Anomalous Diffusion, Lévy Processes, and Fractional Calculus

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Contents

Fractional Brownian motion

The fractional Brownian motion (hereafter fBm) is another model of anomalous diffusion, out of the framework of Lèvy processes.

Definition (fBm)

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, then the (two-sided) fBm with Hurst parameter $H \in (0,1)$ is a Gaussian process $B^H := \{B_t^H\}_{t\in\mathbb{R}}$ on it, having the following properties

(i) $B_0^H = 0$ a.s.

(ii) $\mathbb{E}B_t^H = 0$, $\forall t \in \mathbb{R}$

(iii) $\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, $s, t \in \mathbb{R}$.

Since B^H is Gaussian, it has a continuous modification by the Kolmogorov criterion and considering that $\forall n \in \mathbb{N}$,

$$\mathbb{E}|B_t^H - B_s^H|^n = \frac{2^{n/2}}{\sqrt{\pi}\Gamma((n+1)/2)}|t-s|^{2H}$$

.

Special cases:

- for $H = 1 \Longrightarrow B_t^H = tZ$, for any $t \in \mathbb{R}$, where $Z \sim N(0,1)$
- for $H = 1/2 \Longrightarrow B_t^H = W_t$, for any $t \in \mathbb{R}$, where $\{W_t\}_{t\in\mathbb{R}}$ is the two-sided Brownian motion (or Wiener process)

The n-times c.f. of the fBm reads

$$\mathbb{E}\exp\left\{i\kappa_1 B_{t_1}^H + ... + i\kappa_n B_{t_n}^H\right\} = \exp\left\{-\frac{1}{2}\langle \boldsymbol{\kappa} C_{\boldsymbol{t}}, \boldsymbol{\kappa}\rangle\right\},$$

where $\boldsymbol{\kappa} := (\kappa_1, ...\kappa_n)$, $\boldsymbol{t} := (t_1, ..., t_n)$ and the covariance matrix is $C_{\boldsymbol{t}} := \left\{\mathbb{E} B_{t_i}^H B_{t_j}^H\right\}_{1\le i,j\le n}$

- the fBm is self-similar of index H: for $c > 0$,

$$
\begin{aligned}
\mathbb{E}\exp\left\{ i\sum_{j=1}^{n}\kappa_j B^H_{ct_j} \right\} &= \exp\left\{ -\frac{1}{2}\langle \boldsymbol{\kappa} C_{c\boldsymbol{t}}, \boldsymbol{\kappa}\rangle \right\} \\
&= \exp\left\{ -\frac{c^{2H}}{2}\langle \boldsymbol{\kappa} C_{\boldsymbol{t}}, \boldsymbol{\kappa}\rangle \right\} \\
&= \mathbb{E}\exp\left\{ i\sum_{j=1}^{n}\kappa_j c^H B^H_{t_j} \right\}.
\end{aligned}
$$

$$
\Longrightarrow \{B^H_{ct}\}_{t\in\mathbb{R}} \overset{f.d.d.}{=} \{c^H B^H_t\}_{t\in\mathbb{R}}
$$

Anomalous diffusive behavior of fBm

By the H-self-similarity we get that B_t^H so that the spreading (spatial scale growing) rate is t^H, slower than standard $t^{1/2}$, for $H < 1/2$, and faster, for $H > 1/2$.

Moreover, the fBm is an anomalous diffusion in the MSD sense, for $H \neq 1/2$, i.e. its variance reads by definition

$$var(B_t^H) = |t|^{2H}, \qquad t \in \mathbb{R},$$

so that it is a

- sub-diffusion for $H < 1/2$
- super-diffusion for $H > 1/2$
- diffusion for $H = 1/2$

