{\mathbb{R}^d} p_y(t, y-x)\, dy = \infty$ in some cases.

Idea of the proof.

We divide $\mathcal{K}_A$ into 2 parts: $\mathcal{K}_A = \mathcal{S}_A + \mathcal{L}_A$. Recall

$$\mathcal{K}_A f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} [f(x + a_i(x)w_i) - f(x)] \frac{c_\alpha}{|w|^{1+\alpha}} \, dw.$$

We choose $\delta > 0$ and $\mu : \mathbb{R} \setminus \{0\} \to [0, \infty)$.

$$\mu(w) = \begin{cases} = c_\alpha |w|^{-1-\alpha}, & \text{for } \; w \in (0, \delta], \\ \in (0, c_\alpha |w|^{-1-\alpha}), & \text{for } \; w \in (\delta, 2\delta], \\ = 0, & \text{for } \; w \in [2\delta, \infty). \end{cases}$$

μ is nonincreasing, convex and C^1 on $(0, \infty)$, $\mu(-w) = \mu(w)$.
We define

$$\begin{aligned} \mathcal{S}_A f(x) &= \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} [f(x + a_i(x)w) - f(x)] \, \mu(w) \, dw, \\ \mathcal{L}_A f(x) &= \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} [f(x + a_i(x)w) - f(x)] \left(\frac{c_\alpha}{|w|^{1+\alpha}} - \mu(w) \right) dw. \end{aligned}$$

Idea of the proof. Short jumps.

We have

$$\mathcal{S}_A f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} \left[f(x + a_i(x)w) - f(x) \right] \mu(w)\, dw$$

Put

$$\mathcal{S}_A^y f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} \left[f(x + a_i(y)w) - f(x) \right] \mu(w)\, dw$$

$p_y(t,x) \sim \mathcal{S}_A^y$, $\left(\frac{\partial}{\partial t} - \mathcal{S}_A^y\right) p_y(t,\cdot)(x) = 0$, $\tilde{p}(t,x,y) = p_y(t, y-x)$.
We apply the Levi's method for $\mathcal{S}_A$ and we construct a heat kernel $u(t,x,y)$ and a heat semigroup U_t corresponding to $\mathcal{S}_A$.
For any $f \in C_0(\mathbb{R}^d)$ we have in some week sense
$\left(\frac{\partial}{\partial t} - \mathcal{K}\right) U_t f = 0$.
$U_t f(x) = \int_{\mathbb{R}^d} u(t,x,y) f(y)\, dy$, $f \in L^\infty(\mathbb{R}^d)$.

T. Kulczycki, M. Ryznar, *Gradient estimates of harmonic functions and transition densities for Lévy processes*, Trans. Amer. Math Soc (2016).

Idea of the proof. Long jumps

$\rho(x) = \frac{c_\alpha}{|x|^{1+\alpha}} - \mu(x), \quad x \in \mathbb{R} \setminus \{0\}$,
$\lambda = d \int_{\mathbb{R}} \rho(x)\, dx < \infty$.
$\mathcal{S}_A f(x) = \sum_{i=1}^{d} \int_{\mathbb{R}} [f(x + a_i(x)w) - f(x)]\, \rho(w)\, dw$.
$\mathcal{N}_A f(x) = \sum_{i=1}^{d} \int_{\mathbb{R}} [f(x + a_i(x)w)]\, \rho(w)\, dw$.
For any $t \geqslant 0$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $n \geqslant 1$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define

$$\begin{aligned} \Psi_{0,t} f(x) &= U_t f(x), \\ \Psi_{n,t} f(x) &= \int_0^t U_{t-s}(\mathcal{N}_A(\Psi_{n-1,s} f))(x)\, ds, \quad n \geqslant 1. \end{aligned}$$

$P_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x)$
P_t is the heat semigroup corresponding to $\mathcal{K}_A$. For any $f \in C_0(\mathbb{R}^d)$ we have in some weak sense $\left(\frac{\partial}{\partial t} - \mathcal{K}_A\right) P_t f = 0$.
There exists a function $p(t, x, y)$ such that for any $f \in L^\infty(\mathbb{R}^d)$ we have $P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y)\, dy$.

