On some models of Anomalous Diffusion

Bruno Toaldo
Università degli Studi di Torino
Dipartimento di Matematica *"Giuseppe Peano"*

Anomalous Transport and Anomalous Diffusion
Scuola Normale Superiore, Pisa
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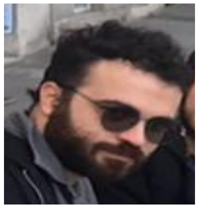


G. Ascione, SSM

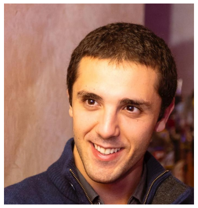
I. Biočić, Zagreb

M. Savov, Sofia

Cedeño-Girón, Turin

E. Scalas, Sapienza

C. Ricciuti, Sapienza



L. Facciaroni, Sapienza

L. Torricelli, Alma Mater

S. Fedotov, Manchester

A. Mijatovic, Warwick

M.C. Bovier, Unito

missing pictures...

young PhD, applied colleagues

Who did the job

Anomalous Diffusion

What is an anomalous diffusion? With the name *Anomalous Diffusion* we refer, in general sense, to random processes such that is:

- diffusion-like processes, they are not step process
- with MSD $\mathbb{E}^x \left\| X(t) - x \right\|^2 \not\approx t$

Why? Typically:

- sticking and trapping effects induced by the medium
- boundaries that accelerate or slow down the evolution
- interaction with the past trajectory!

APPLICATIONS? countless... but just to advertise something I understand: MOTION OF WATER IN THE UNSATURATED SOIL IS SUBDIFFUSIVE [1]

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[1] M.C. Bovier et al. 2025, Adv. Water Resources, M.C. Bovier et al. 2026 (preprint)

How to get anomalous diffusion processes?

There exist *many* different theories

1. *Continous Time Random Walks (CTRWs) and their Scaling limits*
2. *Kinetic processes, i.e., processes with deterministic Hamiltonian subject to random initial conditions and their scaling limits*
3. Stochastic Reflection (processes subject reflection on boundaries)
4. Self-interacting processes (processes interacting with the past trajectory)
5. Many others...

We will speak about the first two approach

CTRWs

Define a discrete-time process in $\mathbb{R}^d \times \mathbb{R}$ (on the prob space $(\Omega, \mathcal{F}, \mathbb{P}^{(x,s)})$

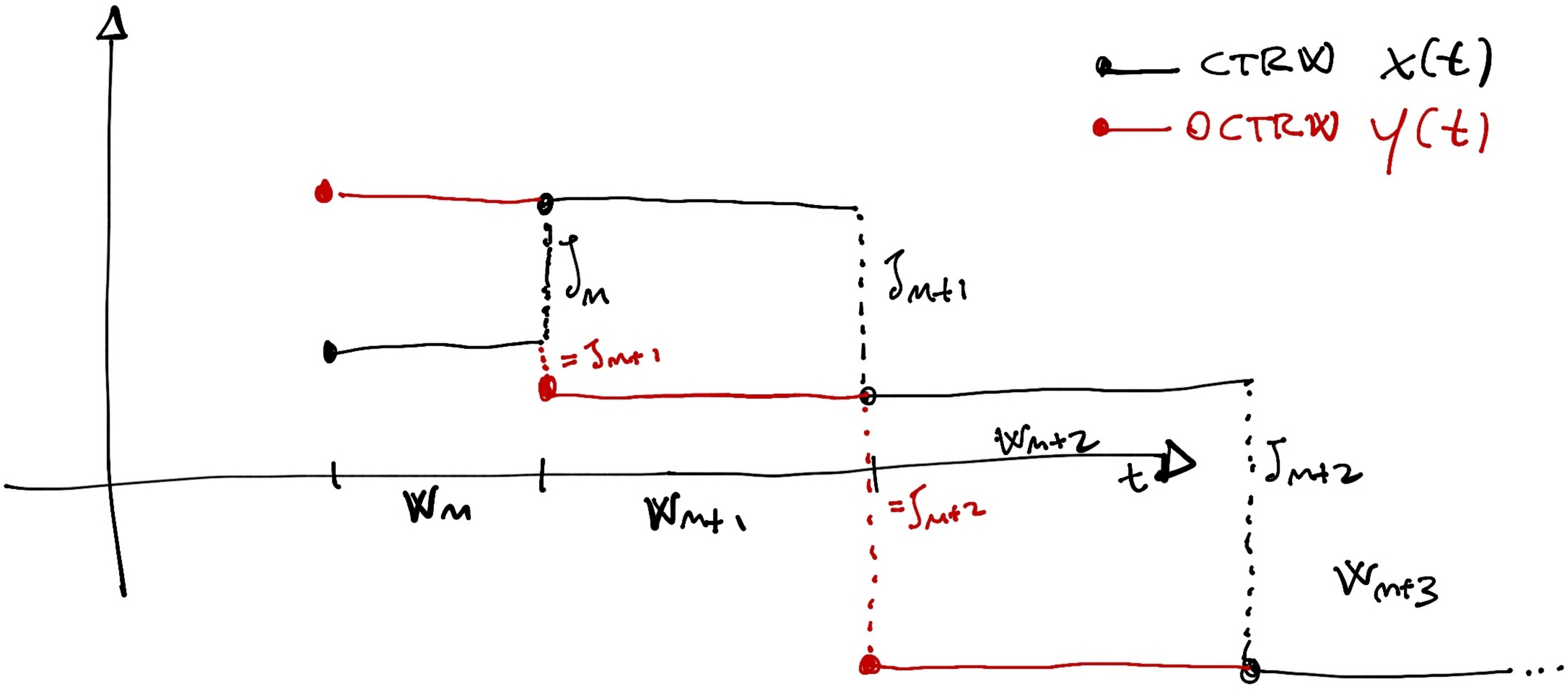
$$(S_n, T_n) \;=\; (M_0, \sigma_0) + \sum_{k=1}^{n} (J_k, W_k).$$

Here S_n is the position of a particle after n jumps, and T_n the times when the particle arrives in that positions (the epoch). Let

$$N_t := \max\{n \in \mathbb{N} : T_n \leq t\} \qquad M_t = \min\{n \in \mathbb{N} : T_n > t\} = N_t + 1 \tag{1}$$

$$\text{CTRW: } X_t := S_{N_t} \qquad\qquad \text{OCTRW: } Y_t := Y_{M_t} = S_{N_t+1}. \tag{2}$$

Temporary assumption (S_n, T_n) is a Markov chain under $\mathbb{P}^{(x,s)}$.



