Anomalous Diffusion, Lévy Processes, and Fractional Calculus

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Aim of the course

We discuss different interplays between

- fractional calculus
  (i.e. theory of non-local operators with non-integer order and their generalizations)

  and

- the theory of stochastic processes, focusing on models of anomalous diffusions
  (i.e. diffusions of particles spreading in a different manner than the traditional one)

Point of view: probability theory

Contents

Standard diffusion and CLT

Standard diffusion $\Longleftrightarrow$ long-time limit of a random walk (RW)

$$S_n := \sum_{j=1}^{n} Y_j, \qquad n \geq 1,$$

under proper rescaling, where the jumps Y_j are i.i.d. (zero-mean) r.v.'s with finite variance, occurring at regularly spaced intervals.

By the central limit theorem (CLT) the rescaled RW

$$n^{-1/2} S_{\lfloor nt \rfloor} = n^{-1/2}(Y_1 + \ldots + Y_{\lfloor nt \rfloor}), \qquad t \geq 0,$$

converges to the Brownian motion B_t, for any $t \geq 0$, as $n \to \infty$, whatever the distribution of the jumps (see details later)

Brownian motion

The Brownian motion $B := \{B_t\}_{t\geq 0}$ is a stochastic process defined by

1. $B_0 = 0$ a.s.
2. it is continuous a.s.
3. it has independent increments over non-overlapping intervals (i.e. $B_{t_2} - B_{t_1} \perp B_{s_2} - B_{s_1}$, $s_1 < s_2 < t_1 < t_2$)
4. $B_t - B_s \sim N(0, |t-s|)$, $s, t \geq 0$.

$\Longrightarrow$ the mean-square displacement (MSD) of the moving particle is

$$var(B_t) = t$$

(standard diffusion)

Real world: anomalous diffusions

In real situations we can have asymmetry, drifts or heavy tails in distributions...

The motions can behave as

- super-diffusions (faster than usual): $\sim t^\alpha$, for $\alpha > 1$
  which can be produced by fractional derivatives in space (in the equation governing the motion) $\Longleftrightarrow$ long power-law jumps
- sub-diffusions (slower than usual): $\sim t^\alpha$, for $\alpha < 1$
  which can be produced by fractional derivatives in time (in the equation governing the motion) $\Longleftrightarrow$ long power-law waiting times between particle jumps

CLT: details

Central Limit Theorem

Let Y_j, $j = 1, ..., n$, be i.i.d. r.v.'s with probability density function $f_Y(\cdot)$, with p-absolute moment $\mu_p = \mathbb{E}|Y|^p := \int |y|^p f_Y(y)dy < \infty$, $p \in \mathbb{N}$, and with Fourier transform $\hat{f}(\kappa) := \mathbb{E}e^{-i\kappa Y} = \int e^{-i\kappa y} f_Y(y)dy$, $\kappa \in \mathbb{R}$.
If $\mu_1 = 0$ and $\mu_2 = \sigma^2$ then

$$n^{-1/2} S_n = n^{-1/2} \sum_{j=1}^{n} Y_j \Longrightarrow Z \sim N(0, \sigma^2)$$

as $n \to \infty$, where $\Longrightarrow$ denotes convergence in distribution.

CLT: details

Proof: If $\hat{f}_Y(\cdot)$ is p times differentiable, by Taylor's series expansion

$$\hat{f}_Y(\kappa) = \sum_{j=0}^{p} \frac{\kappa^j}{j!} \hat{f}^{(j)}(0) + o(\kappa^p) = \sum_{j=0}^{p} \frac{(-i\kappa)^j}{j!} \mu_j + o(\kappa^p)$$

By independence and i.d. of jumps, the normalized sum has FT

$$\mathbb{E} e^{-i\kappa n^{-1/2} S_n} = \left(\hat{f}_Y(n^{-1/2}\kappa)\right)^n = \left(1 - \frac{\kappa^2\sigma^2}{n} + o(n^{-1})\right)^n \to e^{-\kappa^2\sigma^2},$$

as $n \to \infty$, which is the characteristic function of a $N(0, \sigma^2)$. By the Lévy continuity theorem of the FT this is sufficient to prove the CLT.

Convergence of rescaled RW

The CLT can be extended to the convergence (of f.d.d.) of a re-scaled RW.

Corollary

Let $c > 0$ and let $\lfloor a \rfloor = k \in \mathbb{N}$, for $k \leq a < k+1$, then

$$c^{-1/2} S_{\lfloor ct \rfloor} \Longrightarrow B_t, \qquad c \to +\infty,$$

for any $t \geq 0$, where $B_t \sim N(0, 2t)$.

The location of the particle at time t, for large time scale c, converges to the position of a Brownian motion.

Convergence of rescaled RW

Proof:

$$\mathbb{E}e^{-i\kappa c^{-1/2}S_{\lfloor ct \rfloor}} = \left[\left(1 - \frac{\kappa^2}{c} + o(c^{-1}) \right)^{\lfloor ct \rfloor / c} \right]^c \to e^{-\kappa^2 t}, \qquad c \to \infty,$$

which is the characteristic function of a $N(0, 2t)$. Convergence holds again by the continuity theorem.

BM and heat equation

The transition density of the BM, i.e. $p(x,t) := P(B_t \in dx)/dx$, $x \in \mathbb{R}$, $t \geq 0$, satisfies the heat equation

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial^2}{\partial x^2} p(x,t), \qquad x \in \mathbb{R},\ t \geq 0,$$

under the initial and boundary conditions

$$\begin{cases} p(x,0) = \delta_0(x); \\ \lim_{|x|\to\infty} p(x,t) = 0; \\ \lim_{|x|\to\infty} \frac{\partial}{\partial x} p(x,t) = 0. \end{cases}$$

BM and heat equation

Details:

$\delta_{x_0}(\cdot)$ can be defined as a distribution (i.e. a continuous linear functional on the Schwartz space of rapidly decreasing functions at infinity $\mathcal{S}(\mathbb{R})$): $\delta_{x_0}(\phi) = \phi(x_0)$, for $\phi \in S(\mathbb{R})$, or as

$$\int \delta_{x_0}(x)\phi(x)dx = \phi(x_0).$$

Thus $\hat{p}(\kappa, t) = e^{-t\kappa^2}$ satisfies the ODE

$$\frac{d}{dt}\hat{p} = -\kappa^2\hat{p} = (i\kappa)^2\hat{p}, \qquad \hat{p}(\kappa, 0) = 1.$$

Integrating by parts and applying the boundary conditions:

$$(i\kappa)^2\hat{p}(\kappa, t) = \int e^{-i\kappa x}\frac{\partial^2}{\partial x^2}p(x, t)dx.$$

Heat equation

The heat equation describes the temperature variation as heat diffuses through a physical medium $\Longrightarrow$ the BM is the prototype of a very rich class of models called diffusion processes.

