Nonhomogeneous boundary condition for spectral non-local operators

Vanja Wagner
(joint work with Ivan Biočić)

University of Zagreb

Anomalous Transport and Anomalous Diffusion, Pisa, March 17, 2026.

Funded by the European Union

Research was supported by the European Union – NextGenerationEU through the National Recovery and Resilience Plan 2021-2026. Institutional grant of University of Zagreb Faculty of Science IK IA 1.1.3. Impact4Math.

The problem

- Let $D \subset \mathbf{R}^d$ be an open set - prescribe the appropriate boundary condition for the (semi-)linear equation

$$Au = f(x,u) \quad \text{in } D,$$

  where A is a nonlocal operator of regional type.
- $A = \psi(-L_{|D})$ – spectral nonlocal operator connected to the subordinate killed (upon exiting D) (subordinate) Brownian motion
- Nonlocal boundary effects – even linear solutions may blow up near ∂D.

The problem

- Let $D \subset \mathbf{R}^d$ be an open set - prescribe the appropriate boundary condition for the (semi-)linear equation

$$Au = f(x,u) \quad \text{in } D,$$

where A is a nonlocal operator of regional type.

- $A = \psi(-L_{|D})$ – spectral nonlocal operator connected to the subordinate killed (upon exiting D) (subordinate) Brownian motion
- Nonlocal boundary effects – even linear solutions may blow up near ∂D.
- Goal: identify the right “reference blow-up” $P_D^{\psi}\sigma$ and a robust notion of boundary trace (for bounded $C^{1,1}$ sets)

The problem

- Let $D \subset \mathbf{R}^d$ be an open set - prescribe the appropriate boundary condition for the (semi-)linear equation

$$Au = f(x,u) \quad \text{in } D,$$

where A is a nonlocal operator of regional type.

- $A = \psi(-L_{|D})$ – spectral nonlocal operator connected to the subordinate killed (upon exiting D) (subordinate) Brownian motion
- Nonlocal boundary effects – even linear solutions may blow up near ∂D.
- Goal: identify the right "reference blow-up" $P_D^{\psi}\sigma$ and a robust notion of boundary trace (for bounded $C^{1,1}$ sets)

$$\lim_{D \ni y \to z} \frac{u(y)}{P_D^{\psi}\sigma(y)} = \zeta(z), \quad z \in \partial D, \qquad \zeta \in C(\partial D).$$

The problem

- Let $D \subset \mathbf{R}^d$ be an open set - prescribe the appropriate boundary condition for the (semi-)linear equation

$$Au = f(x,u) \quad \text{in } D,$$

where A is a nonlocal operator of regional type.

- $A = \psi(-L_{|D})$ – spectral nonlocal operator connected to the subordinate killed (upon exiting D) (subordinate) Brownian motion
- Nonlocal boundary effects – even linear solutions may blow up near ∂D.
- Goal: identify the right "reference blow-up" $P_D^{\psi}\sigma$ and a robust notion of boundary trace (for bounded $C^{1,1}$ sets)

$$\frac{1}{\varepsilon}\int_{\{\delta_D(x)\le\varepsilon\}} \frac{u(x)}{P_D^{\psi}\sigma(x)}\varphi(x)\,dx \xrightarrow{\varepsilon\downarrow 0} \int_{\partial D}\varphi(y)\zeta(y)\,d\sigma(y), \quad \varphi\in C(\overline{D}), \quad \zeta\in L^1(\partial D).$$

The corresponding process

Base process: The subordinate Brownian motion $X = (B_{S_t})_{t\geq 0}$ with generator $-L$.

The corresponding process

Base process: The subordinate Brownian motion $X = (B_{S_t})_{t\geq 0}$ with generator $-L$. We start with the d-dim BM B and an independent Lévy subordinator S with Laplace exponent $\phi \in \mathcal{BF}$,

$$\mathbf{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad \lambda > 0.$$

Here $\phi : [0,\infty) \to [0,\infty)$ is a complete Bernstein function ($\mathcal{CBF}$), i.e.

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(t)\,dt, \quad \lambda > 0,$$

for some $b \geq 0$ and a completely monotone function $\mu : (0,\infty) \to (0,\infty)$ such that $\int_0^\infty (1 \wedge t)\mu(t)\,dt < \infty$.

The corresponding process

Base process: The subordinate Brownian motion $X = (B_{S_t})_{t\geq 0}$ with generator $-L$. We start with the d-dim BM B and an independent Lévy subordinator S with Laplace exponent $\phi \in \mathcal{BF}$,

$$\mathbf{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad \lambda > 0.$$

Here $\phi : [0,\infty) \to [0,\infty)$ is a complete Bernstein function ($\mathcal{CBF}$), i.e.