Other properties of fBm

- the fBm has stationary, but not independent increments (on disjoint intervals), for $H \neq 1/2$:

$$\begin{aligned} & \mathbb{E}(B_t^H - B_s^H)(B_u^H - B_v^H) \\ = \; & \frac{1}{2}((s-u)^{2H} + (t-v)^{2H} - (t-u)^{2H} - (s-v)^{2H}), \end{aligned}$$

  for $0 \leq u < v < s < t$, which is $\neq 0$, for $H \neq 1/2$.
- its increments are negatively (resp. positively) correlated for $H \in (0, 1/2)$ (resp. $H \in (1/2, 1)$):

$$\begin{aligned} & \mathbb{E}(B_t^H - B_s^H)(B_u^H - B_v^H) \\ = \; & H(2H-1) \int_u^v \int_s^t (x-y)^{2H-2} dx dy, \end{aligned}$$

Other properties of fBm

Thus the fBm is

- anti-persistent for $H \in (0, 1/2)$, i.e. its trajectories have extremely alternating behavior, since two subsequent intervals are negatively correlated
- persistent for $H \in (1/2, 1)$, i.e. its behavior is more regular, since two subsequent intervals are positively correlated
- purely random for $H = 1/2$, since increments are independent (BM case)

The fBm has

- short-memory if $H \in (0, 1/2)$
- long-memory if $H \in (1/2, 1)$
- no memory if $H = 1/2$

Indeed, let

$$r(n) := \mathbb{E}[B_1^H(B_{n+1}^H - B_n^H)] = H(2H-1)\int_0^1 \int_n^{n+1} (x-y)^{2H-2} dx dy,$$

thus $r(n) \sim |n|^{2H-2,}$ as $|n| \to \infty$ and

$$\sum_{n \in \mathbb{Z}} r(n) \sim \sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^{2H-2} \begin{cases} < \infty, & H \in (0, 1/2), \\ = \infty, & H \in (1/2, 1). \end{cases}$$

The fBm can be defined as a stochastic integral w.r.t. the (two-sided) BM with a deterministic integrand.

Theorem (Mandelbrot-Van Ness representation of fBm)

The process $\overline{B}^H := \left\{ \overline{B}_t^H \right\}_{t \geq 0}$ defined as

$$\overline{B}_t^H := C_H \int_{\mathbb{R}} [(t-u)_+^{H-1/2} - (-u)_+^{H-1/2}] dW_u, \quad H \in (0, 1/2) \cup (1/2, 1),$$

where $C_H := \sqrt{2H \sin(\pi H) \Gamma(2H)} / \Gamma(H + 1/2)$ and $(x)_+ := x \mathbb{1}_{x>0}$, has a continuous modification which is the fBm $B^H := \{B_t^H\}_{t \geq 0}$

Proof (sketch):
The M-VN representation holds, as a consequence of the previous corollary, since

- $\overline{B}^H$ is Gaussian, with $\mathbb{E}\overline{B}_t^H = 0$, $\forall t$ (by the linearity of the stochastic integral), and $\overline{B}_0^H = 0$ a.s.
- for $t > 0$ and for $k_H(t,u) := (t-u)_+^{H-1/2} - (-u)_+^{H-1/2}$,

$$\mathbb{E}(\overline{B}_t^H)^2 = C_H^2 \left(\int_{-\infty}^0 k_H^2(t,u)du + \int_0^t (t-u)^{2H-1}du \right) = t^{2H},$$

by Ito isometry, while, for $t < 0$,

$$\mathbb{E}(\overline{B}_t^H)^2 = C_H^2 \left(\int_{-\infty}^t k_H^2(t,u)du + \int_t^0 (-u)^{2H-1}du \right) = (-t)^{2H}$$

Proof (cont'd):

- its autocovariance function is

$$\mathbb{E}(\overline{B}_t^H \overline{B}_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \qquad s, t \in \mathbb{R}.$$

Indeed the increment is, for $h > 0$,

$$\begin{aligned} \overline{B}_{t+h}^H - \overline{B}_t^H &= C_H \int_{-\infty}^{t} (k_H(t+h, u) - k_H(t, u)) dW_u \\ &+ C_H \int_{t}^{t+h} k_H(t+h, u) dW_u =: I_1 + I_2, \end{aligned}$$

since, for $u \in (t, t+h)$, $k_H(t, u) = 0$.
I_1 and I_2 are independent, since W has independent increments over disjoint intervals.