$(W_n \; J_n)$ are not independent

CTRWs' limit

Introduce now a scaling $c > 0$,

$$J_k^c, k \in \mathbb{N}, \qquad W_k^c, k \in \mathbb{N}, \tag{3}$$

so that (jointly)

$$J_k^c \to 0, W_k^c \to 0, \tag{4}$$

and, as $c \to +\infty$,

$$\sum_{i=1}^{[cu]} J_i^c \to M_u \quad \sum_{i=1}^{[cu]} W_i^c \to S_u \tag{5}$$

Brownian case

1. J_k^c are i.i.d. and in the context of CLT, i.e., J_i are i.i.d. with variance 1

$$\sum_{i=1}^{[cu]} J_i^c = \frac{1}{\sqrt{c}} \sum_{i=1}^{[cu]} J_i \to B_u \text{ BM}$$

2. $W_k^c = W_k/c$ where W_k are i.i.d. and have exponential distribution, i.e.,

$$P(W_k > t) = e^{-\theta t} \tag{6}$$

then

$$\frac{1}{c} \sum_{i=1}^{[cu]} W_k \to \theta u \tag{7}$$

then

$$N^c(t) = N(ct), \text{ where } N(t) \text{ is a Poisson process} \tag{8}$$

and furthermore, for the CTRW (which is a Markov chain)

$$\sum_{i=1}^{N^c(t)} J_i^c \to B_{\frac{t}{\theta}} \tag{9}$$

as well as for the OCTRW

$$\sum_{i=1}^{N^c(t)+1} J_i^c \to B_{\frac{t}{\theta}} \tag{10}$$

Note that

$$\sum_{i=1}^{[cu]} W_k^c \to S(u) = \theta u \quad \text{and} \quad \frac{t}{\theta} = \inf\{u \geq 0 : S(u) > t\}. \tag{11}$$

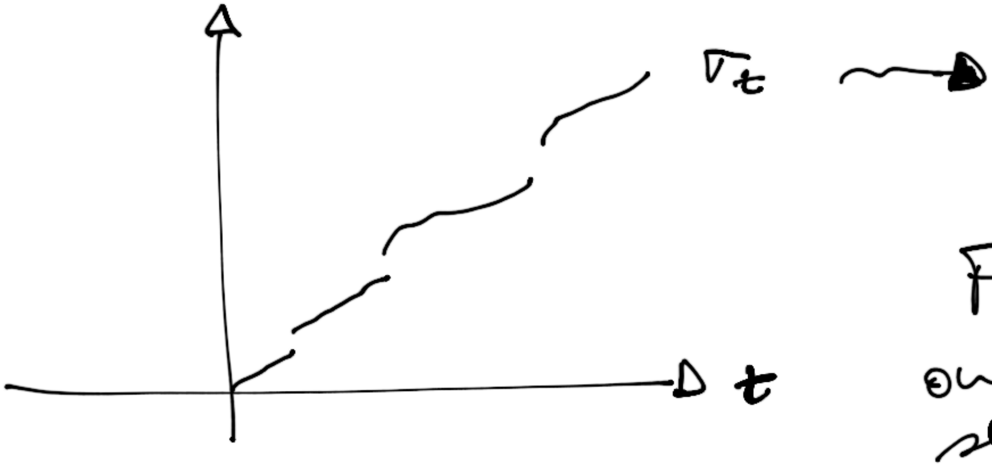
What about, then, if $S(u)$ is a (proper) stochastic process?!

SUBORDINATORS

Define $\sigma = (\sigma_t)_{t \geq 0}$ as a strictly increasing Lévy process with Laplace exponent

$$E e^{-\lambda \sigma(t)} = e^{-t f(\lambda)} \qquad f(\lambda) = \int_0^\infty (1 - e^{-\lambda s}) \, \nu(ds)$$

If $f(\lambda) = \lambda^\alpha$, $\alpha \in (0,1)$ then σ is called the Stable subordinator. The process has infinite activity, i.e., it jumps infinite many (countably) times in any finite interval.



Trajectories are singular functions

Furthermore, the jumps are heavy tailed in the sense:

$$P\left(\sigma(t+s) - \sigma(t) > x\right) \underset{s \to 0}{\sim} s \, \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \qquad \alpha \in (0,1)$$

Mittag-Leffler

Prototype! Fix $\alpha \in (0,1)$. Take a r.v. W s.t.

$$P(W > t) = E_\alpha(-t^\alpha) := \sum_{k=0}^{+\infty} \frac{(-t^\alpha)^k}{\Gamma(\alpha k + 1)} \tag{12}$$

then

$$E_\alpha(-t^\alpha) \sim C \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \text{ as } t \to +\infty \tag{13}$$

and if $e_\alpha(t)$ is a density then

$$e_\alpha(t) \sim C t^{\alpha - 1} \text{ as } t \to 0. \tag{14}$$

Uncoupled example

1. J_k^c are i.i.d. and in the context of CLT, i.e., J_k are i.i.d. with variance 1 and zero mean so that

$$\sum_{i=1}^{[cu]} J_i^c = \frac{1}{\sqrt{c}} \sum_{i=1}^{[cu]} J_i \to B_u \text{ BM}$$

2. $W_k^c = W_k / c^{1/\alpha}$ where W_k are i.i.d. and have Mittag-Leffler distribution, i.e.,

$$P(W_k > t) = E_\alpha(-t^\alpha) \tag{15}$$

then

$$\frac{1}{c^{1/\alpha}} \sum_{i=1}^{[cu]} W_k \to \sigma_u \text{ stable subordinator} \tag{16}$$