Definition (Markov process)

A stochastic process $X := \{X_t\}_{t\geq 0}$ defined on the space $(\Omega, \mathcal{F}, P)$, with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e. $\mathcal{F}_t := \sigma(X_s : s \leq t)$) is a Markov process if

$$P\left[X_t \in A | \mathcal{F}_s\right] = P\left[X_t \in A | X_s\right], \qquad 0 \leq s < t, A \in \mathcal{B}(\mathbb{R}).$$

$\Longrightarrow$ "only the last known position of the process matters".

Diffusion processes

Definition (Diffusion process)

A Markov process X defined on $(\Omega, \mathcal{F}, P)$, with transition density $p(y,t;x,s) := P(X_t \in dy | X_s = x)/dy$, $x, y \in \mathbb{R}$, $0 \leq s < t$, is called a diffusion process if $\forall \varepsilon > 0$:

$$(i) \quad \lim_{t\to s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} p(y,t;x,s)dy = 0$$

$$(ii) \quad \lim_{t\to s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} (y-x)p(y,t;x,s)dy = a(s,x)$$

$$(iii) \quad \lim_{t\to s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} (y-x)^2 p(y,t;x,s)dy = \sigma^2(s,x)$$

for a and σ^2 well-defined functions.

Diffusion processes

(i) prevents the process from having instantaneous jumps
(ii) the drift coefficient $a(x,t)$ is the instantaneous rate of change in the mean of the process (given $X_s = x$), i.e.

$$a(s,x) = \lim_{t \to s} \frac{1}{t-s} \mathbb{E}\left[X_t - X_s | X_s = x\right]$$

(iii) the diffusion coefficient $\sigma^2(s,x)$ is the instantaneous rate of change of the squared fluctuations of the process (given $X_s = x$), i.e.

$$\sigma^2(s,x) = \lim_{t \to s} \frac{1}{t-s} \mathbb{E}\left[(X_t - X_s)^2 | X_s = x\right]$$

BM case: $a(s,x) = 0$ and $\sigma^2(s,x) = \sigma^2$, $\forall s, x$

Anomalous diffusions

As we have seen, the diffusion model describes RW limits with finite variance of jumps. In many real world applications particles follow a heavy-tailed distribution of jumps.

Extended CLT (Pareto case)

Let $S_n = \sum_{j=1}^n Y_j$, where Y_j, for any $j \geq 1$, are i.i.d. r.v. with centered Pareto distribution, i.e. let $Y = X - \mu_1$, for $\mu_1 = \mathbb{E}X = \alpha C^{1/\alpha}/(\alpha-1)$, and $P(X > x) = Cx^{-\alpha}\mathbb{1}_{x \geq C^{1/\alpha}}$, for $C > 0$, $\alpha \in (1,2)$, then

$$n^{-1/\alpha} S_n \Longrightarrow Z, \qquad n \to \infty,$$

where Z is a r.v. with α-stable law, i.e. $\mathbb{E}(e^{-i\kappa Z}) = e^{(i\kappa)^\alpha}$, where $(i\kappa)^\alpha = |\kappa|^\alpha e^{i\alpha sign(\kappa)\pi/2}$, by choosing $C = (\alpha-1)/\Gamma(2-\alpha)$.

Extended CLT (Pareto case)

Proof:

The p-th moment of the Pareto r.v. X reads

$$\mu_p = \mathbb{E}X^p = \begin{cases} \frac{\alpha}{\alpha-p} C^{p/\alpha}, & 0 < p < \alpha, \\ \infty, & p > \alpha. \end{cases}$$

Thus, for $\alpha \in (1,2)$, $\mu_1 = \alpha C^{1/\alpha}/(\alpha - 1)$, while μ_2 is infinite. By some calculations (*)

$$\mathbb{E}e^{-i\kappa X} = 1 - ik\mu_1 + (i\kappa)^\alpha + O(\kappa^2), \qquad \kappa \to 0.$$

Similarly, for $Y = X - \mu_1$,

$$\begin{aligned} \mathbb{E}e^{-i\kappa Y} &= \left[1 - ik\mu_1 + (i\kappa)^\alpha + O(\kappa^2)\right]\left[1 + ik\mu_1 + O(\kappa^2)\right] \\ &= 1 + (i\kappa)^\alpha + O(\kappa^2), \qquad \kappa \to 0. \end{aligned}$$

* Prop. 1.7, Meerschaert-Sikorskii (2011)

Extended CLT (Pareto case)

Then the normalized sum $n^{-1/\alpha} S_n = n^{-1/\alpha} \sum_{j=1}^{n} Y_j$ has FT

$$\left(\hat{f}(n^{-1/\alpha}\kappa)\right)^n = \left(1 + \frac{(i\kappa)^\alpha}{n} + O(n^{-2/\alpha})\right)^n \to e^{(i\kappa)^\alpha},$$

as $n \to \infty$, since $2/\alpha > 1$, for $\alpha < 2$.
The continuity theorem for FT yields the extended CLT, in the Pareto case.

Note: the α-stable law reduces to $N(0,2)$, for $\alpha = 2$, since $\mathbb{E}(e^{-i\kappa Z}) = e^{(i\kappa)^2}$

RW with Pareto jumps

Corollary

Let $S_n = \sum_{j=1}^{n} Y_j$, where Y_j, for any $j \geq 1$, are i.i.d. r.v. with centered Pareto distribution, then, $\forall t \geq 0$,

$$c^{-1/\alpha} S_{\lfloor ct \rfloor} = c^{-1/\alpha} \sum_{j=1}^{\lfloor ct \rfloor} Y_j \Longrightarrow Z_t, \qquad c \to \infty,$$

where $\{Z_t\}_{t \geq 0}$ is an α-stable process, i.e. $\mathbb{E}(e^{-i\kappa Z_t}) = e^{(i\kappa)^\alpha t}$, $\kappa \in \mathbb{R}$.