Futher results.

$$\mathcal{K}_{A,b}f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} (f(x + wa_i(x)) - f(x))\nu_i(w)\, dw + b(x)\nabla f(x),$$

where $A(x) = (a_{i,j}(x))$ is a $d \times d$ matrix, $a_i(x) = (a_{1i}(x), \ldots, a_{di}(x))$.

$$dX_t = A(X_{t-})\, dZ_t + b(X_t)dt,$$

where Z_t has independent symmetric pure jump coordinates with Lévy measures $\nu_i(w)$. We assume some weak scaling conditions. For example let $Z_t^{(i)}$ be one-dimensional standard symmetric α_i-stable process, say $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \alpha_d$. If we assume some balance conditions:

$$\alpha_1 \geqslant (3/2)\alpha_d,$$

then the above results hold (for the case $b \equiv 0$).

Related papers.

$$\mathcal{K}_{A,b}f(x) = \sum_{i=1}^{d} P.V. \int_{\mathbb{R}} (f(x + wa_i(x)) - f(x))\nu_i(w)\, dw + b(x)\nabla f(x),$$

where $A(x) = (a_{i,j}(x))$ is a $d \times d$ matrix, $a_i(x) = (a_{1i}(x), \dots, a_{di}(x))$.

1. T. Kulczycki, M. Ryznar, *Semigroup properties of solutions of SDEs driven by Lévy processes with independent coordinates*, Stoch. Proc. Appl. (2020)

2. T. Kulczycki, M. Ryznar, P. Sztonyk *Strong Feller property for SDEs driven by multiplicative cylindrical stable noise*, Potential Anal. (2021)

3. Z.-Q. Chen, Z. Hao, X. Zhang, *Hölder regularity and gradient estimates for SDEs driven by cylindrical α-stable processes* Electron. J. Probab. (2020)

4. T. Kulczycki, A. Kulik, M. Ryznar, *On weak solution of SDE driven by inhomogeneous singular Lévy noise*, Trans. Amer. Math Soc (2022)

5. T. Kulczycki, A. Kulik, M. Ryznar, *Drift reduction method for SDEs driven by inhomogeneous singular Lévy noise*, Bernoulli (2024)

On radial harmonic functions.

We want to study some class of harmonic functions with respect to the solution of

$$dX_t = A(X_{t-})\, dZ_t.$$

Here we assume that Z_t has independent α- stable standard symmetric coordinates, $0 < \alpha < 2$.
We assume that

$$\begin{aligned} &x \to A(x) \text{ is continuous,} \\ &\sup_x |A(x)| < \infty, \\ &A(x) \text{ is invertible for any } x \in \mathbb{R}^d. \end{aligned}$$

According to already mentioned result by R. Bass, Z.-Q. Chen (2006), the unique weak solution exists for every initial condition $X_0 = x \in \mathbb{R}^d$.

Let $\mathbb{E}^x$, $\mathbb{P}^x$ - expectation and probability of the process X starting from $x \in \mathbb{R}^d$.

$$\tau_D = \inf\{t \geqslant 0 : \ X_t \notin D\} - \text{the first exit time}$$

of an open set $D \subset \mathbb{R}^d$ by X.

u Borel and bounded is *harmonic* with respect to X in a bounded open set $D \subset \mathbb{R}^d$ if

$$u(x) = \mathbb{E}^x(u(X_{\tau_D})) \quad \text{for every } x \in D.$$

u Borel and bounded is *superharmonic* with respect to X in a bounded open set $D \subset \mathbb{R}^d$ if

$$u(x) \geqslant \mathbb{E}^x(u(X_{\tau_D})) \quad \text{for every } x \in D.$$

u is *subharmonic* if we have $\leqslant$ in the above inequality.