Denote L_t to be the inverse of a stable subordinator, i.e., for $t \geq 0$,

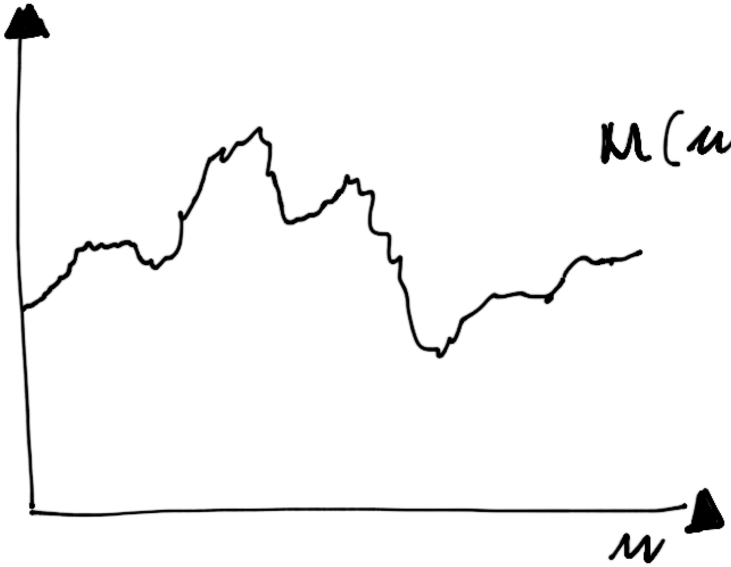
$$L_t = \inf\{s > 0 : \sigma_s > t\} = \sup\{s \geq 0 : \sigma_s \leq t\} \tag{17}$$

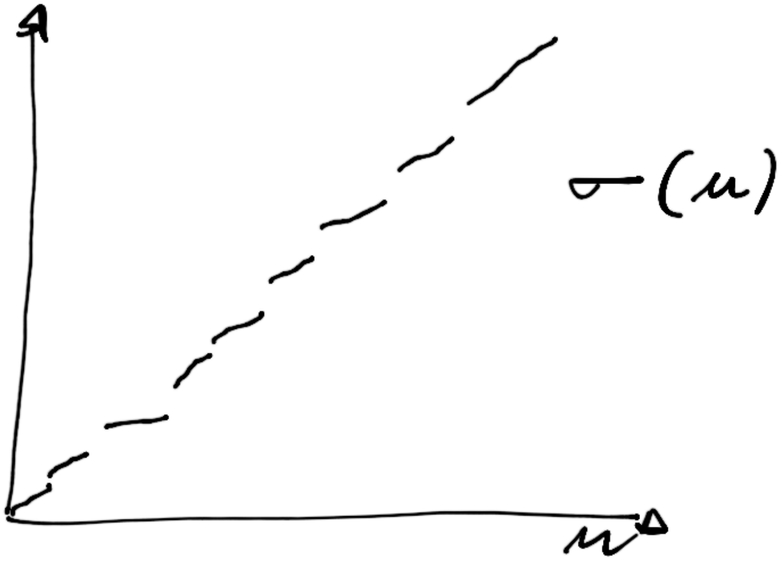
Then we have that

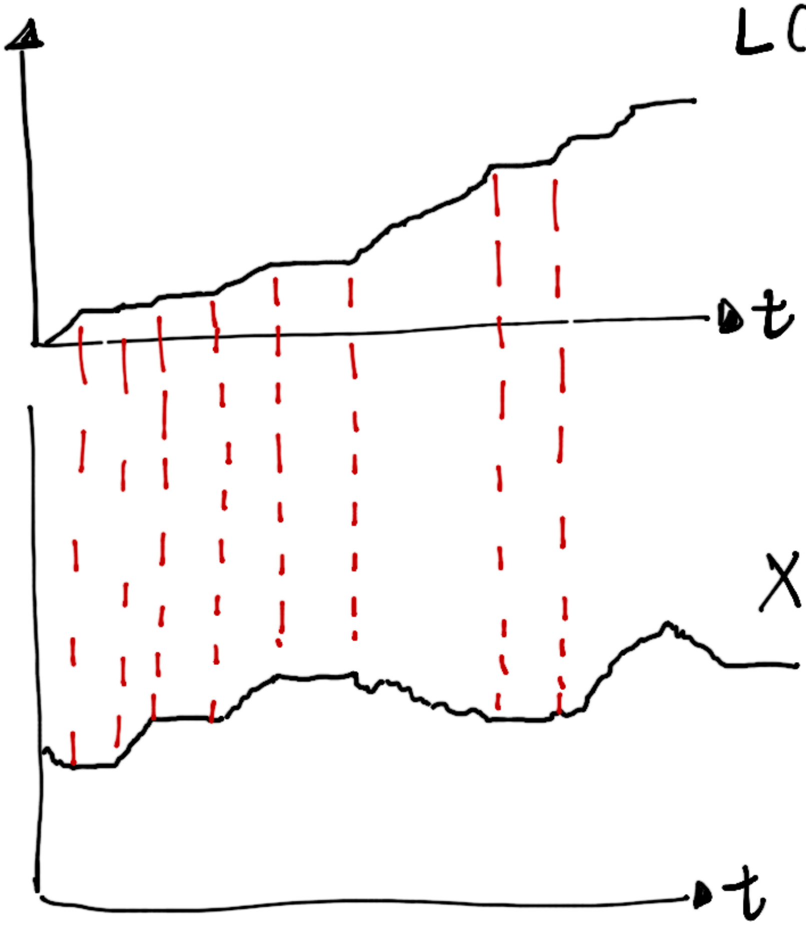
$$\frac{1}{\sqrt{c}} \sum_{i=1}^{N(c^{1/\alpha}t)} J_k \longrightarrow B_{L_t} \longleftarrow \frac{1}{\sqrt{c}} \sum_{i=1}^{N(c^{1/\alpha}t)+1} J_k \tag{18}$$

Note that

1. $\mathbb{E}^x \left\| B_{L_t} \right\|^2 \sim Ct^\alpha$ as $t \to +\infty$ where $\alpha \in (0,1)$... SUBDIFFUSION!