Space-fractional diffusion equation

The FT $\hat{p}(\kappa, t) = e^{t(i\kappa)^\alpha}$, $\alpha \in (1, 2]$, can be inverted in closed form only for $\alpha = 2$ (Gaussian case). Clearly it solves the ODE

$$\frac{d}{dt}\hat{p} = (i\kappa)^\alpha \hat{p}, \qquad \hat{p}(\kappa, 0) = 1. \tag{1}$$

Let us define the Riesz fractional derivative of order $\alpha \in (0, 2]$ as the operator $\frac{d^\alpha}{dx^\alpha}$ such that

$$\mathcal{F}\left(\frac{d^\alpha}{dx^\alpha} f(x); \kappa\right) = (i\kappa)^\alpha \mathcal{F}\left(f(x); \kappa\right) = (i\kappa)^\alpha \hat{f}(\kappa),$$

for any $f \in \mathcal{S}$, where $\mathcal{F}$ denotes the FT.

For the n-th derivative: $\mathcal{F}\left(\frac{d^n}{dx^n} f(x); \kappa\right) = (i\kappa)^n \hat{f}(\kappa)$, $f \in \mathcal{S}$

Space-fractional diffusion equation

Thus, by inverting the FT in eq. (1), the solution to the space-fractional diffusion equation

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial^\alpha}{\partial x^\alpha}p(x,t), \qquad x \in \mathbb{R}, t \geq 0, \alpha \in (1,2),$$

under the initial and boundary conditions

$$\begin{cases} p(x,0) = \delta_0(x); \\ \lim_{|x|\to\infty} p(x,t) = 0; \\ \lim_{|x|\to\infty} \frac{\partial}{\partial x}p(x,t) = 0, \end{cases}$$

coincides with the transition density of an α-stable process $Z := \{Z_t\}_{t\geq 0}$.

Super-diffusions

The stable pdf $p(\cdot, t)$ is positively skewed, with heavy power-law tail (*):

$$p(x,t) = Ax^{-\alpha-1} + o(x^{-\alpha-1}), \qquad x \to \infty,$$

for some $A > 0$ (depending on C, t, α), contrary to the Gaussian case.

For $\alpha \in (1, 2)$, the space fractional diffusion equation models the spreading of a cloud of particles with perform jumps with power-law distribution and infinite variance. $\Longrightarrow$ super-diffusion.

* Property 1.2.15, Samorodnitski-Taqqu (1994)

Super-diffusions

Since

$$\hat{p}(\kappa, ct) = e^{ct(i\kappa)^\alpha} = e^{t(i\kappa c^{1/\alpha})^\alpha} = \hat{p}(c^{1/\alpha}\kappa, t)$$

the process is self-similar of index $1/\alpha$, i.e.

$$Z_{ct} \overset{f.d.d.}{=} c^{1/\alpha} Z_t$$

where $\overset{f.d.d.}{=}$ indicates equality of the finite-dimensional distributions. For the one-dimensional case

$$p(x, ct) = c^{-1/\alpha} p(c^{-1/\alpha} x, t), \qquad x \in \mathbb{R}, c > 0, t > 0.$$

Thus the spreading (spatial scale growing) rate is $t^{1/\alpha}$, faster than $t^{1/2}$ of the standard diffusion (for $\alpha < 2$)

Space-f.d.e. with scale and drift

Let us now introduce, in the limiting process, the drift parameter $v \in \mathbb{R}$ and the scale parameter $D > 0$

$$D^{1/\alpha} Z_t + vt, \qquad \alpha \in (1,2),$$

so that

$$\hat{p}(\kappa, t) = \mathbb{E}\left(e^{-i\kappa(vt + D^{1/\alpha} Z_t)}\right) = e^{-i\kappa vt + Dt(i\kappa)^\alpha},$$

by self-similarity. This FT satisfies the ODE

$$\frac{d}{dt}\hat{p} = \left[-i\kappa v + D(i\kappa)^\alpha\right]\hat{p}, \qquad \hat{p}(\kappa, 0) = 1.$$

Space-f.d.e. with scale and drift

By inverting the FT, we get

$$\frac{\partial}{\partial t}p(x,t) = \left[-v\frac{\partial}{\partial x} + D\frac{\partial^\alpha}{\partial x^\alpha}\right]p(x,t),$$

which is thus the governing equation of the process

$$D^{1/\alpha}Z_t + vt$$

modeling the spreading of a particle at a super-diffusive rate $t^{1/\alpha}$, from the center of mass $x = vt$.

Two-sided space-f.d.e.

We extend the previous model by considering a two-sided Pareto distribution for the jumps of the RW.

Extended CLT (two-sided Pareto case)

Let $S_n = \sum_{j=1}^{n} Y_j$, where Y_j, for any $j \geq 1$, are i.i.d. r.v. with $Y = X - \mu$. Let $P(X > x) = pCx^{-\alpha}\mathbb{1}_{x \geq C^{1/\alpha}}$ and $P(X < -x) = qCx^{-\alpha}\mathbb{1}_{x \leq -C^{1/\alpha}}$, for $\alpha \in (1,2)$, $p, q \in [0,1]$ such that $p + q = 1$. Then

$$c^{-1/\alpha} S_{\lfloor ct \rfloor} \Longrightarrow Z_t, \qquad c \to \infty,$$

where Z_t is a two-sided α-stable process, i.e.

$$\mathbb{E}(e^{-i\kappa Z_t}) = e^{t[pD(i\kappa)^\alpha + qD(-i\kappa)^\alpha]}, \qquad \kappa \in \mathbb{R}, \tag{2}$$

for $D > 0$, depending on C and α.

Two-sided space-fde

The FT (2) satisfies the ODE

$$\frac{d}{dt}\hat{p} = \left[pD(i\kappa)^\alpha + qD(-i\kappa)^\alpha\right]\hat{p}, \qquad \hat{p}(\kappa,0) = 1.$$

Let us define the operator $\frac{d^\alpha}{d(-x)^\alpha}$ such that

$$\mathcal{F}\left(\frac{d^\alpha}{d(-x)^\alpha}f(x);\kappa\right) = (-i\kappa)^\alpha \mathcal{F}\left(f(x);\kappa\right) = (-i\kappa)^\alpha \hat{f}(\kappa),$$

for any $f \in \mathcal{S}$.
Thus, taking the inverse FT, $p(x,t)$ satisfies, for $\alpha \in (1,2)$,

$$\frac{\partial}{\partial t}p(x,t) = pD\frac{\partial^\alpha}{\partial x^\alpha}p(x,t) + qD\frac{\partial^\alpha}{\partial(-x)^\alpha}p(x,t), \qquad x \in \mathbb{R}, t \geq 0,$$

under the usual initial and boundary conditions.