Proof (cont'd):
Moreover, by stationarity of increments of W, we can shift t to being equal to zero, i.e.

$$I_1 = \int_{-\infty}^{0} (k_H(h,u) - k_H(0,u))dW_u$$
$$I_2 = \int_{0}^{h} k_H(h,u)dW_u,$$

so that, for any $t \in \mathbb{R}$, $h > 0$, $\overline{B}^H_{t+h} - \overline{B}^H_t \stackrel{d}{=} \overline{B}^H_h$ and

$$\mathbb{E}(\overline{B}^H_{t+h} - \overline{B}^H_t)^2 = \mathbb{E}(\overline{B}^H_h)^2 = h^{2H}.$$

Finally, the auto-covariance function follows by recalling that

$$\mathbb{E}(\overline{B}^H_t \overline{B}^H_s) = \frac{1}{2}[\mathbb{E}(\overline{B}^H_s)^2 + \mathbb{E}(\overline{B}^H_t)^2 - \mathbb{E}(\overline{B}^H_t - \overline{B}^H_s)^2]$$

Let $(\Omega, \mathcal{F}, P) = (\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu)$ (where $\mathcal{S}' := \mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space, $\mathcal{C}_\sigma(\mathcal{S}')$ is the cylinders' σ-field, and $\mu(\cdot)$ is the Gaussian measure) be the white noise space.

Then we define there the variable $X_f(\omega) := \langle \omega, f \rangle$ (for $f \in \mathcal{S}$ and $\omega \in \mathcal{S}'$) that assigns a random value to a function f which coincides with the Wiener integral $\int f(s) \mathrm{d}W_s$.

We model these generalized random variables in time, by choosing $f \in L^2(\mathbb{R})$ as follows:

If $f = \mathbb{1}_{[0,t)}$, then

$$\langle \omega, \mathbb{1}_{[0,t)} \rangle = B_t(\omega), \quad t > 0;$$

If $f = \mathbb{1}_{[s,t)}$ with $s < t$, then

$$\langle \omega, \mathbb{1}_{[s,t)} \rangle = \langle \omega, \mathbb{1}_{[0,t)} \rangle - \langle \omega, \mathbb{1}_{[0,s)} \rangle = B_t(\omega) - B_s(\omega), \quad t > 0;$$

Classical white-noise analysis

Theorem (Bochner-Minlos)

On the nuclear triple $\mathcal{S} := \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})' =: \mathcal{S}'$, the white noise measure is defined as the unique probability measure μ on $\big(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}')\big)$ such that

$$\mathrm{e}^{-\frac{\langle \xi, \xi \rangle}{2}} = \int_{\mathcal{S}'} \mathrm{e}^{\mathrm{i}\langle \omega, \xi \rangle} \, \mu(\mathrm{d}\omega), \quad \xi \in \mathcal{S}.$$

Thus, we have constructed a probability space denoted by

$$\big(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu\big).$$

The corresponding Hilbert space is denoted by

$$L^2(\mu) := L^2(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu).$$

Hida et al. (1993)

- The r.v. X_f may be defined for elements $f \in L^2(\mathbb{R})$. In that case, $X_f(\omega)$ is defined only for μ-a.e. $\omega \in \mathcal{S}'$ as an element of $L^2(\mu)$;
- the moments and covariance of $X_f = \langle \omega, f \rangle$, $f, g \in L^2(\mathbb{R})$ are:

$$\begin{cases} \mathbb{E}_\mu[X_f^{2n}] = \int_{\mathcal{S}'} \langle \omega, f \rangle^{2n} d\mu(\omega) = \frac{(2n)!}{2^n n!} (f, f)^n, \\ \mathbb{E}_\mu[X_f^{2n+1}] = \int_{\mathcal{S}'} \langle \omega, f \rangle^{2n+1} d\mu(\omega) = 0 \\ \mathbb{E}_\mu[X_f X_g] = \int_{\mathcal{S}'} \langle \omega, f \rangle \langle \omega, g \rangle d\mu(\omega) = (f, g) \end{cases}$$

- Hence $X_f(\omega) = \langle \omega, f \rangle \sim N(0, \|f\|_{L^2}^2)$.