$L(t) = u, \qquad \sigma(u-) \leq t < \sigma(u),$

$X(t) = M(u), \qquad \sigma(u-) \leq t < \sigma(u).$

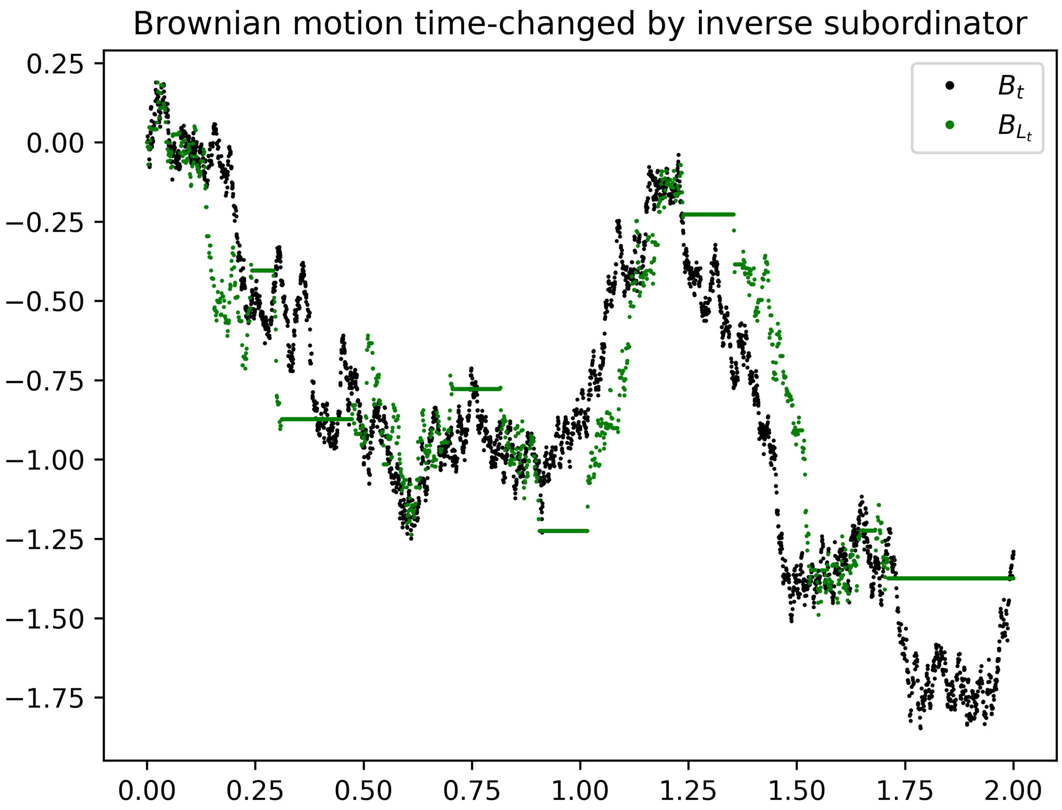


Figure 1: Exact sampling from the finite dimensional distribution (Biočić, Cedeño-Girón and T., Annals Appl. Probab. (2026)); Biočić et al. (2026+)

Associated fractional equation

Let M_t, $t \geq 0$, be a Markov process associated with a semigroup of operators T_t, $t \geq 0$, on some Banach space, with generator $(G, \mathsf{Dom}(G))$. Define

$$q(x,t) \;=\; \mathbb{E}^x u(M_{L(t)}). \tag{19}$$

Then $q(x,t)$ satisfies[2]

$$\partial_t^\alpha q(x,t) \;=\; Gq(x,t), \qquad q(x,0) = u(x) \in \mathsf{Dom}(G). \tag{20}$$

This is an abstract Cauchy problem and, for $\alpha \in (0,1)$,

$$\partial_t^\alpha u(t) \;=\; \frac{d}{dt}\int_0^t (q(s) - q(0))\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{21}$$

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[2] Baeumer and Meerschaert, FCAA, 2000

2nd Uncoupled example

1. J_k^c are i.i.d. and with distribution

$$P(J_1 \in dx) \;=\; dx \int_0^{+\infty} \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} e_\alpha(s) ds \tag{22}$$

   where $e_\alpha(\cdot)$ is the Mittag-Leffler density, i.e., J_k are i.i.d. with INFINITE EXPECTATION. Then

$$\sum_{i=1}^{[cu]} J_i^c = \frac{1}{c^{1/2\alpha}} \sum_{i=1}^{[cu]} J_i \to B_{\sigma(u)} \text{ SUBORDINATE BM}$$

2. $W_k^c = W_k / c$ where W_k are i.i.d. and have exponential distribution, i.e. $P(W_k > t) = e^{-\theta t}$, then

$$\frac{1}{c} \sum_{i=1}^{[cu]} W_k \to S(u) = \theta u \tag{23}$$

Then

$$L_t = \inf\{u > 0 : S(u) > t\} = \frac{t}{\theta} \tag{24}$$

and finally

$$\sum_{i=1}^{N(ct)} \frac{J_i^c}{c^{1/2\alpha}} \longrightarrow B_{\sigma(t/\theta)} \longleftarrow \sum_{i=1}^{N(ct)+1} \frac{J_i^c}{c^{1/2\alpha}}. \tag{25}$$

1. $B \circ \sigma$ has jumps!
2. $\mathbb{E}\left\|B_{\sigma(t/\theta)}\right\|^2 = +\infty$. SUPERDIFFUSION
3. $q(x,t) = \mathbb{E}^x u(B_{\sigma(t)})$ satisfies $\underbrace{\partial_t q(x,t) = -(-\Delta)^\alpha q(x,t)}_{\text{Markovian... all transitions, all finite dimensional}}$.

Coupled example

Suppose that the space-time jumps W_n, J_n are not independent and admit the joint density

$$f_{J,W}(x,s) \;=\; \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} e_\alpha(s) \mathbb{1}_{[x\in\mathbb{R}^2, s>0]} \tag{26}$$

where e_α is the density of the Mittag-Leffler distribution. Then we have

$$\left(\frac{1}{c^{\frac{1}{2\alpha}}} \sum_{i=1}^{[cu]} J_i, \frac{1}{c^{\frac{1}{\alpha}}} \sum_{i=1}^{[cu]} W_i \right) \to (M_u, \sigma_u), \tag{27}$$