Clearly, if $D \subset \mathbb{R}^d$ is a bounded open set and $g : D^c \to \mathbb{R}$ is Borel and bounded function then u given by

$$u(x) = \begin{cases} \mathbb{E}^x(g(X_{\tau_D})), & \text{for} \quad x \in D, \\ g(x), & \text{for} \quad x \in D^c \end{cases}$$

is harmonic with respect to X on D. The function g may be treated as Dirichlet exterior datum for u.

An ultimate objective is to find or estimate the distribution of X_{τ_D}. Explicit formulas are known only in the case when X is the isotropic stable process and D is a ball or a half-space.

In our case we are able to estimate the distribution of $|X_{\tau_{B(z_0,R)}} - z_0|$, where $B(z_0, r)$ is a ball of the radius r and center z_0. Without loosing generality we may assume that $z_0 = 0$

Our method consists of constructing so called *barrier functions* and applying maximum principle.

For $d = 2$ we have

$$\mathcal{K}_A f(x) = \sum_{i=1}^{2} P.V. \int_{\mathbb{R}} \frac{[f(x + a_i(x)w) - f(x)]\, c_\alpha \, dw}{|w|^{1+\alpha}}.$$

where $a_i(x) = (a_{1i}(x), a_{2i}(x))$.
Let for $a \in \mathbb{R}^d, a \neq 0$

$$\begin{aligned} \mathcal{K}_a f(x) &= P.V. \int_{\mathbb{R}} \frac{[f(x + aw) - f(x)]\, c_\alpha \, dw}{|w|^{1+\alpha}} \\ &= P.V. |a|^\alpha \int_{\mathbb{R}} \frac{[f(x + (a/|a|)w) - f(x)]\, c_\alpha \, dw}{|w|^{1+\alpha}} \\ &= |a|^\alpha \mathcal{K}_{a/|a|} f(x) \end{aligned}$$

If $f(x)$ is rotation invariant then the above property shows that

$$\mathcal{K}_A f(x) = \sum_{i=1}^{2} \mathcal{K}_{a_i(x)} f(x) \leqslant 0,\ |x| < r \quad \text{provided } \mathcal{K}_{e_1} f(x) \leqslant 0,\ |x| < r$$

Put

$$h(y) = \begin{cases} (r^2 - |x|^2)^{\alpha/2-1}, & \text{for} \quad y \in B(0,r), \\ 0 & \text{for} \quad y \in B^c(0,r). \end{cases}$$

One can show that for any $y \in B(0,r)$ we have

$$\mathcal{K}_{e_1} h(y) = 0. \tag{1}$$

Indeed, for $v > 0$ let $s_v(u) = (v^2 - |u|^2)_+^{\alpha/2-1}, u \in \mathbb{R}$, then it is well known that the function s_v is α-harmonic in the interval $(-v, v)$, so

$$p.v. \int_{\mathbb{R}} [s_v(u+w) - s_v(u)] \, \frac{dw}{|w|^{1+\alpha}} = 0, \quad u \in (-v, v).$$

Next, for $x = (x_1, x_2)$, $|x| < r$ we have $|x_1| < \sqrt{r^2 - |x_2|^2}$, and

$$\begin{aligned}
&\mathcal{K}_{e_1} h(x) \\
= \;& p.v.\ \mathcal{A}_\alpha \int_{\mathbb{R}} \left[(r^2 - |x + we_1|^2)_+^{\alpha/2-1} - (r^2 - |x|^2)_+^{\alpha/2-1} \right] \frac{dw}{|w|^{1+\alpha}} \\
= \;& \int_{\mathbb{R}} \left[(r^2 - x_2^2 - (x_1 + w)^2)_+^{\alpha/2-1} - (r^2 - x_2^2 - x_1^2)_+^{\alpha/2-1} \right] \frac{\mathcal{A}_\alpha dw}{|w|^{1+\alpha}} \\
= \;& \mathcal{A}_\alpha \int_{\mathbb{R}} \left[s_{\sqrt{r^2 - |x_2|^2}}(x_1 + w) - s_{\sqrt{r^2 - |x_2|^2}}(x_1) \right] \frac{dw}{|w|^{1+\alpha}} = 0.
\end{aligned}$$