Two-sided super-diffusion

- it models the spreading of a cloud of particles with power-law jumps in both directions
- the two-sided density of Z_t spreads at super-diffusive rate $t^{1/\alpha}$ in both directions
- the weights p and q represent the relative likelihood of positive/negative jumps (resp.)
- the parameter $\beta = p - q$ indicates whether the pdf is positively skewed ($\beta > 0$) or negatively skewed ($\beta < 0$) or symmetric ($\beta = 0$)

Plots of trajectories

Brownian motion $\Longleftrightarrow$ Space-fractional diffusion

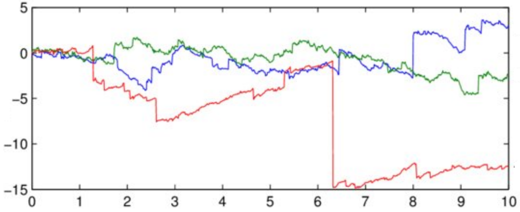

Figure: space-fractional diffusion trajectories, for $\alpha = 1.3$ (red), $\alpha = 1.5$ (blue), $\alpha = 1.8$ (green)

Further extensions

Let $R : [A, \infty) \to (0, \infty)$, $A > 0$, be a Borel measurable function, then it is regularly varying of index ρ (RV(ρ)) if

$$\lim_{x \to \infty} \frac{R(\lambda x)}{R(x)} = \lambda^\rho, \qquad \lambda > 0.$$

Extended CLT (regularly varying case)

Let Y_j be i.i.d. r.v.'s with $V_0(x) = P(|Y| > x)$, $x > 0$. Then

$$a_n \sum_{n=1}^{n} Y_j - b_n \Longrightarrow Z, \qquad n \to \infty,$$

where Z is a stable r.v. ($\alpha \in (0, 2)$), for some $a_n \in \mathbb{R}^+$ and $b_n \in \mathbb{R}$, $n \geq 1$, iff $V_0(\cdot)$ is RV($-\alpha$) and

$$\lim_{x \to \infty} \frac{P(Y > x)}{V_0(x)} = p, \qquad 0 \leq p \leq 1.$$

Extended CLT (RV case)

Proof: see (*) Theorem 4.5

Details on a_n, b_n: see (*) Prop. 4.16

Pareto case: $V_0(x) = P(|Y| > x) = x^{-\alpha}$, for $x \geq 1$, is RV$(-\alpha)$, $a_n = n^{-1/\alpha}$, $b_n = 0$.

RW convergence: the previous result extends to

$$a_n S_{\lfloor nt \rfloor} - \frac{\lfloor nt \rfloor}{n} b_n \Longrightarrow Z_t, \qquad n \to \infty,$$

where Z_t is an α-stable process ($\alpha \in (0,2)$), for any $t \geq 0$.

(*) Meerschaert-Sikorskii (2011)

References

- Meerschaert, M. and Sikorskii, A. (2011), *Stochastic Models for Fractional Calculcus*, De Gruyter, Berlin-Boston.
- Samorodnitsky, G. and Taqqu, M.S. (1994), *Stable Non-Gaussian Random Processes. Stochastic Models with Infinite Variance.* Chapman and Hall, New York.

Contents

Infinite divisibility

Definition (Convolution of measures)

Let $\mathcal{M}(\mathbb{R}^d)$ be the set of all Borel probability measures in $\mathbb{R}^d$, $d \geq 1$ and let $\mu_i \in \mathcal{M}(\mathbb{R}^d)$, $i = 1, 2$, $A \in \mathcal{B}(\mathbb{R}^d)$. Then we define the convolution of measures as

$$\begin{aligned} (\mu_1 \star \mu_2)(A) &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_A(x+y)\mu_1(dx)\mu_2(dy) \\ &= \int_{\mathbb{R}^d} \mu_1(A-x)\mu_2(dx), \end{aligned}$$

where $A - x = \{y - x, y \in A\}$ and $\mathbb{1}_A(x+y) = \mathbb{1}_{A-x}(y)$.

Easy to prove that $\mu_1 \star \mu_2 \in \mathcal{M}(\mathbb{R}^d)$.
Let the n-fold convolution $\mu^{\star n} := \mu \star \cdots \star \mu$ (n times)

Infinite divisibility

Properties of convolutions:

- $\forall f \in B_b(\mathbb{R}^d)$ (bounded Borel-measurable function)

$$\int_{\mathbb{R}^d} f(z)(\mu_1 \star \mu_2)(dz) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y)\mu_1(dx)\mu_2(dy)$$

  (indeed it holds for $f = \mathbb{1}_A$, and thus, by linearity, for any elementary function and can be extended by approximation to any function)

- If X_1 and X_2 are independent r.v.'s with distribution μ_1 and μ_2, resp., then $\forall f \in B_b(\mathbb{R}^d)$,

$$\mathbb{E}f(X_1 + X_2) = \int_{\mathbb{R}^d} f(z)(\mu_1 \star \mu_2)(dz),$$

  so that, for $f = \mathbb{1}_A$, $P(X_1 + X_2 \in A) = (\mu_1 \star \mu_2)(A)$

Infinite divisibility

Definition (Infinite divisibility)

Let X be a r.v. taking values in $\mathbb{R}^d$ with law μ_X, we say that X is infinitely divisible (ID) if, $\forall n \in \mathbb{N}$, there exist i.i.d. r.v.'s $Y_1^{(n)}, \cdots, Y_n^{(n)}$ s.t.

$$X \stackrel{d}{=} Y_1^{(n)} + \cdots + Y_n^{(n)}.$$

Infinite divisibility

Theorem

The following are equivalent:

(i) X is ID;

(ii) μ_X has a convolution n-th root $\mu_X^{1/n} \in \mathcal{M}(\mathbb{R}^d)$ s.t. $(\mu_X^{1/n})^{\star n} = \mu_X$, which is itself a r.v., for any $n \in \mathbb{N}$;

(iii) $\Phi_X(\kappa) := \mathbb{E}e^{i\kappa X} = \mathcal{F}(\mu_X; -\kappa)$ has a n-th root $\Phi_X^{1/n}(\cdot)$ which is itself the characteristic function of a r.v., $n \in \mathbb{N}$.