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(t)dt, \quad \lambda > 0,$$

for some $b \geq 0$ and a completely monotone function $\mu : (0,\infty) \to (0,\infty)$ such that $\int_0^\infty (1 \wedge t)\mu(t)dt < \infty$.

Define $X_t = B_{S_t}$ – the semigroup of X given via Bochner subordination of B by S.

The corresponding process

Base process: The subordinate Brownian motion $X = (B_{S_t})_{t\geq 0}$ with generator $-L$. We start with the d-dim BM B and an independent Lévy subordinator S with Laplace exponent $\phi \in \mathcal{BF}$,

$$\mathbf{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad \lambda > 0.$$

Here $\phi : [0,\infty) \to [0,\infty)$ is a complete Bernstein function ($\mathcal{CBF}$), i.e.

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(t)\,dt, \quad \lambda > 0,$$

for some $b \geq 0$ and a completely monotone function $\mu : (0,\infty) \to (0,\infty)$ such that $\int_0^\infty (1 \wedge t)\mu(t)\,dt < \infty$.

Define $X_t = B_{S_t}$ – the semigroup of X given via Bochner subordination of B by S. The corresponding generator $-L$ of X satisfies

$$\widehat{Lu}(\xi) = \phi(|\xi|^2)\,\widehat{u}(\xi), \quad \xi \in \mathbf{R}^d.$$

$\longrightarrow$ this motivates the notation $L = \phi(-\Delta)$, special case $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0,2)$.

The corresponding process

The integral representation for $b = 0$, $u \in C_b^2(\mathbf{R}^d)$

$$\phi(-\Delta)u(x) = \text{P.V.} \int_{\mathbf{R}^d} (u(x) - u(y))\, j_\phi(|y-x|)\, dy.$$

The corresponding process

The integral representation for $b = 0$, $u \in C_b^2(\mathbf{R}^d)$

$$\phi(-\Delta)u(x) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} (u(x) - u(y))\, j_\phi(|y-x|)\, dy.$$

The corresponding process

The integral representation for $b = 0$, $u \in C_b^2(\mathbf{R}^d)$

$$\phi(-\Delta)u(x) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} (u(x) - u(y))\, j_\phi(|y-x|)\, dy.$$

Killing upon exiting D: the killed process X^D with generator $-L_{|D}$ is defined as

$$X_t^D = \begin{cases} X_t, & t < \tau_D \\ \partial, & t \geq \tau_D, \end{cases}$$

where $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$.
$\longrightarrow$ $L_{|D}$ is the Dirichlet operator corresponding to L ($u = 0$ on D^c)

The corresponding process

The integral representation for $b = 0$, $u \in C_b^2(\mathbf{R}^d)$

$$\phi(-\Delta)u(x) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} (u(x) - u(y))\, j_\phi(|y-x|)\, dy.$$

Killing upon exiting D: the killed process X^D with generator $-L_{|D}$ is defined as

$$X_t^D = \begin{cases} X_t, & t < \tau_D \\ \partial, & t \geq \tau_D, \end{cases}$$

where $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$.
$\longrightarrow$ $L_{|D}$ is the Dirichlet operator corresponding to L ($u = 0$ on D^c)

Subordinate again: $Y_t = (X^D)_{T_t}$, where T is a subordinator with Laplace exponent $\psi \in \mathcal{CBF}$, independent of B and S.

The corresponding process

The integral representation for $b=0$, $u \in C_b^2(\mathbf{R}^d)$

$$\phi(-\Delta)u(x) = \lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon} (u(x)-u(y))\, j_\phi(|y-x|)\,dy.$$

Killing upon exiting D: the killed process X^D with generator $-L_{|D}$ is defined as

$$X_t^D = \begin{cases} X_t, & t<\tau_D \\ \partial, & t\geq \tau_D, \end{cases}$$

where $\tau_D = \inf\{t\geq 0 : X_t \notin D\}$.
$\longrightarrow$ $L_{|D}$ is the Dirichlet operator corresponding to L ($u=0$ on D^c)

Subordinate again: $Y_t = (X^D)_{T_t}$, where T is a subordinator with Laplace exponent $\psi \in \mathcal{CBF}$, independent of B and S.

The process Y is called the **subordinate killed subordinate Brownian motion** with generator denoted by $-\psi(-L_{|D})$.

Main examples

Let $\alpha, \beta \in (0,2)$.