.

$$\lambda(y) = \begin{cases} (r^2 - |y|^2)^{\alpha/2}, & \text{for} \quad y \in B(0,r), \\ 0, & \text{for} \quad y \in B^c(0,r) \end{cases}$$

By similar arguments, for $x \in B(0,r)$ we have

$$\mathcal{K}_{e_1}\lambda(x) = -\tilde{\mathcal{A}}_\alpha,$$

This leads to the following estimate (M.R. 2025). For $z \in \mathbb{R}^d$, $r > 0$, we have

$$\mathbb{E}^x \tau_{B(z,r)} \approx (r^2 - |x-z|^2)^{\alpha/2}, \quad \text{for } x \in B(z,r).$$

For any $r > 0$ and $\varepsilon \in (0, r/4]$ let us define the following auxiliary class of functions $\mathcal{A}_1(r, \varepsilon)$.
(i)

$$\theta(v) = \begin{cases} \dfrac{1}{r^2 - v}, & \text{for} \quad v \in [0, (r-\varepsilon)^2], \\ C_0, & \text{for} \quad v \in [(r - \varepsilon + K)^2, r^2), \end{cases}$$

for some $K \in (0, \varepsilon/4]$ and $C_0 > 0$,
(ii) $\theta \in C^2[0, r^2)$,
(iii) θ is strictly increasing on $[(r-\varepsilon)^2, (r - \varepsilon + K)^2)$,
(iv) $\max\{\theta''(v) : \ v \in [0, r^2)\} = \theta''((r-\varepsilon)^2)$. Let for a given function $\theta \in \mathcal{A}_1(r, \varepsilon)$

$$f_\theta(x) = \begin{cases} (r^2 - |x|^2)^{\alpha/2} \theta(|x|^2), & \text{for} \quad x \in B(0, r), \\ 0, & \text{for} \quad x \in B^c(0, r). \end{cases}$$

Now for any $b > 0$ and any function $\theta \in \mathcal{A}_1(r, \varepsilon)$ let us define an auxiliary function $f_{b,\theta}$ by the formula

$$f_{b,\theta}(x) = \frac{b\eta r^{1-\alpha/2}}{\varepsilon^{\alpha/2}} f_\theta(x) + g(x), \;\; x \in \mathbb{R}^d,$$

where

$$g(y) = \begin{cases} 1, & \text{for} \qquad\qquad |y| \in (r+\varepsilon, r+\varepsilon+\eta), \\ 0, & \text{otherwise.} \end{cases}$$

Proposition

There exist a constant $b = b(\alpha) > 0$ depending only on α and a function $\theta \in \mathcal{A}_1(r, \varepsilon)$ such that

$$\mathcal{K}_A f_{b,\theta}(x) \leqslant 0 \quad \textit{for any } x \in B(0, r).$$

Hence $f_{b,\theta}$ is superharmonic wrt. X.

$$u(x) = \begin{cases} \mathbb{E}^x(g(X_{\tau_D})), & \text{for} \quad x \in B(0,r), \\ g(x), & \text{for} \quad x \in B^c(0,r) \end{cases}$$

Next, $u_1(x) = f_{b,\theta}(x) - u(x)$ is superharmonic in $B(0,r)$ and

$$u_1(x) = 0, x \in B^c(0,r).$$

$$u_1(x) \geqslant u_1(X_{\tau_{B(0,r)}}) = 0, x \in B(0,r).$$

Conclusion:

$$u(x) \leqslant f_{b,\theta}(x), x \in B(0,r).$$

We have for $x \in B(0,r)$

$$f_{b,\theta}(x) = \frac{b\eta r^{1-\alpha/2}(r^2 - |x|^2)^{\alpha/2}\theta(|x|^2)}{\varepsilon^{\alpha/2}} \approx \frac{b\eta(r-|x|)^{\alpha/2}}{\varepsilon^{\alpha/2}(\varepsilon + r - |x|)}.$$