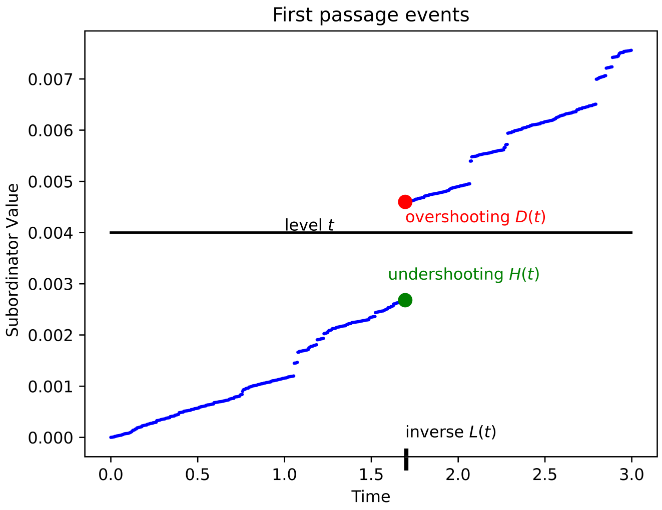
where σ_u is a stable subordinator

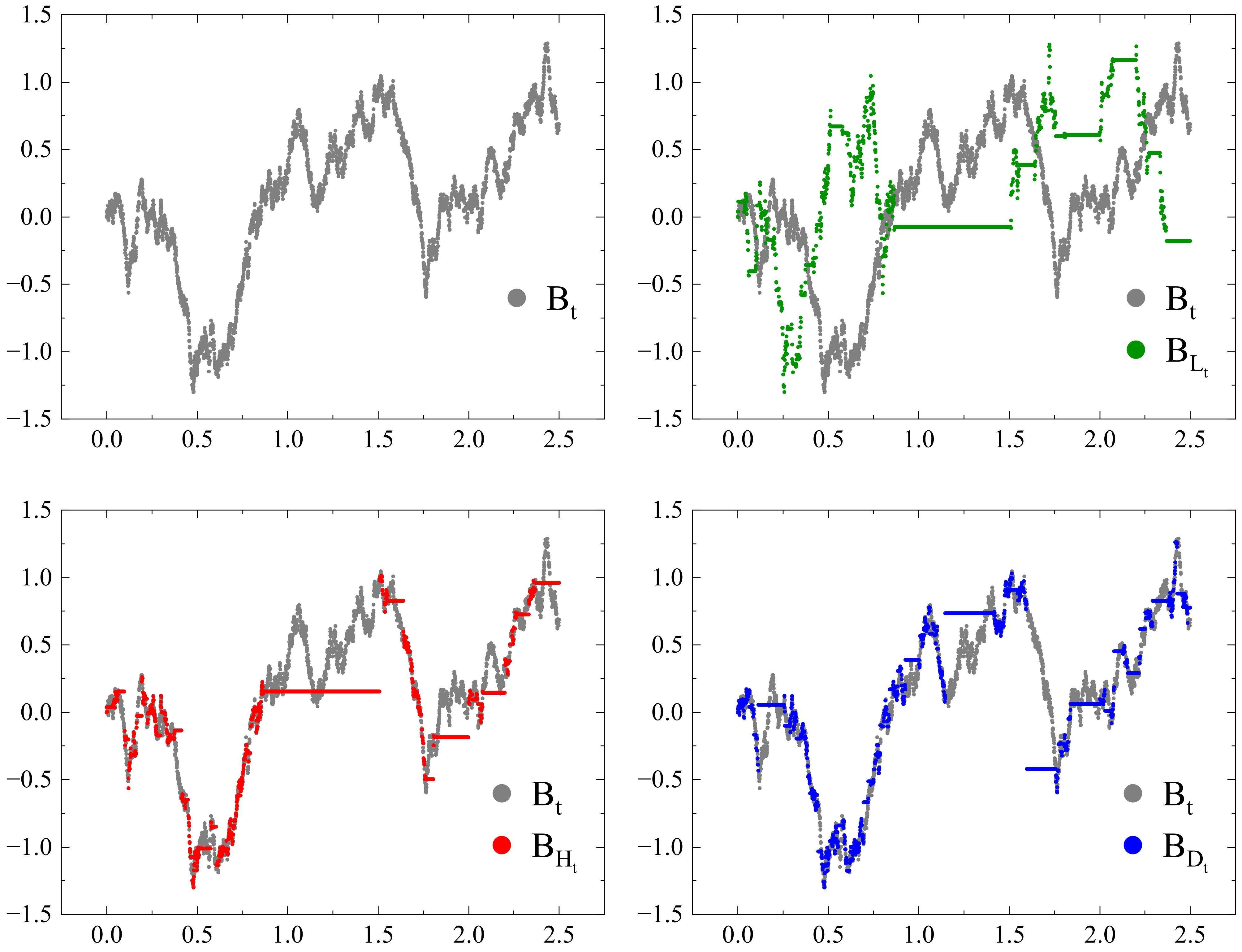
$$(M_u, \sigma_u) = (B_{\sigma_u}, \sigma_u). \tag{28}$$

It is clear that these processes are not independent and jump simultaneously! CAREFULL WHEN TAKING INVERSE!

Being L_t the inverse stable subordinator, L_t is a **jump time of** $B \circ \sigma$

$$\sum_{i=1}^{N(c^{1/\alpha}(t))} \frac{J_i}{c^{1/2\alpha}} \longrightarrow (B_{\sigma_{L_t}-})^+ = (B_{H_t})^+ \qquad \sum_{i=1}^{N(c^{1/\alpha}(t))+1} \frac{J_i}{c^{1/2\alpha}} \longrightarrow B_{\sigma_{L_t}} = B_{D_t} \tag{29}$$




1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.5
1.0
1.5
2.0
2.5
B_t
B_{L_t}
B_{H_t}
B_{D_t}

1. $\mathbb{E}^x \left\| B_{H_t} \right\|^2 \sim t$ ... DIFFUSIVE
2. $\mathbb{E}^x \left\| B_{D_t} \right\|^2 = +\infty$ ... SUPERDIFFUSIVE

Let $q^-(x,t) = \mathbb{E}^x u(B_{H_t})$ and $q^+(x,t) = \mathbb{E}^x u(B_{D_t})$, then[3]

$$(\partial_t - \Delta)^\alpha_- q^-(x,t) = 0, \qquad q^-(x,0) = u \in \mathsf{Dom}(\Delta), \tag{30}$$

$$(\partial_t - \Delta)^\alpha_+ q^+(x,t) = 0, \qquad q^+(x,0) = u \in \mathsf{Dom}(\Delta), \tag{31}$$

where

$$(\partial_t - \Delta)^\alpha_- f(x,t) \;=\; (\partial_t - \Delta)^\alpha f(x,t) + f(x,0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{32}$$

$$(\partial_t - \Delta)^\alpha_+ f(x,t) = (\partial_t - \Delta)^\alpha f(x,t) + \int_t^{+\infty} P_s f(x,0)\frac{\alpha s^{-\alpha-1}}{\Gamma(1-\alpha)} ds \tag{33}$$

$$(\partial_t - \Delta)^\alpha f(x,t) = \int_0^{+\infty} \left(P_s f(x,t-s) \mathbb{1}_{[s<t]} - f(x,t) \right) \frac{\alpha s^{-\alpha-1} ds}{\Gamma(1-\alpha)} \tag{34}$$

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[3] Ascione, Scalas, T. and Torricelli (submitted); Biočić and T. (submitted)

Another coupled example: space dependent!