- Dirichlet fractional Laplacian: $\psi(\lambda) = \lambda$, $\phi(\lambda) = \lambda^{\beta/2}$, $A = ((-\Delta)^{\beta/2})_{|D}$
  $\rightarrow$ the killed β-stable Lévy process

Main examples

Let $\alpha, \beta \in (0,2)$.

- Dirichlet fractional Laplacian: $\psi(\lambda) = \lambda$, $\phi(\lambda) = \lambda^{\beta/2}$, $A = ((-\Delta)^{\beta/2})_{|D}$
  $\rightarrow$ the killed β-stable Lévy process

- Spectral fractional Laplacian: $\psi(\lambda) = \lambda^{\alpha/2}$, $\phi(\lambda) = \lambda$, $A = (-\Delta_{|D})^{\alpha/2}$
  $\rightarrow$ $\alpha/2$-stable subordinate killed Brownian motion

Main examples

Let $\alpha, \beta \in (0,2)$.

- Dirichlet fractional Laplacian: $\psi(\lambda) = \lambda$, $\phi(\lambda) = \lambda^{\beta/2}$, $A = ((-\Delta)^{\beta/2})_{|D}$
  $\to$ the killed β-stable Lévy process
- Spectral fractional Laplacian: $\psi(\lambda) = \lambda^{\alpha/2}$, $\phi(\lambda) = \lambda$, $A = (-\Delta_{|D})^{\alpha/2}$
  $\to$ $\alpha/2$-stable subordinate killed Brownian motion
- Interpolated fractional Laplacian: $\psi(\lambda) = \lambda^{\alpha/2}$ and $\phi(\lambda) = \lambda^{\beta/2}$,
  $A = (((-\Delta)^{\beta/2})_{|D})^{\alpha/2}$
  $\to$ $\alpha/2$-stable subordinate killed β-stable Lévy process

Usual scaling condition on $\mathscr{CBF}$ ψ (and ϕ)

In order to obtain two-sided (heat kernel, jumping density, Green function, etc.) estimates, we assume a weak scaling condition on ψ and/or ϕ:

A $\varphi \in \mathscr{BF}$ satisfies (WSC) if there exist $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0,1)$ such that

$$a_1 \lambda^{\delta_1} \le \frac{\varphi(\lambda t)}{\varphi(t)} \le a_2 \lambda^{\delta_2}, \quad \lambda \ge 1,\ t > 0.$$

Usual scaling condition on $\mathscr{CBF}$ ψ (and ϕ)

In order to obtain two-sided (heat kernel, jumping density, Green function, etc.) estimates, we assume a weak scaling condition on ψ and/or ϕ:

A $\varphi \in \mathscr{BF}$ satisfies (WSC) if there exist $a_1, a_2 > 0$ and $\delta_1, \delta_2 \in (0,1)$ such that

$$a_1 \lambda^{\delta_1} \le \frac{\varphi(\lambda t)}{\varphi(t)} \le a_2 \lambda^{\delta_2}, \quad \lambda \ge 1,\ t > 0.$$

Recall that for $(-\Delta)^{\beta/2}$ we have the jumping density of the form

$$j(x,y) = \frac{C_{d,\beta}}{|y-x|^{d+\beta}},\ x, y \in \mathbf{R}^d.$$

Under (WSC) for ϕ, the subordinate Brownian motion X with generator $-\phi(-\Delta)$ has the jumping density

$$j_\phi(x,y) = j_\phi(|x-y|) \asymp \frac{\phi(|x-y|^{-2})}{|x-y|^d},\ x, y \in \mathbf{R}^d.$$

Integral representation of the regional operators $A = \psi(-\phi(-\Delta)_{|D})$

The variations of this model can be observed through the integral representation of the operator

$$Au(x) = \text{P.V.} \int_D \big(u(x) - u(y)\big) J(x,y)\, dy + \kappa(x)\, u(x), \qquad u \in C_b^2(D),$$

- $J(x,y)$ is the **jumping density** of Y – due to the interaction with the domain, it is not (in general) radial,
- κ is the **killing density** (potential).

Integral representation of the regional operators $A = \psi(-\phi(-\Delta)_{|D})$

The variations of this model can be observed through the integral representation of the operator

$$Au(x) = \text{P.V.}\int_D \big(u(x) - u(y)\big) J(x,y)\, dy + \kappa(x)\, u(x), \qquad u \in C_b^2(D),$$

- $J(x,y)$ is the **jumping density** of Y – due to the interaction with the domain, it is not (in general) radial,
- κ is the **killing density** (potential).