Hence

$$P^x(r+\varepsilon < |X(\tau_{B(0,r)})| < r+\varepsilon+\eta) \leqslant C\frac{b\eta(r-|x|)^{\alpha/2}}{\varepsilon^{\alpha/2}(\varepsilon + r - |x|)}$$

For any fixed $r > 0$ and $\varepsilon \in (0, r/4]$ let us define

$$\Theta(v) = \begin{cases} \dfrac{1}{r^2 - v}, & \text{for} \quad v \in [0, (r-\varepsilon)^2], \\ \frac{1}{r^2 - v}\left[1 - \left(\frac{v-(r-\varepsilon)^2}{q}\right)^3\right], & \text{for} \quad v \in ((r-\varepsilon)^2, r^2), \end{cases}$$

where $q = r^2 - (r-\varepsilon)^2$. It is easy to check that $\Theta \in C^2[0, r^2)$. Next, for the already defined function Θ, let

$$F_\Theta(x) = \begin{cases} (r^2 - |x|^2)^{\alpha/2}\Theta(|x|^2), & \text{for} \quad x \in B(0, r), \\ 0, & \text{for} \quad x \in B^c(0, r). \end{cases}$$

$F_{b,\Theta}(x) = \frac{b\eta r^{1-\alpha/2}}{\varepsilon^{\alpha/2}} F_\Theta(x) + g(x)$ is subharmonic in $B(0, r)$.

Theorem (T. Kulczycki, M.R. (2026))

For each fixed $z \in \mathbb{R}^d$, $r > 0$, $x \in D = B(z, r)$ and any Borel set $W \subset [r, \infty)$ put

$$\kappa_D^x(W) = P^x(|X(\tau_D) - z| \in W).$$

The measure κ_D^x is a probability measure on $[r, \infty)$ which is abs. cont. f_D^x - the density of κ_D^x satisfies $\kappa_D^x(W) = \int_W f_D^x(y)\, dy$ for any Borel set $W \subset [r, \infty)$. Moreover

$$f_D^x(y) \approx \frac{\delta^{\alpha/2}(x) r^{\alpha/2}}{(y - r)^{\alpha/2} y^{\alpha/2} (y + \delta(x) - r)}, \ y \geqslant r$$

and $\delta(x) = r - |x - z|$.

Denote $B = B(z, r)$ and

$$P = \{w \in \mathbb{R}^d : R \leqslant |w - z| \leqslant R + \eta\}, 0 < \eta < r < (4/5)R$$

. By the I-W formula we have

$$\mathbb{P}^x(X_{\tau_B} \in P) = \int_B G_B(x, dy)\nu(y, P),$$

where $G_B(x, dy)$ is the Green measure of the ball B and $\nu(y, dz)$ is the jumping kernel of the process X.

$$\mathbb{P}^x(X_{\tau_B} \in P) = \int_B G_B(x, dy)\nu(y, P),$$

where $G_B(x, dy)$ is the Green measure of B and $\nu(y, dz)$ is the jumping kernel of X. For all $y \in B$

$$C_2 \frac{\eta}{R^{1+\alpha}} \leqslant \nu(y, P) \leqslant C_1 \frac{\eta}{R^{1+\alpha}}.$$

$$\mathbb{P}^x(X_{\tau_B} \in P) \approx G_B(x, B) \frac{\eta}{R^{1+\alpha}}.$$

Since

$$G_B(x, B) = \mathbb{E}^x \tau_B \approx (r^2 - |x - z|^2)^{\alpha/2}$$

the proof is completed.