Standard and anomalous diffusion of energy in chains of coupled oscillators

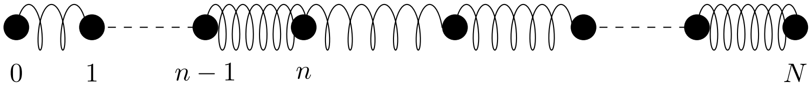
Marielle Simon

Anomalous transport and Anomalous diffusion, Pisa 2026

One-dimensional chain of oscillators

Microscopic model

- $1d$ chain of $N + 1$ **unpinned coupled oscillators**



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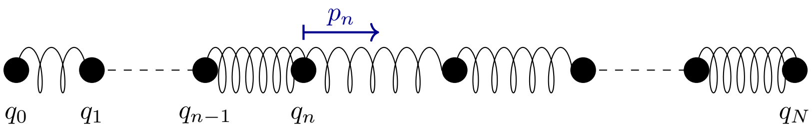

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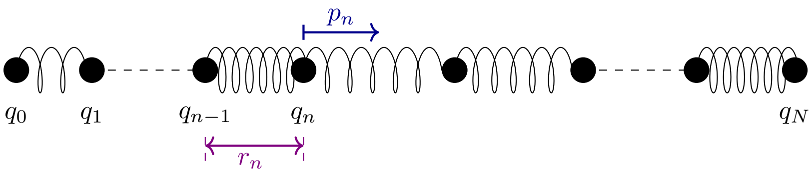

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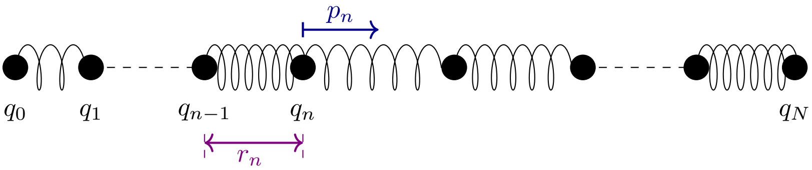

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- **Dynamics** on the configurations of particles

$$(\mathbf{r}, \mathbf{p}) = (r_1, \ldots, r_N, p_0, \ldots, p_N) \in \mathbb{R}^N \times \mathbb{R}^{N+1}$$

Total energy = Hamiltonian

$$\mathcal{H}_N := \frac{p_0^2}{2} + \sum_{n=1}^{N} \mathcal{E}_n \qquad \text{with} \qquad \mathcal{E}_n := \frac{p_n^2}{2} + \underbrace{V(r_n)}_{\text{interaction}} + \underbrace{\cancel{W(q_n)}}_{\text{no pinning}}$$

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3. Langevin **thermostats**: enforce $\mathbb{E}[p_0^2] = T_-$ and $\mathbb{E}[p_N^2] = T_+$.

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☞ **Periodic** case: $p_0 = p_N,\ r_0 = r_N$ $\Rightarrow$ **conservation of**

$$\sum_{n=1}^{N} \frac{p_n^2}{2} + V(r_n) \ \textbf{(energy)} \qquad \text{and} \qquad \sum_{n=1}^{N} r_n \ \textbf{(volume)}$$

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HARD question with a **long** history!

- Quite **few results** for a generic choice of V
- Rigorous results with the **harmonic choice** $V(r) = \frac{r^2}{2}$

Harmonic potential and periodic b.c.

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where
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Understanding the behavior of **heat**

! In the **harmonic** case there are much **more** conservation laws

$$\mathcal{R}_N = \sum r_n, \qquad \mathcal{H}_N = \sum \mathcal{E}_n, \qquad \text{but also } \mathcal{P}_N = \sum p_n, \ldots$$

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▷ **Too many** conservation laws! We add a **stochastic noise** which

- keeps **a few** conservation laws: **energy**, **volume**, (**momentum**)
- models the effect of **nonlinearity** in V'
- allows us to prove convergences rigorously

The velocity-flip noise & **standard** heat diffusion

Stochastic perturbation which does not preserve momentum

Property of the **velocity-flip noise**: preserves $\mathcal{H}_N$ and $\mathcal{R}_N$

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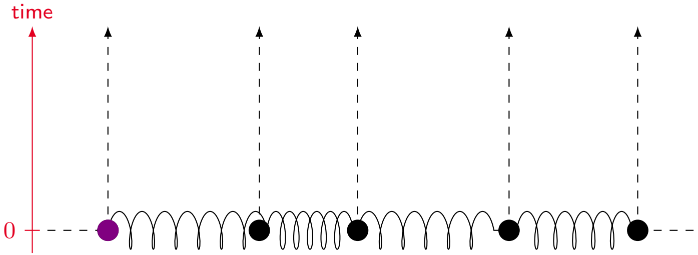