Interior behaviour of jumps: $D_r := \{x \in D : \delta_D(x) > r\}$

- ϕ satisfies (WSC), $\psi(t) = t \to J(x,y) = j_\phi(|x-y|)$, $x, y \in D$;
- ψ satisfies (WSC), $\phi(t) = t \to J(x,y) \asymp j_\psi(|x-y|)$, $x, y \in D_r$;
- ψ, ϕ satisfy (WSC) $\to J(x,y) \asymp j_{\psi\circ\phi}(|x-y|)$, $x, y \in D_r$.

Boundary behaviour of J in the spectral/interpolated case

Here, the jumping kernel J is obtained by averaging the heat kernel p_D of X^D against the Lévy density ν of the subordinator T:

$$J(x,y) = \int_0^\infty p_D(t,x,y)\,\nu(t)\,dt, \qquad \kappa(x) = \int_0^\infty \mathbf{P}_x(\tau_D \le t)\,\nu(t)\,dt.$$

Boundary behaviour of J in the spectral/interpolated case

Here, the jumping kernel J is obtained by averaging the heat kernel p_D of X^D against the Lévy density ν of the subordinator T:

$$J(x,y) = \int_0^\infty p_D(t,x,y)\,\nu(t)\,dt, \qquad \kappa(x) = \int_0^\infty \mathbf{P}_x(\tau_D \le t)\,\nu(t)\,dt.$$

- In the spectral subordinate Laplacian setting (ψ satisfies (WSC), $\phi(t) = t$) the boundary behaviour is inherited from the boundary decay of harmonic functions for the Dirichlet Laplacian

$$J(x,y) \asymp \left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right) j_\psi(|x-y|), \quad x,y \in D.$$

Boundary behaviour of J in the spectral/interpolated case

Here, the jumping kernel J is obtained by averaging the heat kernel p_D of X^D against the Lévy density ν of the subordinator T:

$$J(x,y) = \int_0^\infty p_D(t,x,y)\,\nu(t)\,dt, \qquad \kappa(x) = \int_0^\infty \mathbf{P}_x(\tau_D \le t)\,\nu(t)\,dt.$$

- In the spectral subordinate Laplacian setting (ψ satisfies (WSC), $\phi(t)=t$) the boundary behaviour is inherited from the boundary decay of harmonic functions for the Dirichlet Laplacian

$$J(x,y) \asymp \left(\frac{\delta_D(x)}{|x-y|}\wedge 1\right)\left(\frac{\delta_D(y)}{|x-y|}\wedge 1\right) j_\psi(|x-y|), \quad x,y\in D.$$

- In the interpolated case (ψ,ϕ satisfy (WSC)) the boundary decay is of nonlinear order, with a more complex structure

$$\frac{J(x,y)}{j_{\psi\circ\phi}(|x-y|)} \asymp \begin{cases} \left(\frac{(\delta_D(x)\wedge\delta_D(y))^\beta}{|x-y|^\beta}\wedge 1\right)^{\frac12}\left(\frac{(\delta_D(x)\vee\delta_D(y))^\beta}{|x-y|^\beta}\wedge 1\right)^{\frac12-\frac{\alpha}{2}}, & \alpha<1, \\ \left(\frac{(\delta_D(x)\wedge\delta_D(y))^\beta}{|x-y|^\beta}\wedge 1\right)^{\frac12}\log\left(1+\frac{(\delta_D(x)\vee\delta_D(y))^\beta\wedge|x-y|^\beta}{(\delta_D(x)\wedge\delta_D(y))^\beta\wedge|x-y|^\beta}\right), & \alpha=1, \\ \left(\frac{(\delta_D(x)\wedge\delta_D(y))^\beta}{|x-y|^\beta}\wedge 1\right)^{1-\frac{\alpha}{2}}, & \alpha>1. \end{cases}$$

The operator $\psi(-L_{|D})$ - spectral approach

Let $\{\varphi_k\}$ be an ONB in $L^2(D)$ of eigenfunctions for $-\phi(-\Delta)_{|D}$,

$$-\phi(-\Delta)_{|D}\varphi_k = \lambda_k \varphi_k \text{ in } D$$
$$\varphi_k|_{\partial D} = 0.$$

The operator $\psi(-L_{|D})$ - spectral approach

Let $\{\varphi_k\}$ be an ONB in $L^2(D)$ of eigenfunctions for $-\phi(-\Delta)_{|D}$,

$$-\phi(-\Delta)_{|D}\varphi_k = \lambda_k\varphi_k \text{ in } D$$
$$\varphi_k|_{\partial D} = 0.$$