Recall that for a continuous time random walks one starts with

$$(S_n, T_n) = (M_0, \sigma_0) + \sum_{i=1}^{n} (J_i, W_i) \tag{35}$$

and S_n, T_n is a Markov chain. Assume the transition probabilities

$$\mu_{x,s}(dy, dw) \; = \; e_{\alpha(x)}(w) p_x(dy) dw \tag{36}$$

where $p_x(\cdot)$ has mean zero and variance one. Then re-scale as

$$\mu^c_{x,s}(dy, dw) \; = \; \frac{1}{c^{1/\alpha(x)}} e_{\alpha(x)} \left(w \, c^{1/\alpha(x)} \right) p_x(dy/\sqrt{c}) dw, \tag{37}$$

... and send $c \to +\infty$.

Then the limit process is, under some technical assumptions, a Feller process (B_u, σ_u) generated by [4]

$$Gh(x,t) \;=\; \frac{1}{2}\Delta_x h(x,t) + \int_0^{+\infty} (h(x,t+s) - h(x,t))\, \frac{\alpha(x) s^{-\alpha(x)-1}}{\Gamma(1-\alpha(x))} ds. \tag{38}$$

Furthermore, one has that [5] this limit process is Markov additive and such that

$$\mathbb{E}^{(x,0)} e^{-\lambda \sigma(t)} \;=\; \mathbb{E}^{(x,0)} e^{-\int_0^t \lambda^{\alpha(B_s)} ds}. \tag{39}$$

It follows that B_{L_t} is such that

$$X(t) = B_s \qquad\qquad \underbrace{\sigma_{s-} \leq t < \sigma_s}_{\text{dependent on } B_S\text{, i.e., position dependent}} \tag{40}$$

and the governing equation here as the form

$$\partial_t^{\alpha(x)} q(x,t) \;=\; \frac{1}{2}\Delta q(x,t) \tag{41}$$

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[4] Straka 2018, Kolokoltsov 2022

[5] Savov and T., AIHP, 2021

Anomalous aggregation

Theorem 1 (Savov and T. AIHP, 2021). *Let* $\alpha : \mathbb{R} \mapsto (0,1)$, $\alpha^* = \min_{x\in\mathbb{R}} \alpha(x) > 0, \max_{x\in\mathbb{R}} \alpha(x) < 1$ *and*

$$1 > \lim_{x\to+\infty} \alpha(x) = \alpha_I > \alpha^*,\ 1 > \lim_{x\to-\infty} \alpha(x) = \alpha_J > \alpha^*.$$

Also let there exist β_0 *small enough such that for all* $\beta_0 \geq \beta$, *the set* $A_\beta = \{x \in \mathbb{R} : \alpha(x) < \alpha^* + \beta < 1\}$ *is bounded and satisfies* $0 < \ell(A_\beta) < \infty$ *and also* $\ell(\partial A_\beta) = 0$. *Then,*

1. *if* $2\alpha^* < \min\{\alpha_I, \alpha_J\}$ *we have that for any* $0 \leq \beta \leq \beta_0$

$$\lim_{t\to+\infty} \frac{\int_0^t \mathbb{1}_{[B(L(s))\in A_\beta]} ds}{t} = 1,\ \mathbb{P}^{(x,0)}\text{-}a.s.; \tag{42}$$

2. *and if* $2\alpha^* > \min\{\alpha_I, \alpha_J\}$, *for any* $K > 0$,

$$\lim_{t\to+\infty} \frac{\int_0^t \mathbb{1}_{\left[B(L(s))\in A_\beta^c \cap [-K,K]^c\right]} ds}{t} = 1,\ \mathbb{P}^{(x,0)}\text{-}a.s.. \tag{43}$$

The main theorem[6]

Introduce a scale parameter $c > 0$ and denote J_n^c, W_n^c the rescaled waiting times and the jumps. Let

$$(S^c_{[cu]}, T^c_{[cu]}) \Longrightarrow (M_u, \sigma_u), \quad \text{as } c \to +\infty,$$

in the space of of càdlàg functions (with J_1 topology), where (M_u, σ_u) is a Feller process on $\mathbb{R}^{d+1}$. Note that σ is increasing, and the dependency between A and σ are "arbitrary" (driven by the dependence between jumps and waiting times of the Markov chain). Straka and Henry (SPA '15.) showed that CTRW and OCTRW converge as $c \to +\infty$:

$$X_t^c = S^c_{N_t^c} \Longrightarrow (M_{L_t-})^+ := \lim_{\delta \downarrow 0} \lim_{h \uparrow 0} M_{L_{t+\delta}+h},$$

$$Y_t^c = S^c_{N_t^c+1} \Longrightarrow M_{L_t}.$$

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[6] Many authors, but the final version is Straka and Henry (2015), SPA