Proof:

(i) $\Longrightarrow$ (ii) if X is ID, then $\mu_X^{1/n} = \mu_Y$

Infinite divisibility

(ii) $\Longrightarrow$ (iii) if $\mu_Y = \mu_X^{1/n}$, then, since $f(z) = e^{i\langle\kappa, z\rangle} \in B_b(\mathbb{R}^d)$, by the property of convolutions,

$$\begin{aligned}
\Phi_X(\kappa) &= \int_{\mathbb{R}^d} e^{i\langle\kappa,z\rangle}(\mu_X^{1/n})^{\star n}(dz) \\
&= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{i\langle\kappa, y_1+\ldots+y_n\rangle} \mu_X^{1/n}(dy_1)\ldots\mu_X^{1/n}(dy_n) \\
&= (\Phi_Y(\kappa))^n
\end{aligned}$$

(iii) $\Longrightarrow$ (i) If $\Phi_X(\kappa)$ has a n-th root $\Phi_X^{1/n}(\cdot)$ which is the c.f. of a r.v. Y, then, by choosing n independent copies of Y, we have that

$$\Phi_X(\kappa) = \underbrace{\Phi_Y(\kappa)\cdots\Phi_Y(\kappa)}_{n} = \mathbb{E}e^{i\langle\kappa, Y_1^{(n)}+\ldots+Y_n^{(n)}\rangle},$$

so that $X \overset{d}{=} Y_1^{(n)} + \ldots + Y_n^{(n)}$.

Infinite divisibility

Examples of ID laws $(d = 1)$

- Gaussian: $X \sim N(m, \sigma^2)$

$$\Phi_X(\kappa) = \exp\left\{-\frac{\kappa^2\sigma^2}{2} + im\kappa\right\} = \left(\exp\left\{-\frac{\kappa^2\sigma^2}{2n} + \frac{im\kappa}{n}\right\}\right)^n$$

  thus $X \stackrel{d}{=} Y_1^{(n)} + \cdots + Y_n^{(n)}$, with $Y_i^{(n)} \sim N(m/n, \sigma^2/n)$, $i = 1, ..., n$.

- Compound Poisson (CP): $N \sim Poi(\lambda)$ and $W = \sum_{n=1}^{N} X_n$, for i.i.d. $X \sim \mu$

$$\Phi_Y(\kappa) = \exp\{\lambda(\hat{\mu}(-\kappa) - 1)\} = \left(\exp\left\{\frac{\lambda}{n}(\hat{\mu}(-\kappa) - 1)\right\}\right)^n$$

  thus $W \stackrel{d}{=} W_1^{(n)} + \cdots + W_n^{(n)}$, with $W_i^{(n)} \sim CP(\lambda/n, \mu)$, $i = 1, ..., n$.

Lévy Khintchine representation

Theorem (Lévy-Khintchine)

A probability measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is ID if and only if its c.f. $\Phi(\kappa) := \hat{\mu}(-\kappa)$ is of the form $\Phi(\kappa) = e^{\psi(\kappa)}$, where

$$\psi(\kappa) = i\langle \kappa, a\rangle - \frac{1}{2}\langle \kappa, A\kappa\rangle + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{i\langle \kappa, x\rangle} - 1 - \frac{i\langle \kappa, x\rangle}{1 + \|x\|^2} \right) \nu(dx),$$

$a \in \mathbb{R}^d$, A is a non-negative definite symmetric matrix (i.e. $\langle x, Ax\rangle \geq 0$, $\forall x \in \mathbb{R}^d$) and $\nu(\cdot)$ is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$, i.e.

$$\int_{\mathbb{R}^d \setminus \{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty.$$

Lévy Khintchine representation

Proof: see Thm. 1.2.14 (*)

- The Lévy measure is s.t. $\nu(-\varepsilon, \varepsilon)^c < \infty$, $\forall \varepsilon > 0$
- The triplet (a, A, ν) is unique
- The function $\psi(\cdot)$ is called Lévy exponent

(*) Applebaum (2009)

Lévy Khintchine representation

Corollary

Every ID law is the weak limit of Compound Poisson laws.

Proof: Let Y be ID with c.f. $\Phi(\kappa) = e^{\psi(\kappa)}$, then $Y \stackrel{d}{=} X_1^{(n)} + ... + X_n^{(n)}$ where $X_i^{(n)}$ has c.f. $\Phi_n(\kappa) = e^{\psi(\kappa)/n}$ s.t. $\Phi(\kappa) = (\Phi_n(\kappa))^n$.
Now let $Z_n \sim CP$ with parameter $\lambda = n$ and c.f. of the jumps equal to $\Phi_n(\kappa)$, then

$$\mathbb{E}e^{i\kappa Z_n} = \exp\left\{n(\Phi_n(\kappa) - 1)\right\}.$$

Fix $\kappa \in \mathbb{R}$ and write

$$\Phi_n(\kappa) - 1 = \exp\left\{\frac{\psi(\kappa)}{n}\right\} - 1 = \sum_{j=1}^{\infty} \frac{\psi(\kappa)^j}{n^j}\frac{1}{j!} = \frac{\psi(\kappa)}{n} + O(n^{-2}),$$

so that $\mathbb{E}e^{i\kappa Z_n} = e^{\psi(\kappa)+O(n^{-1})} \to \Phi(\kappa)$, as $n \to \infty$.

Theorem (Alternative L-K representation)

Let $d = 1$, $R > 0$ and $B_R := \{x \in \mathbb{R} : |x| < R\}$, then if X has an ID law μ with c.f. $\Phi(\kappa) = e^{\psi_0(\kappa)}$, then, for any R,

$$\psi_0(\kappa) = i\kappa a_0 - \frac{\kappa^2 A}{2} + \int_{\mathbb{R}} (e^{i\kappa y} - 1 - i\kappa y \mathbb{1}_{B_R}(y))\nu(dy).$$

(i) Moreover, if $\int_{B_R} |y|\nu(dy) < \infty$, then the Lévy exponent is

$$\psi_1(k) = i\kappa a_1 - \frac{\kappa^2 A}{2} + \int_{\mathbb{R}} (e^{i\kappa y} - 1)\nu(dy)$$

(ii) if $\int_{B_R^c} |y|\nu(dy) < \infty$, then the Lévy exponent is

$$\psi_2(k) = i\kappa a_2 - \frac{\kappa^2 A}{2} + \int_{\mathbb{R}} (e^{i\kappa y} - 1 - i\kappa y)\nu(dy).$$

Alternative L-K representation

Proof (sketch): The integral

$$\delta_0 = \int_{\mathbb{R}} \left(\frac{y}{1+y^2} - y\mathbb{1}_{B_R}(y) \right) \nu(dy) < \infty,$$

since the integrand is bounded and $O(y^3)$, as $y \to 0$. Then we put $a_0 = a - \delta_0$ to get the L-K formula;
Moreover,

(i) we get $\psi_1(\cdot)$ by putting $a_1 = a_0 - \int_{B_R} |y|\nu(dy)$.