Here $\lambda_j \asymp \phi(j^{2/d})$, $j \in \mathbf{N}$, [Chen, Song, 2005], and by [KKLL, 2019]

$$\|\varphi_j\|_{L^\infty(D)} \le C_1\lambda_j^{k-1},$$
$$\|\varphi_j\|_{C^V(D)} \le C_2\lambda_j^k,$$
$$\left\|\frac{\varphi_j}{V(\delta_D)}\right\|_{C^\varepsilon(D)} \le C_3\lambda_j^k.$$

and the principal eigenfunction φ_1 can be chosen such that $\varphi_1 > 0$. Then by [Biswas, Lőrinczi, 2021] it holds that

$$\varphi_1 \asymp V(\delta_D) \quad \text{in } D.$$

The operator $\psi(-L_{|D})$ - spectral approach

Let $\{\varphi_k\}$ be an ONB in $L^2(D)$ of eigenfunctions for $-\phi(-\Delta)_{|D}$,

$$-\phi(-\Delta)_{|D}\varphi_k = \lambda_k\varphi_k \text{ in } D$$
$$\varphi_k|_{\partial D} = 0.$$

Here $\lambda_j \asymp \phi(j^{2/d})$, $j \in \mathbf{N}$, [Chen, Song, 2005], and by [KKLL, 2019]

$$\|\varphi_j\|_{L^\infty(D)} \le C_1\lambda_j^{k-1},$$
$$\|\varphi_j\|_{C^V(D)} \le C_2\lambda_j^{k},$$
$$\left\|\frac{\varphi_j}{V(\delta_D)}\right\|_{C^\varepsilon(D)} \le C_3\lambda_j^{k}.$$

and the principal eigenfunction φ_1 can be chosen such that $\varphi_1 > 0$. Then by [Biswas, Lőrinczi, 2021] it holds that

$$\varphi_1 \asymp V(\delta_D) \quad \text{in } D.$$

Define $\psi(-L_{|D})u = \sum_{k\ge1}\psi(\lambda_k)\,\widehat{u}_k\,\varphi_k$ with

$$\mathscr{D}(\psi(-L_{|D})) = \left\{u = \sum_{j=1}^{\infty}\widehat{u}_j\varphi_j \in L^2(D) : \sum_{j=1}^{\infty}\psi(\lambda_j)^2|\widehat{u}_j|^2 < \infty\right\}.$$

Green function of Y

- Green function of Y:

$$G_D^\psi(x,y) = \int_0^\infty p^Y(t,x,y)\,dt = \int_0^\infty p_D(t,x,y)u(t)dt,$$

  where u denotes the density of the potential measure of T.

- Green potential of f:

$$G_D^\psi f(x) = \int_D G_D^\psi(x,y)f(y)\,dy = \mathbf{E}_x\left[\int_0^\infty f(Y_t)\,dt\right].$$

- G_D^ψ is the inverse of $\psi(-L_{|D})$ (in L^2)

$$G_D^\psi f = \sum_{j=1}^\infty \frac{1}{\psi(\lambda_j)}\widehat{f}_j\varphi_j, \quad f \in L^2(D).$$

Boundary behaviour of the Green function

Boundary decay is given in terms of the renewal function $V(t) \asymp \frac{1}{\sqrt{\phi(t^{-2})}} (= t^{\beta/2})$ associated with X,

$$G_D^{\psi}(x,y) \asymp \left(\frac{V(\delta_D(x))}{V(|x-y|)} \wedge 1 \right) \left(\frac{V(\delta_D(y))}{V(|x-y|)} \wedge 1 \right) \frac{1}{|x-y|^d \, \psi(V(|x-y|)^{-2})}.$$

Boundary behaviour of the Green function

Boundary decay is given in terms of the renewal function $V(t) \asymp \frac{1}{\sqrt{\phi(t^{-2})}} (= t^{\beta/2})$ associated with X,

$$G_D^\psi(x,y) \asymp \left(\frac{V(\delta_D(x))}{V(|x-y|)} \wedge 1\right)\left(\frac{V(\delta_D(y))}{V(|x-y|)} \wedge 1\right) \frac{1}{|x-y|^d \, \psi(V(|x-y|)^{-2})}.$$

Definition of the Poisson kernel (modified Martin kernel):

$$P_D^\psi(x,z) := -\partial_V G_D^\psi(x,z) = \lim_{D \ni y \to z} \frac{G_D^\psi(x,y)}{V(\delta_D(y))}, \qquad x \in D,\ z \in \partial D.$$

Boundary behaviour of the Green function

Boundary decay is given in terms of the renewal function $V(t) \asymp \frac{1}{\sqrt{\phi(t^{-2})}} (= t^{\beta/2})$ associated with X,

$$G_D^{\psi}(x,y) \asymp \left(\frac{V(\delta_D(x))}{V(|x-y|)} \wedge 1\right)\left(\frac{V(\delta_D(y))}{V(|x-y|)} \wedge 1\right) \frac{1}{|x-y|^d \psi(V(|x-y|)^{-2})}.$$