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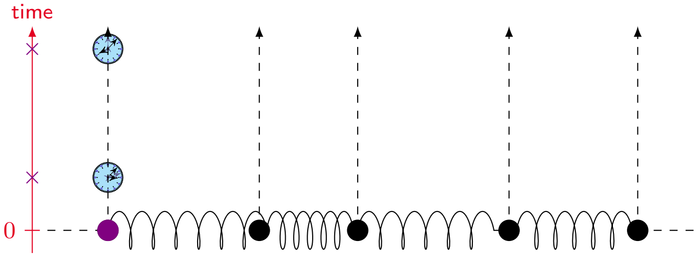

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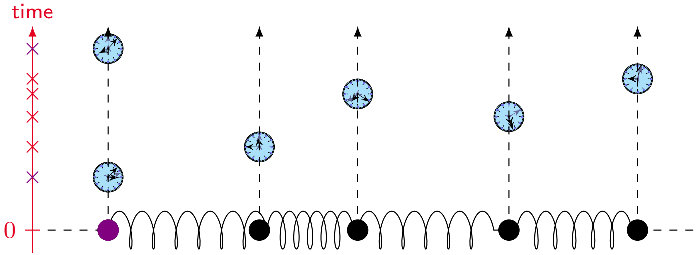

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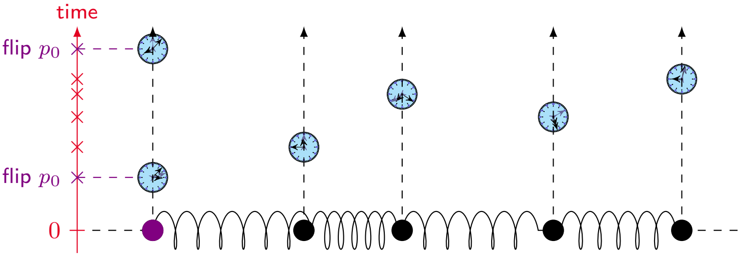

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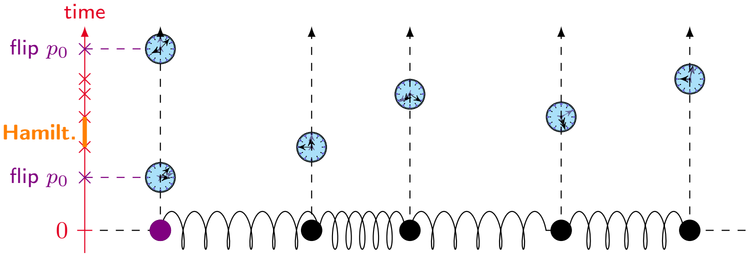

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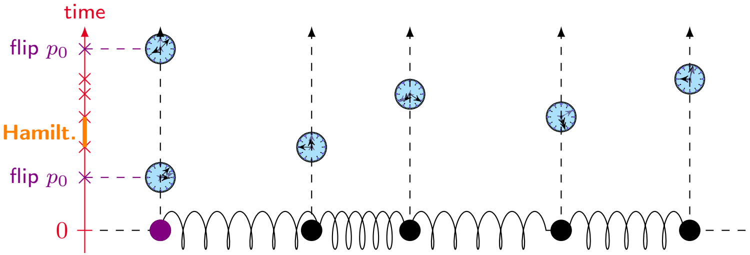

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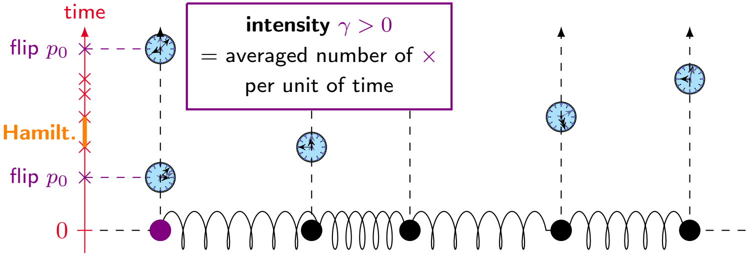

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Full dynamics: Hamiltonian ✚ Flip noise

Microscopic dynamics with periodic b.c. $(N \equiv 0)$

$$\mathrm{d}r_n(t) = (p_n(t) - p_{n-1}(t))\,\mathrm{d}t$$

$$\mathrm{d}p_n(t) = \underbrace{(r_{n+1}(t) - r_n(t))\,\mathrm{d}t}_{\text{hamiltonian}} \underbrace{-\,2p_n(t^-)\,\mathrm{d}\mathcal{N}_n(\gamma t)}_{\text{flip of intensity } \gamma}.$$

Then $\big\{(r_n(tN^2), p_n(tN^2))\big\}_{t\geqslant 0}$ is a **Markov process** with generator $\mathcal{L}_N$ on $\mathbb{R}^N \times \mathbb{R}^N$ which has only two **conserved quantities** $\sum \mathcal{E}_n$ and $\sum r_n$.