(ii) we get $\psi_2(\cdot)$ by putting $a_2 = a_0 + \int_{B_R^c} |y|\nu(dy)$.

Note:
In the case of Lévy process with $\mathbb{E}e^{i\kappa Z_t} = e^{t\psi(\kappa)}$, if $A = 0$ and $R = 1$, then condition
(i) $\Longrightarrow$ finite variation
(ii) $\Longrightarrow$ infinite variation

Alternative L-K representation

Note:
in this alternative expression, it is easy to prove that the L-K formula holds for the c.f. of an ID law, under condition (i).
Indeed, if we denote by $\{U_n\}_{n\in\mathbb{N}}$ a sequence of sets in $\mathcal{B}(\mathbb{R}^d)$, monotonically decreasing to $\{0\}$ and, $\forall\kappa\in\mathbb{R}^d$, $n\in\mathbb{N}$,

$$\psi_n(\kappa) := i\langle a - \int_{U(n)^c\cap B_1(0)} y\nu(dy), \kappa\rangle - \frac{1}{2}\langle\kappa, A\kappa\rangle + \int_{U(n)^c}(e^{i\langle\kappa,y\rangle}-1)\nu(dy)$$

where the first integral I is finite by the Lévy measure's condition and $\{U(n)^c\}\to\mathbb{R}^d\setminus\{0\}$, as $n\to\infty$.
The function $\psi_n(\cdot)$ is the Lévy exponent of a convolution of $X_1\sim N(m,A)$, with $m=a-I$, and $X_2\sim CP(1,\nu)$. Moreover, $\lim_{n\to\infty} e^{\psi_n(\kappa)} = e^{\psi_1(\kappa)}$ (given in (i)).

Lévy processes

Definition (Lévy process)

The stochastic process $X := \{X_t\}_{t \geq 0}$, defined on $(\Omega, \mathcal{F}, P)$, is a Lévy process iff

(i) $X_0 = 0$, a.s.

(ii) its increments are stationary (i.e. $X_{t_{j+1}} - X_{t_j} \overset{d}{=} X_{t_{j+1} - t_j}$) and independent (i.e. $(X_{t_{j+1}} - X_{t_j}) \perp X_{t_j}$), for $j = 1, ..., n-1$ and $0 \leq t_1 \leq ... \leq t_n$.

(iii) is stochastically continuous, i.e. $\forall a > 0$, $s \geq 0$

$$\lim_{t \to s} P(|X_t - X_s| > a) = 0.$$

Lévy processes

Theorem

If X is a Lévy process then X_t, $\forall t \geq 0$,

(i) is ID

(ii) has Lévy symbol $\eta(t,\kappa) := \log \Phi_{X_t}(\kappa) = t\eta(1,\kappa) = t\psi(\kappa)$.

Proof:

(i) Let $Y_k^{(n)} := X_{kt/n} - X_{(k-1)t/n}$, $1 \leq k \leq n$, then, by def. of Lévy process, $Y_k^{(n)} \overset{d}{=} X_{(k-(k-1))t/n} = X_{t/n}$ are i.i.d. and, moreover, $Y_k^{(n)} \perp Y_{k-1}^{(n)}$, for any $k = 1, ..., n$.

Finally $X_t = \sum_{k=0}^{n} [X_{kt/n} - X_{(k-1)t/n}] \overset{d}{=} \sum Y_k^{(n)}$

(ii) see Thm. 1.2.17 (*)

(*) Applebaum (2009)

One-sided stable law

Let $\alpha \in (0,2)$ and let X be a (one-dimensional) ID r.v.with triplet $(a, 0, \nu)$, with $a \in \mathbb{R}$, $c > 0$, and

$$\nu(dy) = \begin{cases} c\alpha y^{-\alpha-1} dy, & y > 0 \\ 0, & y \leq 0, \end{cases}$$

which is a Lévy measure since

$$\int_{\mathbb{R}} (y^2 \wedge 1)\nu(dy) = c\alpha \int_0^1 y^{1-\alpha} dy + c\alpha \int_1^\infty y^{-1-\alpha} dy < \infty.$$

Then X has a one-sided α-stable law.

One-sided stable law with $\alpha \in (0,1)$

For $\alpha \in (0,1)$,

$$\int_{B_R} |y|\nu(dy) = c\alpha \int_0^R y^{-\alpha} dy = \frac{c\alpha}{1-\alpha} R^{1-\alpha}.$$

Thus, by the alternative L-K repr. (i),

$$\begin{aligned} \Phi(\kappa) = e^{\psi_1(\kappa)} &= \exp\left\{ i\kappa a + c\alpha \int_0^{+\infty} (e^{i\kappa y} - 1) y^{-\alpha-1} dy \right\} \\ &=: \exp\{i\kappa a + cI(\alpha)\} \end{aligned}$$

The integral $I(\alpha)$ can be approximated, as $s \to 0^+$, by

$$I_s(\alpha) := \alpha \int_0^{+\infty} (e^{(i\kappa - s)y} - 1) y^{-\alpha-1} dy$$

by the dominated convergence theorem, since $|e^{(i\kappa-s)y} - 1| \leq 2$, $y > 0$ (see details in Applebaum (2009)).