Come back to the Markovian CTRWs assumptions

Recall that a CTRW needs

$$(S_n, T_n) = (S_0, T_0) + \sum_{i=1}^{n} (J_i, W_i). \tag{44}$$

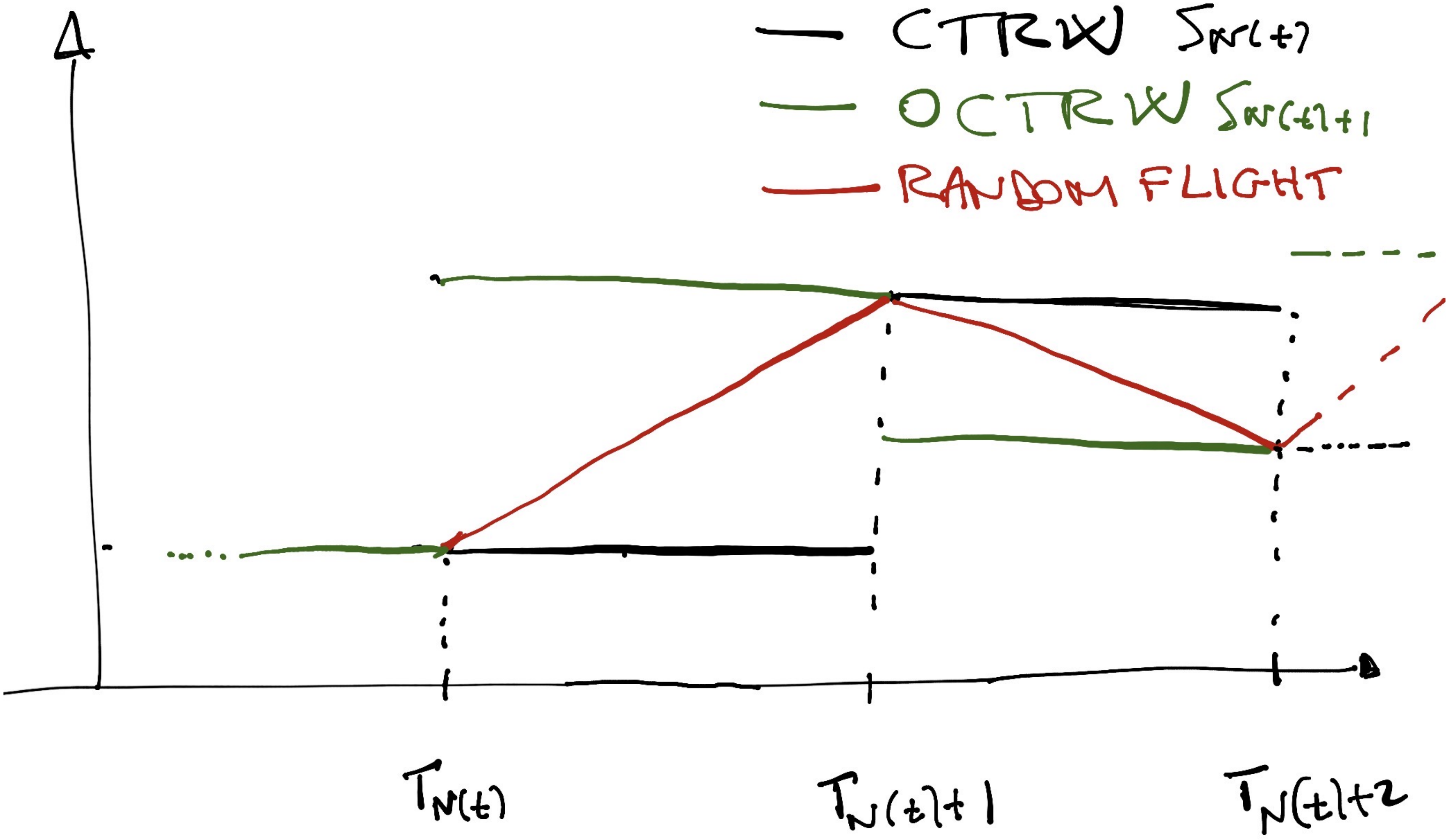
If we want that $S_{N(t)}$ is Markovian we put

$$P(W_1 > t_1, \dots, W_n > t_n | S_0 = x_0, \dots, S_{n-1} = x_{n-1}) = \prod_{k=1}^{n} e^{-\theta(x_{k-1}) t_{k-1}} \tag{45}$$

If you take now

$$\mathcal{X}(t) = \mathsf{LIN}\left[\begin{pmatrix} S_{N(t)} \\ T_{N(t)} \end{pmatrix}, \begin{pmatrix} S_{N(t)+1} \\ T_{N(t)+1} \end{pmatrix} \right] \tag{46}$$

what do you get?


CTRW $S_{N(t)}$
OCTRW $S_{N(t)+1}$
RANDOM FLIGHT
$T_{N(t)}$
$T_{N(t)+1}$
$T_{N(t)+2}$

Let's go to three dimensions

Let $\theta(x) = \theta > 0$, and the jumps are (uniformly oriented in $\mathbb{R}^3$), i.e.,

$$J_i = cv_iW_i, \text{ where } c > 0,\ v_i \text{ are i.i.d. uniform r.v.'s on } S^2 = \{u \in \mathbb{R}^3 : |u| = 1\} \tag{47}$$

$$\text{then } t \mapsto \mathcal{X}(t) \text{ is a continuous random function } \mathcal{X}(t) \ = \ \mathcal{X}(0) + c\int_0^t v_{N(s)}ds, \tag{48}$$

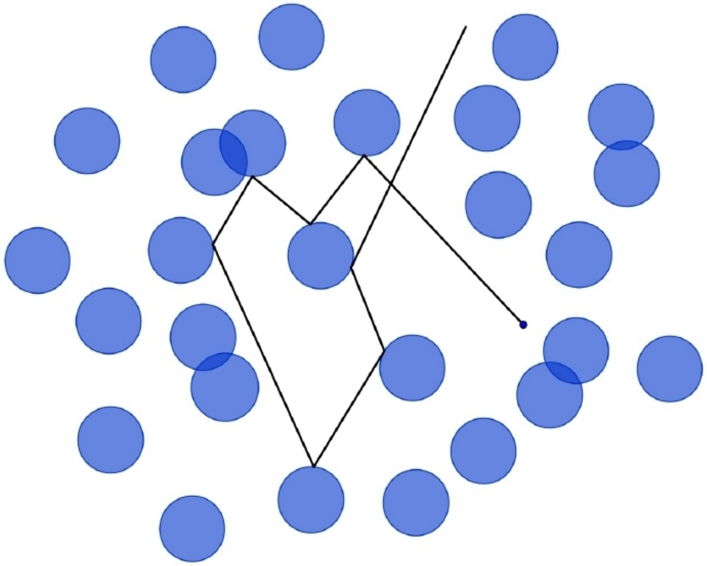
is called the MARKOVIAN RANDOM FLIGHT! The family $P_th(x,v) = \mathbb{E}^{x,v}h(\mathcal{X}(t), v_{N(t)})$, $t \geq 0$, forms a semigroup on $C_0(\mathbb{R}^3 \times S^2)$ generated by

$$Ah(x,v) \ = \ cv \cdot \nabla_x h(x,v) + \theta\int_{S_2}(h(x,y) - h(x,v))\mu(dy). \tag{49}$$

$$q(x,v,t) := P_th(x,v) \text{ solves } \partial_t q(x,v,t) = Aq(x,v,t), \quad q(x,v,0) = h(x,v) \tag{50}$$

Kinetic Processes...