Definition of the Poisson kernel (modified Martin kernel):

$$P_D^{\psi}(x,z) := -\partial_V G_D^{\psi}(x,z) = \lim_{D \ni y \to z} \frac{G_D^{\psi}(x,y)}{V(\delta_D(y))}, \qquad x \in D,\ z \in \partial D.$$

- use the G_D^{ψ} representation via the potential density u, DCT via HKE for p_D

$$\lim_{D \ni y \to z} \int_0^{\infty} \frac{p_D(t,x,y)}{V(\delta_D(y))} u(t)\,dt = \int_0^{\infty} \lim_{D \ni y \to z} \frac{p_D(t,x,y)}{V(\delta_D(y))} u(t)\,dt$$

- existence (and joint continuity) of $\partial_V p_D(t,x,z) := \lim_{D \ni y \to z} p_D(t,x,y)/V(\delta_D(y))$ follows from the spectral representation + regularity of eigenfunctions

$$p_D(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y).$$

Representation of harmonic functions

Intuition: $P_D^{\psi}(\cdot, z)$ encodes how harmonic functions build up from boundary data.

Representation: For every nonnegative $\psi(-L_{|D})$-harmonic function h there exists a finite measure ζ on ∂D such that

$$h(x) = P_D^{\psi}\zeta(x).$$

Probabilistic equivalence: h is $\psi(-L_{|D})$-harmonic (distributional) iff it satisfies the mean-value property for the process Y (after modification on a null set), i.e. if for all $U \subset\subset D$ it holds that

$$h(x) = \mathbf{E}_x[h(Y_{\tau_U})], \quad x \in U,$$

where $\tau_U = \inf\{t > 0 : Y_t \notin U\}$ and we assume the expectation is finite.

Representation of harmonic functions

Intuition: $P_D^{\psi}(\cdot, z)$ encodes how harmonic functions build up from boundary data.

Representation: For every nonnegative $\psi(-L_{|D})$-harmonic function h there exists a finite measure ζ on ∂D such that

$$h(x) = P_D^{\psi}\zeta(x).$$

Probabilistic equivalence: h is $\psi(-L_{|D})$-harmonic (distributional) iff it satisfies the mean-value property for the process Y (after modification on a null set), i.e. if for all $U \subset\subset D$ it holds that

$$h(x) = \mathbf{E}_x[h(Y_{\tau_U})], \quad x \in U,$$

where $\tau_U = \inf\{t > 0 : Y_t \notin U\}$ and we assume the expectation is finite.

→ This is what makes the boundary condition natural: prescribing ζ is prescribing the boundary measure in the harmonic representation.

Proof via subordination

Recall $\psi \in \mathscr{CBF} \subset \mathscr{SBF}$. Introduce the conjugate Bernstein function of ψ:

$$\psi^*(\lambda) := \frac{\lambda}{\psi(\lambda)}, \quad \lambda > 0.$$

Harmonic representation:

- $h \in L^1(D, \rho(\delta_D))$ is harmonic in D with respect to $\psi(-L_{|D})$ $\iff G_D^{\psi^*} h$ is harmonic in D with respect to $L_{|D}$
- for $K_D(x,z) = \lim_{D \ni y \to z} \frac{G_D(x,y)}{V(\delta_D(y))}$, the modified Martin kernel in D of the Lévy process X

$$\int_D G_D^{\psi^*}(x,\xi) P_D^{\psi}(\xi,z)\, d\xi = K_D(x,z),$$

which implies

$$G_D^{\psi^*} P_D^{\psi} \zeta = K_D \zeta, \;\; \zeta \in \mathscr{M}(\partial D).$$

The reference blow-up function $P_D^\psi\sigma$

$P_D^\psi(x,z)$ is jointly continuous on $D\times\partial D$ and

$$P_D^\psi(x,z)\asymp\frac{V(\delta_D(x))}{V(|x-z|)^2\,|x-z|^d\,\psi(V(|x-z|)^{-2})}.$$

Let σ be the $(d-1)$-Hausdorff measure on ∂D and define

$$P_D^\psi\sigma(x):=\int_{\partial D}P_D^\psi(x,z)\,d\sigma(z).$$

Sharp boundary behaviour:

$$P_D^\psi\sigma(x)\asymp\frac{1}{\delta_D(x)\,V(\delta_D(x))\,\psi(V(\delta_D(x))^{-2})}.$$

Meaning: $P_D^\psi\sigma$ gives the *universal explosion rate* for positive $\psi(-L_{|D})$-harmonic functions near ∂D. So the "Dirichlet datum" is best formulated as a ratio $u/(P_D^\psi\sigma)$.