One-sided stable law with $\alpha \in (0,1)$

By parts ($u = e^{(i\kappa - s)y} - 1 \simeq O(y)$, as $y \to 0$)

$$\begin{aligned} I_s(\alpha) &= -\left[(e^{(i\kappa-s)y} - 1)y^{-\alpha}\right]_0^\infty + (i\kappa - s)\int_0^{+\infty} e^{(i\kappa-s)y} y^{-\alpha} dy \\ &= -\Gamma(1-\alpha)(s - i\kappa)^\alpha \to -\Gamma(1-\alpha)(-i\kappa)^\alpha, \qquad s \to 0^+. \end{aligned}$$

Thus, for $\alpha \in (0,1)$ and $a = 0$, the c.f. reads

$$\Phi(\kappa) = e^{-c\Gamma(1-\alpha)(-i\kappa)^\alpha}, \qquad \text{(minus !)}.$$

Note that the expected value (as well as the variance) is infinite, since

$$\left.\frac{d}{d\kappa}\Phi(\kappa)\right|_{\kappa=0} = -c\alpha\Gamma(1-\alpha)(-i)^\alpha \kappa^{\alpha-1}\Big|_{\kappa=0}$$

diverges for $\alpha \in (0,1)$.

Properties of stable laws: stability

Definition (α-stability)

The law of a r.v. X satisfies a stability property of index α, if, for any $n \geq 2$, there exist $a_n > 0$ and $b_n \in \mathbb{R}$, such that

$$X_1 + ... + X_n \stackrel{d}{=} a_n X + b_n,$$

where $X_1, ..., X_n$ are independent copies of X.

- the parameter α must be in $(0, 2]$
- α-stability implies ID

One-sided stable law with $\alpha \in (0, 1)$

It is easy to check that the law of Z with c.f.

$$\Phi_Z(\kappa) = \mathbb{E}e^{\kappa Z} = e^{D(-i\kappa)^\alpha}$$

is α-stable.
Indeed, for $S_n = Z_1 + ... + Z_n$, we have that

$$\Phi_{S_n}(\kappa) = (\Phi_Z(\kappa))^n = \left(e^{D(-i\kappa)^\alpha}\right)^n = e^{D(-in^{1/\alpha}\kappa)^\alpha} = \Phi_X(n^{1/\alpha}\kappa)$$

and thus

$$S_n = \sum_{j=1}^{n} Z_j \stackrel{d}{=} n^{1/\alpha} X.$$

This can be proved analogously for the case $\alpha \in (1, 2)$ and for the two-sided stable law.

Properties of stable laws: heavy tails

The one-sided stable law is positively skewed, with heavy power-law tail (*):

$$\overline{\mu}(x) := P(X > x) = Ax^{-\alpha} + o(x^{-\alpha}), \qquad x \to \infty, \tag{3}$$

for some $A > 0$ (depending on c, α), contrary to the Gaussian case.

Indeed, integrating by parts, we can write the Laplace transform

$$\begin{aligned} \int_0^\infty e^{-sx}\overline{\mu}(x)dx &= -\frac{1}{s}\left[e^{-sx}\overline{\mu}(x)\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-sx}\frac{d}{dx}\overline{\mu}(x)dx \\ &= \frac{1}{s}(1-\mathbb{E}e^{-sX}) = \frac{1}{s}(1-e^{-c\Gamma(1-\alpha)s^\alpha}) \\ &\sim c\Gamma(1-\alpha)s^{\alpha-1}, \qquad s \to 0. \end{aligned}$$

* Property 1.2.15, Samorodnitski-Taqqu (1994)

Properties of stable laws: heavy tails

By a well-known Tauberian theorem (*), for $\rho > 0$,

$$\int_0^\infty e^{-sx}\overline{\mu}(x)dx \sim s^{-\rho}L(1/s) \qquad s \to 0,$$

if and only if

$$P(X > x) \sim \frac{1}{\Gamma(\rho)} x^{\rho-1} L(x), \qquad x \to \infty,$$

for a slowly varying function $L : (0,\infty) \to (0,\infty)$ (i.e. s.t. $\lim_{\lambda\to\infty} L(\lambda x)/L(\lambda) = 1$, $x > 0$).

Therefore, we get the asymptotics in (3), by choosing $\rho = 1 - \alpha > 0$, for $\alpha \in (0,1)$, and $L(x)$ =const.

* Theorem 13.5.4, Feller (1971)

One-sided stable process with $\alpha \in (0,1)$

Thus a one-sided α-stable process with $\alpha \in (0,1)$ has c.f.

$$\Phi(\kappa) = e^{-c\Gamma(1-\alpha)t(-i\kappa)^{\alpha}}.$$

Recall:
we have obtained, by the extended CLT (with Pareto distributed jumps), the weak limit of the scaled RW as the process $\{Z_t\}_{t\geq 0}$ with $\mathbb{E}e^{-\kappa Z_t} = e^{tD(i\kappa)^{\alpha}}$, for $\alpha \in (1,2)$, $D > 0$, $(i\kappa)^{\alpha} = |\kappa|^{\alpha} e^{i\alpha sign(\kappa)\pi/2}$.

Thus here there is a minus on the r.h.s. of the space-fde satisfied by its transition function $p(x,t)$:

$$\frac{\partial}{\partial t} p(x,t) = -D \frac{\partial^{\alpha}}{\partial x^{\alpha}} p(x,t), \qquad \alpha \in (0,1),$$

where, for any $f \in \mathcal{S}$, $\mathcal{F}\left(d^{\alpha}/dx^{\alpha} f(x); \kappa\right) = (i\kappa)^{\alpha} \hat{f}(\kappa)$.
$\Longrightarrow$ the solution to the space-fde is still a Lévy (and thus markovian) process

One-sided stable law with $\alpha \in (1,2)$

For $\alpha \in (1,2)$, we have that

$$\int_{B_R^c} |y|\nu(dy) = c\alpha \int_R^\infty y^{-\alpha} dy = \frac{c\alpha}{\alpha-1} R^{1-\alpha}.$$

Thus, by the alternative L-K repr. (ii),

$$\begin{aligned}\Phi(\kappa) = e^{\psi_2(\kappa)} &= \exp\left\{ i\kappa a + c\alpha \int_0^{+\infty} (e^{i\kappa y} - 1 - i\kappa y) y^{-\alpha-1} dy \right\} \\ &=: \exp\left\{ i\kappa a + cJ(\alpha) \right\}\end{aligned}$$