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Initial measure μ_0^N: Given some profiles $\mathbf{r}_{\mathrm{ini}}(\cdot)$ and $\mathbf{e}_{\mathrm{ini}}(\cdot)$

$$\frac{1}{N}\sum_{n=1}^{N} G\big(\tfrac{n}{N}\big)\mathbb{E}\big[r_n(0)\big] \xrightarrow[N\to\infty]{} \int_0^1 G(x)\mathbf{r}_{\mathrm{ini}}(x)\mathrm{d}x$$

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Typical example of initial measure

Let $\mathbf{r}_{\text{ini}} : [0,1] \to \mathbb{R}$ and $\mathbf{e}_{\text{ini}} : [0,1] \to (0,+\infty)$ be **continuous**.

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i.e. the **local Gibbs measure**, which satisfies (indeed)

$$\frac{1}{N}\sum_{n=1}^{N} G(\tfrac{n}{N})\mathbb{E}\big[r_n(0)\big] \xrightarrow[N\to\infty]{} \int_0^1 G(x)\mathbf{r}_{\text{ini}}(x)\mathrm{d}x$$

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THEOREM **[Komorowski, Olla, S. 2018]**

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Towards anomalous diffusion

A new **momentum-preserving** stochastic noise

We replace the **stochastic noise** with the following:

▷ Each couple labelled $n, n+1$ **exchange their velocities** $p_n \leftrightarrow p_{n+1}$ at exponential independent random times of intensity γ.

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THEOREM **[Jara, Komorowski, Olla, 2015]**

Starting with a local equilibrium measure s.t. $\mathbb{E}[r_n(0)] = \mathbb{E}[p_n(0)] = 0$,

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Convergence of time correlations: let $\mathcal{P}_t = \exp(t\mathcal{L})$ (semi-group)

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? Prove the following **convergence**, with $\mathcal{P}_t = \exp(t\mathcal{L})$

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Towards an interpolation

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$\sigma = 1$ Some **Interpolation Lévy** process $\mathcal{L}_\gamma$ — time scale

$$\begin{cases} \mathcal{L}_\gamma \xrightarrow[\gamma\to 0]{} \mathcal{L}^{\mathrm{f}} \\ \sqrt{\gamma}\mathcal{L}_\gamma \xrightarrow[\gamma\to +\infty]{} \mathcal{L}^{\mathrm{s}} \end{cases} \qquad \alpha = \tfrac{3}{2}$$

The skew fractional $\mathcal{L}^{\mathrm{f}}$ and the interpolation $\mathcal{L}_\gamma$ (with $c \equiv 1$)

- The **extension problem.** Let $G \in \mathcal{S}(\mathbb{R})$ (Schwarz) and define

$u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ as the solution to the boundary-value problem

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- At the **microscopic** level, the **energy** fluctuations

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The **skew fractional** $\mathcal{L}^{\mathrm{f}}$ and the **interpolation** $\mathcal{L}_\gamma$ (with $c \equiv 1$)

- The **extension problem.** Let $G \in \mathcal{S}(\mathbb{R})$ (Schwarz) and define

$u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ as the solution to the boundary-value problem

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Therefore

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From microscopic to macroscopic

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evolve in the **hyperbolic time scale** according to the generalized Orstein-Uhlenbeck process driven by the operator

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To sum up!

- The volume correlation field (**hyperbolic** time scale) **act as a fast variable** for the evolution of the energy (**superdiffusive** $\frac{3}{2}$ time scale)

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Anything else about $\mathcal{L}_\gamma$?

The interpolation Lévy process

▷ $\mathcal{L}_\gamma$ has the **Fourier** representation

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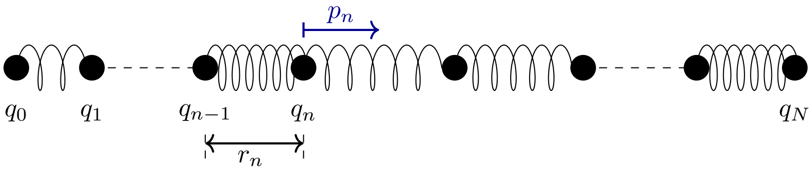
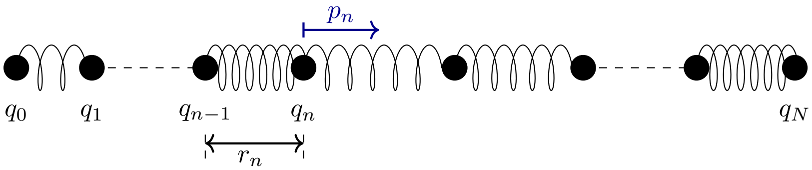
Conclusion