A weak boundary trace

For $\zeta \in L^1(\partial D)$ and $\varphi \in C(\overline{D})$:

$$\frac{1}{t}\int_{\{\delta_D(x)\le t\}} \frac{u(x)}{P_D^{\psi}\sigma(x)}\varphi(x)\,dx \xrightarrow[t\downarrow 0]{} \int_{\partial D}\varphi(z)\zeta(z)\,d\sigma(z).$$

Bonus (pointwise): if z is a continuity point of ζ then

$$\lim_{D\ni x\to z}\frac{u(x)}{P_D^{\psi}\sigma(x)} = \zeta(z),$$

and the convergence is uniform if $\zeta \in C(\partial D)$.

Result: $u = P_D^{\psi}\zeta$ is a $\psi(-L_{|D})$-harmonic function satisfying this boundary condition

A weak boundary trace

For $\zeta \in L^1(\partial D)$ and $\varphi \in C(\overline{D})$:

$$\frac{1}{t} \int_{\{\delta_D(x) \le t\}} \frac{u(x)}{P_D^{\psi}\sigma(x)} \varphi(x)\, dx \xrightarrow[t\downarrow 0]{} \int_{\partial D} \varphi(z)\zeta(z)\, d\sigma(z).$$

Bonus (pointwise): if z is a continuity point of ζ then

$$\lim_{D \ni x \to z} \frac{u(x)}{P_D^{\psi}\sigma(x)} = \zeta(z),$$

and the convergence is uniform if $\zeta \in C(\partial D)$.

Result: $u = P_D^{\psi}\zeta$ is a $\psi(-L_{|D})$-harmonic function satisfying this boundary condition $\longrightarrow$ this is a clean trace-like boundary operator adapted to nonlocal blow-up.

Linear Dirichlet problem: representation and uniqueness

Consider

$$\begin{cases} \psi(-L_{|D})\,u = \lambda & \text{in } D, \\ \dfrac{u}{P_D^{\psi}\sigma} = \zeta & \text{on } \partial D, \end{cases}$$

with the integrability condition $\int_D V(\delta_D(x))\,|\lambda|(dx) + |\zeta|(\partial D) < \infty$.

Solution: there is a unique weak-dual solution and it has the explicit form

$$u = G_D^{\psi}\lambda + P_D^{\psi}\zeta.$$

Linear Dirichlet problem: representation and uniqueness

Consider

$$\begin{cases} \psi(-L_{|D})u = \lambda & \text{in } D, \\ \dfrac{u}{P_D^\psi \sigma} = \zeta & \text{on } \partial D, \end{cases}$$

with the integrability condition $\int_D V(\delta_D(x))\,|\lambda|(dx) + |\zeta|(\partial D) < \infty$.

Solution: there is a unique weak-dual solution and it has the explicit form

$$u = G_D^\psi \lambda + P_D^\psi \zeta.$$

$$\iff \int_D u(x)\psi(-L_{|D})\varphi(x)\,dx = \int_D \varphi(x)\lambda(dx) - \int_{\partial D} \frac{\partial}{\partial V}\varphi(z)\zeta(dz)$$

for all $\varphi \in \{\eta \in C^V(D) : \exists \xi \in C_c^\infty(D)$ such that $\psi(-L_{|D})\eta = \xi$ in D, $\eta_{|\partial D} \equiv 0\}$.

Linear Dirichlet problem: representation and uniqueness

Consider

$$\begin{cases} \psi(-L_{|D})u = \lambda & \text{in } D, \\ \dfrac{u}{P_D^{\psi}\sigma} = \zeta & \text{on } \partial D, \end{cases}$$

with the integrability condition $\int_D V(\delta_D(x))\,|\lambda|(dx) + |\zeta|(\partial D) < \infty$.

Solution: there is a unique weak-dual solution and it has the explicit form

$$u = G_D^{\psi}\lambda + P_D^{\psi}\zeta.$$

$$\iff \int_D u(x)\psi(-L_{|D})\varphi(x)\,dx = \int_D \varphi(x)\lambda(dx) - \int_{\partial D} \frac{\partial}{\partial V}\varphi(z)\zeta(dz)$$

for all $\varphi \in \{\eta \in C^V(D) : \exists \xi \in C_c^{\infty}(D) \text{ such that } \psi(-L_{|D})\eta = \xi \text{ in } D, \eta_{|\partial D} \equiv 0\}$.