FBm as generalized process in the white noise space

Thank to the M-VN representation, we can write the fractional Brownian motion with Hurst parameter $H \in (0,1)$ as:

$$\overline{B}_t^H = C_H \int_{\mathbb{R}} \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}W_u = \langle \omega, \mathcal{M}_-^H \mathbb{1}_{[0,t)} \rangle,$$

where $\mathcal{M}_-^H$ is a fractional operator of Riemann-Liouville type, i.e.

$$\mathcal{M}_-^H f := \begin{cases} K_H \mathcal{D}_-^{1/2-H} f, & H \in (0,1/2) \\ f, & H = 1/2 \\ K_H \mathcal{I}_-^{H-1/2} f, & H \in (1/2,1), \end{cases}$$

for $K_H := C_H \Gamma(H+1/2) = \sqrt{2H \sin(\pi H) \Gamma(2H)}$, and we denoted by $\mathbb{1}_{(a,b)}(\cdot)$, $a, b \in \mathbb{R}$, the general indicator function, i.e.

$$\mathbb{1}_{(a,b)}(t) = \begin{cases} 1, & a \leq t < b, \\ -1, & b \leq t < a, \\ 0, & \text{otherwise.} \end{cases}$$

For $\beta \in (0,1)$ and $f \in \mathcal{S}$, the (right-sided) Riemann-Liouville fractional derivative is defined as

$$\mathcal{D}_{-}^{\beta} f(x) := -\frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^\infty f(t)(t-x)^{-\beta} \mathrm{d}t, \quad x \in \mathbb{R},$$

and the (right-sided) Riemann-Liouville fractional integral is defined as

$$\mathcal{I}_{-}^{\beta} f(x) := \frac{1}{\Gamma(\beta)} \int_x^\infty f(t)(t-x)^{\beta-1} \mathrm{d}t, \quad x \in \mathbb{R}.$$

Thus it is easy to check that

$$\mathcal{M}_{-}^{H} \mathbb{1}_{[0,t)}(u) = C_H \left[(t-u)_{+}^{H-1/2} - (-u)_{+}^{H-1/2} \right],$$

for $t \in \mathbb{R}$, and that $\mathcal{M}_{-}^{H} \mathbb{1}_{[0,t)}(\cdot) \in L^2(\mathbb{R})$.

Thus, for $t > 0$ and $H \in (0, 1/2) \cup (1/2, 1)$, we have that

$$\begin{aligned}\overline{B}_t^H &= \langle \omega, \mathcal{M}_-^H \mathbb{1}_{[0,t)} \rangle = \int_{\mathbb{R}} \mathcal{M}_-^H \mathbb{1}_{[0,t)} \mathrm{d}W_u \\ &= C_H \int_{\mathbb{R}} \left[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}W_u.\end{aligned}$$

Indeed, the autocovariance function is given by

$$\begin{aligned}&\mathbb{E}_\mu[\overline{B}_t^H \overline{B}_s^H] = \int_{\mathcal{S}'} \langle \omega, \mathcal{M}_-^H \mathbb{1}_{[0,t)} \rangle \langle \omega, \mathcal{M}_-^H \mathbb{1}_{[0,s)} \rangle d\mu(\omega) \\ &= (\mathcal{M}_-^H \mathbb{1}_{[0,t)}, \mathcal{M}_-^H \mathbb{1}_{[0,s)}) \\ &= c \int_{\mathbb{R}} [(t-x)_+^{H-1/2} - (-x)_+^{H-1/2}][(s-x)_+^{H-1/2} - (-x)_+^{H-1/2}] dx \\ &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).\end{aligned}$$

where $c := 1/\Gamma(H + 1/2)^2$

Thus we have the following effect by the fractional operators:

$$\begin{cases} H \in (0,1/2),\ \langle \omega, \mathcal{D}_-^{1/2-H} 1\!\!1_{[0,t)} \rangle & \to \text{short-memory/sub-diffusion} \\ H = 1/2,\ \langle \omega, 1\!\!1_{[0,t)} \rangle & \to \text{no memory/standard diffusion} \\ H \in (1/2,1),\ \langle \omega, \mathcal{I}_-^{H-1/2} 1\!\!1_{[0,t)} \rangle & \to \text{long-memory/super-difusion} \end{cases}$$