1) Purely harmonic chain $\rightarrow$ **transport** of energy phonons

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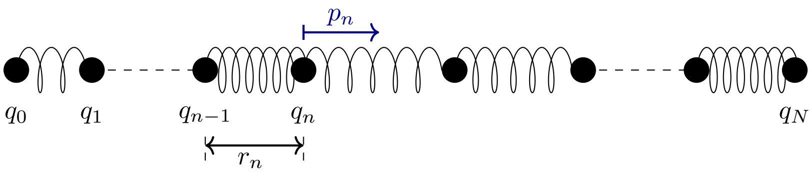

1) Purely harmonic chain $\rightarrow$ **transport** of energy phonons

2) Add **stochastic FLIP noise** $\rightarrow$ **diffusion of total energy**

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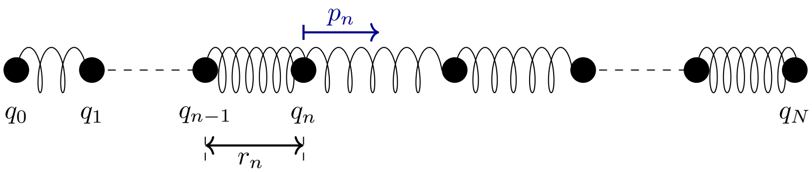

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3) Add **stochastic EXCHANGE noise** $\rightarrow$ $\frac{3}{2}$**–superdiffusion**

$$\partial_t \mathbf{e}(t,x) = -\frac{\kappa}{\sqrt{\gamma}}|\partial_{xx}|^{\frac{3}{4}}\mathbf{e}(t,x)$$

Thank you for your attention!

▷ **Superdiffusion of energy in a chain of harmonic oscillators with noise**

M. Jara, T. Komorowski, S. Olla

Communications in Mathematical Physics 339 (2015)

▷ **Macroscopic evolution of mechanical and thermal energy in a harmonic chain with random flip of velocities**

T. Komorowski, S. Olla, M. Simon

Kinetic and Related Models 11 (2018)

▷ **Interpolation process between standard diffusion and fractional diffusion**

C. Bernardin, P. Gonçalves, M. Jara, M. Simon

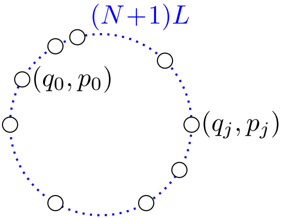
Annales de l'Institut Henri Poincaré - Probabilités et Statistiques 26 (2018)

▷ **From normal diffusion to superdiffusion of energy in the evanescent flip noise limit**

C. Bernardin, P. Gonçalves, M. Jara, M. Sasada, M. Simon

Journal of Statistical Physics 159 (2015)

About the periodic boundary conditions $q_0 = q_N$ and $p_0 = p_N$



A particle configuration.

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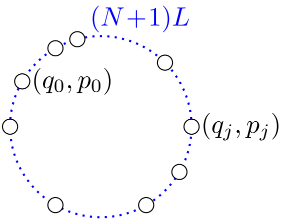

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▷ With $L \in \mathbb{R}$ being the "average algebraic distance"

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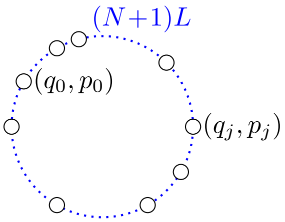

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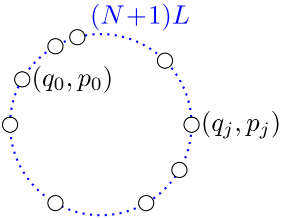

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- Then $r_{N+j} = r_j$ is also **periodic**, and

$$\sum_{n=1}^{N} r_n(0) = \sum_{n=1}^{N} r_n(t)$$ **(conservation of volume)**

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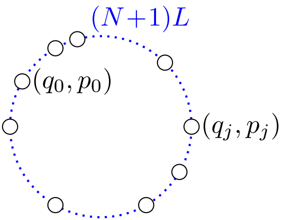

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Periodic b.c. $\{r_n, p_n\}_{n=1,\dots,N}$ following $(\star)$ with $r_0 = r_N$ and $p_0 = p_N$