Interpretation:

- $G_D^{\psi}\lambda$ is the "interior forcing" part and is annihilated at the boundary (in a weak sense).
- $P_D^{\psi}\zeta$ is the "harmonic extension" that carries the boundary trace.

Why weighted spaces appear

Two natural weighted spaces $L^1(D, w(x)\,dx)$ show up:

- $w = V(\delta_D)$ $\to$ ensures the finiteness of Green potentials
  $\to$ appropriate space for weak-dual solutions

- $w = \rho(\delta_D) \to$ appropriate space for distributional solutions since it controls $\psi(-L_{|D})\xi$ for $\xi \in C_c^\infty(D)$:

$$\psi(-L_{|D})\xi(x) \asymp \text{P.V.}\int_{\operatorname{supp}\xi} J(x,y)\,dy \asymp \rho(\delta_D).$$

Recall that the jump kernel $J(x,y)$ *vanishes at the boundary* in a way dictated by ψ and V (i.e., ϕ), so $\psi(-L_{|D})\xi$ has a characteristic boundary profile $\rho(\delta_D$.

This is what makes distributional formulations and boundary traces compatible in optimal L^1 scales.

Semilinear problem: general existence theory

We study

$$\begin{cases} \psi(-L_{|D})u = f(x,u) & \text{in } D, \\ \dfrac{u}{P_D^{\psi}\sigma} = \zeta & \text{on } \partial D, \end{cases}$$

under a very flexible growth envelope:

$$|f(x,t)| \le q(x)\,\Lambda(|t|),$$

where q is locally bounded and Λ is nondecreasing.

Main existence results (informal):

- **Nonpositive f:** if $q\,\Lambda(P_D^{\psi}\zeta) \in L^1(D, V(\delta_D)dx)$, then there exists $u \in C(D) \cap L^1(D, V(\delta_D)dx)$.
- **Nonnegative monotone f:** a smallness-type condition yields existence via monotone iterations.
- **Signed / nonmonotone:** existence under sublinear growth (or small parameter).

Interpolated fractional Laplacian: explicit exponents

Going back to the special case

$$\psi(\lambda) = \lambda^{\alpha/2}, \qquad \phi(\lambda) = \lambda^{\beta/2}, \qquad \alpha, \beta \in (0,2).$$

Then $V(t) \asymp t^{\beta/2}$ and the reference blow-up becomes

$$P_D^{\psi}\sigma(x) \asymp \delta_D(x)^{-1-\beta/2+\alpha\beta/2}.$$

For the power-type nonlinearity

$$f(x,t) = \pm\,\delta_D(x)^{\theta}|t|^{p}$$

the critical exponent (existence / nonexistence threshold) is equal to

$$p_* = \frac{1 + \frac{2\theta}{2+\beta}}{1 - \frac{\beta\alpha}{2+\beta}},$$

i.e. a solution exists (with the correct boundary behaviour) iff $p < p_*$.

How to interpret the critical exponent (intuition)

Heuristic: near ∂D, a solution behaves like the boundary profile

$$u(x) \asymp P_D^{\psi}\sigma(x) \asymp \delta_D(x)^{-1-\beta/2+\alpha\beta/2}.$$

Plugging this into the forcing $f(x,u) \sim \delta_D(x)^{\theta}|u|^p$ predicts an integrability condition in the Green potential

$$\int_D G_D^{\psi}(\cdot,y)\,\delta_D(y)^{\theta+p(-1-\beta/2+\alpha\beta/2)}\,dy < \infty,$$

and the borderline case gives exactly p_*.

References

Cho, Kim, Song, Vondraček: *Heat kernel estimates for subordinate Markov processes and their applications*, Journal of Differential Equations, 2022.

Abatangelo, Gómez-Castro, Vázquez: *Singular boundary behaviour and large solutions for fractional elliptic equations*, Journal of the London Mathematical Society. Second Series, 2023.

Huynh, Nguyen: *Compactness of Green operators with applications to semilinear nonlocal elliptic equations*, Journal of Differential Equations, 2025.

Biočić, W.: *Large solutions for subordinate spectral Laplacian*, Nonlinear Differential Equations and Applications, 2026.

Biočić: *Representation of harmonic functions with respect to subordinate Brownian motion*, Journal of Mathematical Analysis and Applications, 2022.

Klimsiak, Rozkosz: *Dirichlet problem for semilinear partial integro-differential equations: the method of orthogonal projection*, 2025.

https://web.math.pmf.unizg.hr/kcpsc/

Lectures: Bruno Toaldo (University of Turin), VW, Jaeyun Yi (SLMath, Berkeley)