One-sided stable law with $\alpha \in (1,2)$

We approximate $J(\alpha)$ by

$$\begin{aligned} J_s(\alpha) &:= \alpha \int_0^{+\infty} \underbrace{\left(e^{(i\kappa - s)y} - 1 - (i\kappa - s)y\right)}_{u \simeq O(y^2),\, y\to 0} y^{-\alpha-1} dy \\ &= (i\kappa - s)\int_0^{+\infty} (e^{(i\kappa-s)y} - 1) y^{-\alpha} dy \\ &= \frac{i\kappa - s}{\alpha - 1}\int_0^{+\infty} (e^{(i\kappa-s)y} - 1) y^{-(\alpha-1)-1}(\alpha-1) dy, \end{aligned}$$

where $\alpha - 1 \in (0,1)$. Thus

$$\begin{aligned} J_s(\alpha) &= \frac{i\kappa - s}{\alpha - 1} I_s(\alpha - 1) = \frac{s - i\kappa}{\alpha - 1}\Gamma(1 - (\alpha - 1))(s - i\kappa)^{\alpha - 1} \\ &= \frac{\Gamma(2-\alpha)}{\alpha - 1}(s - i\kappa)^{\alpha} \to \frac{\Gamma(2-\alpha)}{\alpha - 1}(-i\kappa)^{\alpha}, \qquad s \to 0. \end{aligned}$$

Thus, by dominated convergence theorem, $J_s(\alpha) \to J(\alpha)$.

One-sided stable law with $\alpha \in (1,2)$

Thus, for $a = 0$, the c.f. reads

$$\Phi(\kappa) = e^{\frac{c\Gamma(2-\alpha)}{\alpha-1}(-i\kappa)^{\alpha}}, \qquad \alpha \in (1,2).$$

Note:

- the expected value is finite, since

$$\left.\frac{d}{d\kappa}\Phi(\kappa)\right|_{\kappa=0} = c\alpha\frac{\Gamma(2-\alpha)}{\alpha-1}(-i)^{\alpha}\kappa^{\alpha-1}\Big|_{\kappa=0} = 0$$

- the variance is infinite since

$$\left.\frac{d^2}{d\kappa^2}\Phi(\kappa)\right|_{\kappa=0} = c\alpha\Gamma(2-\alpha)(-i)^{\alpha}\kappa^{\alpha-2}\big|_{\kappa=0} = \infty.$$

One-sided stable process with $\alpha \in (1, 2)$

Thus a one-sided α-stable process with $\alpha \in (1, 2)$ has c.f.

$$\Phi(\kappa) = e^{\frac{ct\Gamma(2-\alpha)}{\alpha-1}(-i\kappa)^\alpha}.$$

Thus it coincides with the weak limit of the scaled RW as the process $\{Z_t\}_{t\geq 0}$ with $\mathbb{E}e^{-\kappa Z_t} = e^{tD(i\kappa)^\alpha}$, for $\alpha \in (1, 2)$, $D = c\Gamma(2-\alpha)/(\alpha - 1) > 0$.

Its transition function $p(x, t)$ satisfies the space-fde (with plus sign):

$$\frac{\partial}{\partial t}p(x,t) = D\frac{\partial^\alpha}{\partial x^\alpha}p(x,t), \qquad \alpha \in (1, 2).$$

Pareto law $\Longleftrightarrow$ space-fde

We have then connected the coefficients α and D in the space-fde with the parameters of the Pareto law of jumps (and the Lèvy measure):

- order of fractional derivative α coincides with power law index of jumps
- diffusivity coefficient

$$D = \begin{cases} c\Gamma(1-\alpha), & \alpha \in (0,1) \\ \frac{c\Gamma(2-\alpha)}{\alpha-1}, & \alpha \in (1,2) \end{cases}$$

coincides with the scale parameter of jumps' distribution

Two-sided stable law

Let $\alpha \in (0, 2)$ and let X be a (one-dimensional) ID r.v.with triplet $(a, 0, \nu)$, with $a \in \mathbb{R}$ and

$$\nu(dy) = \begin{cases} pc\alpha y^{-\alpha-1} dy, & y > 0 \\ qc\alpha |y|^{-\alpha-1} dy, & y < 0, \end{cases}$$

with $p, q \in (0, 1)$ and $p + q = 1$.

Then X has a two-sided α-stable law.

- the weights p and q represent the relative likelihood of positive/negative jumps (resp.)
- the parameter $\beta = p - q$ indicates whether the pdf is positively skewed ($\beta > 0$) or negatively skewed ($\beta < 0$) or symmetric ($\beta = 0$)

Two-sided stable process with $\alpha \in (0, 1)$

Similarly to the one-sided case, for $\alpha \in (0,1)$ and $a = 0$, the c.f. of the two-sided stable law reads

$$\Phi(\kappa) = e^{-pc\Gamma(1-\alpha)(-i\kappa)^\alpha - qc\Gamma(1-\alpha)(i\kappa)^\alpha}, \qquad \text{minus!}$$

so that the FT of the transition density of the corresponding process $\{Z_t\}_{t\geq 0}$ is equal to

$$\hat{p}(\kappa, t) = e^{-pct\Gamma(1-\alpha)(i\kappa)^\alpha - qct\Gamma(1-\alpha)(-i\kappa)^\alpha}.$$

Thus its FT inverse satisfies the space-fde (with minus)

$$\frac{\partial}{\partial t} p(x,t) = -pD\frac{\partial^\alpha}{\partial x^\alpha} p(x,t) - qD\frac{\partial^\alpha}{\partial(-x)^\alpha} p(x,t), \qquad \alpha \in (0,1),$$

with $D = c\Gamma(1-\alpha) > 0$.

Two-sided stable process with $\alpha \in (1, 2)$

For $\alpha \in (1, 2)$ and $a = 0$, the c.f. of the two-sided stable law reads reads

$$\Phi(\kappa) = e^{pc\frac{\Gamma(2-\alpha)}{\alpha-1}(-i\kappa)^\alpha + qc\frac{\Gamma(2-\alpha)}{\alpha-1}(i\kappa)^\alpha},$$

so that the FT of the transition density of the corresponding process $\{Z_t\}_{t\geq 0}$ is equal to

$$\hat{p}(\kappa, t) = e^{pct\frac{\Gamma(2-\alpha)}{\alpha-1}(i\kappa)^\alpha + qct\frac{\Gamma(2-\alpha)}{\alpha-1}(-i\kappa)^\alpha}.$$

This its FT inverse satisfies the space-fde

$$\frac{\partial}{\partial t}p(x,t) = pD\frac{\partial^\alpha}{\partial x^\alpha}p(x,t) + qD\frac{\partial^\alpha}{\partial(-x)^\alpha}p(x,t), \qquad \alpha \in (1,2),$$

with $D = c\Gamma(2-\alpha)/(\alpha - 1) > 0$.

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