G. Gallavotti. Rigorous Theory of the Boltzmann Equation in the Lorentz Gas. Nota interna n. 358, Istituto di Fisica, Universit'a di Roma, 1972.



- Motion among reflecting obstacles of radius R
- The centers of the obstacles are distributed according to a Poisson point process.

Spohn, H.: The Lorentz process converges to a random flight. Commun. Math. Phys. 60, 277–290 (1978)

Let $t \mapsto (X^{R,\rho}(t),\ V^{R,\rho}(t))$, be the position - velocity process of the particle at time $t > 0$. This is not Markovian, due to the presence of re-collisions which create a **long memory** in the evolution. $R^2\rho \to$ cost $\in (0, +\infty)$ **Boltzmann-Grad limit - The mean flight time stays finite**.

Spohn generalized (IIF) to very general Point processes such that

1. For any collection of disjoint Borel sets $B_1, B_2, \ldots, B_n$, the scaled variables $R^2\mathcal{N}_R(B_1)$, $R^2\mathcal{N}_R(B_2)$, . . . , $R^2\mathcal{N}_R(B_n)$ are independent in the limit $R \to 0$.
2. For any Borel set B, the scaled number of obstacles centers $R^2\mathcal{N}_R(B)$ converges in probability to a constant, as $R \to 0$.

The mean flight time stays finite

Many improvements on this theory exists by: Pulvirenti, Boldrighini, Basile, Saffirio, Simonella, Nota (Math Phys) and Toth, Markloff (Probability) and many others...

Brownian motion in the limit

Under the limit (diffusive limit)

$$\theta \to +\infty \qquad c \to +\infty \qquad \frac{c^2}{\theta} \to \gamma \tag{51}$$

the Markovian Random Flight converges to a diffusive process

$$\mathcal{X}(t) \to B(\gamma t). \tag{52}$$

It is also possible to express this as a scaling limit

$$\frac{1}{\sqrt{c}}\mathcal{X}(ct) \to B\left(\frac{t}{\theta}\right) \tag{53}$$

Change CTRW model

Now change the CTRW model as follows:[7]

$$P(W_1 > t_1, \ldots, W_n > t_n | S_0 = x_0, \ldots, S_{n-1} = x_{n-1}) = \prod_{k=1}^{n} e^{-\theta t_{k-1}} \;=\; e^{-\theta \sum_{k=1}^{n} t_{k-1}} \tag{54}$$

becomes

$$P(W_1 > t_1, \ldots, W_n > t_n | S_0 = x_0, \ldots, S_{n-1} = x_{n-1}) = E_\alpha \left(- \left(\theta \sum_{k=1}^{n} t_k \right)^{\alpha} \right) \tag{55}$$

to introduce heavy tailed flight times.

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[7] Facciaroni, Ricciuti, Scalas and T., Submitted (2025)

The process

$$\mathcal{X}^\alpha(t) \;=\; \mathcal{X}(0) + \int_0^t v_{N(s)} ds \tag{56}$$

is a random flight such that

1. INFINITE MEAN DISPLACEMENTS
2. $\mathbb{E}^x \left\|\mathcal{X}^\alpha(t)\right\|^2 \sim Ct^{2-\alpha}$, $\alpha \in (0,1)$ SUPERDIFFUSIVE
3. $\mathcal{X}^\alpha(t) \to B(\mathcal{L}t)$ (fdd) as $c^2/\theta \to 1$ where $\mathcal{L}$ is a Lamperti r.v.

The r.v. $\mathcal{L}$ is such that $\mathcal{L} \overset{d}{=} \mathcal{L}^{-1}$, $\mathbb{E}\mathcal{L} = +\infty$.

The process $t \mapsto B(\mathcal{L}t)$ is a randomly scaled process, it is continuous, not Markovian and $\mathbb{E}^x \left\|B(\mathcal{L}t)\right\|^2 = +\infty$, i.e., SUPERDIFFUSIVE.

Boltzmann-Grad limit?

We cannot start with a random field of obstacles inducing finite mean displacements. We must rule out the Spohn case.

Definition 1. *A point process* Π *is said to be a Mittag-Leffler point process of parameters* $\rho \in (0,\infty)$ *and* $\alpha \in (0,1]$ *if, for any collection of mutually disjoint, finite Borel sets* $\{B_j,\ j \in \{1,\dots,n\}\}$*, and any choice of non-negative integers* $\{k_j,\ j \in \{1,\dots,n\}\},\ n \geq 1$*, we have*

$$P\left(\bigcap_{j=1}^{n} (\mathcal{N}(B_j) = k_j)\right) = (-1)^k E_\alpha^{(k)}\left(-\left(\sum_{j=1}^{n} \rho |B_j|\right)^\alpha\right) \prod_{j=1}^{n} \frac{(\rho|B_j|)^{k_j}}{k_j!} \qquad (57)$$

where $k := \sum_{j=1}^{n} k_j$.

Infinite mean flight time

In our model

1. The field of obstacles is isotropic!
2. The number of centers in disjoint sets are not independent r.v.'s
3. If we run a particle as in the Gallavotti model, the mean free flight time is infinite (power law)
4. Taken one center, the distance from the nearest center is Mittag-Leffler distributed
5. $X^{\rho,R}(t) \longrightarrow \mathcal{X}^{\alpha}(t)$ as $R^2\rho \to$ cost

Fractional equations?

The diffusive of the obtained random flight is the randomly scaled Brownian motion

$$t \mapsto B(\mathcal{L}t) \tag{58}$$

It turns out that

$$B(\mathcal{L}t) \overset{d}{=} B(\sigma(L(t)))\text{(only 1-dim!!)} \tag{59}$$

where σ and L are independent stable subordinator and inverse stable subordinator. This means that it is also governed by

$$\partial_t^\alpha q(x,t) = -(-\Delta)^\alpha q(x,t) \qquad q(x,0) = u(x). \tag{60}$$

But of course they are very different processes.

References

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