Grey Brownian motion

By means of the previous theory, we can define an extension of fBm in the so-called grey noise space.

Recall that, by the Bochner-Minlos theorem, for any CM * function $F(\cdot)$ on $\mathcal{S}(\mathbb{R})$, there exists a unique functional $\Phi(\cdot)$ such that $\Phi(\xi) := F(\|\xi\|^2) \in L^2(\mathbb{R})$ and

$$\Phi(\xi) = \int_{\mathcal{S}'} e^{i\langle x,\xi\rangle} d\mu(x), \qquad \xi \in \mathcal{S},$$

for a unique probability measure μ.

As we have seen, for $F(x) = e^{-x}$, the measure given by

$$\Phi(\xi) = \mathrm{e}^{-\frac{(\xi,\xi)}{2}} = \int_{\mathcal{S}'} \mathrm{e}^{\mathrm{i}\langle\omega,\xi\rangle}\, \mu(\mathrm{d}\omega), \quad \xi \in \mathcal{S},$$

is the white noise measure (Gaussian case).

(*) i.e. C^∞ and $F(x) = a + bx + \int_0^{+\infty} e^{-xt}\mu(dt), \qquad a, b \geq 0$

If, instead, we choose the CM function $F(t) = E_\beta(-t)$, for $\beta \in (0,1)$, where $E_\beta(x) := \sum_{j=0}^{\infty} x^j/\Gamma(\beta j + 1)$ is the Mittag-Leffler function, then

$$\Phi(\xi) = E_\beta(-(\xi,\xi)^2) = \int_{\mathcal{S}'} \mathrm{e}^{\mathrm{i}\langle \omega, \xi\rangle}\, \mu_\beta(\mathrm{d}\omega), \quad \xi \in \mathcal{S},$$

where μ_β is the ML-grey noise measure and $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_\beta)$ the ML-grey noise space.

For $\beta = 1 \implies E_\beta(-x) = e^{-x} \implies \mu_\beta$ reduces to the Gaussian measure.

We define on the grey noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_\beta)$ the generalized process

$$X(\cdot, \varphi) := \langle \cdot, \varphi \rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing. Then

- if $\varphi = 1\!\!1_{[0,t)} \implies$ grey Brownian motion

$$X_\beta(\omega, t) := \left\langle \omega, 1\!\!1_{[0,t)} \right\rangle, \qquad \omega \in \mathcal{S}'(\mathbb{R}),$$

- if $\varphi = \mathcal{M}_-^H 1\!\!1_{[0,t)} \implies$ generalized grey Brownian motion

$$X_\beta^H(\omega, t) := \left\langle \omega, 1\!\!1_{[0,t)} \right\rangle, \qquad \omega \in \mathcal{S}'(\mathbb{R}).$$

Schneider (1988), Grothaus-Jahnert (2015), Mura-Taqqu (2008)

Then, for any $\kappa \in \mathbb{R}$, the c.f. of the generalized grey Brownian motion is given by

$$\begin{aligned}\mathbb{E}e^{i\kappa\left\langle \omega, \mathcal{M}_{-}^{H}\mathbb{1}_{[0,t)}\right\rangle} &= \int_{\mathcal{S}'(\mathbb{R})} e^{i\kappa\left\langle \omega, \mathcal{M}_{-}^{H}\mathbb{1}_{[0,t)}\right\rangle} d\mu_{\beta}(\omega) \\ &= E_{\beta}\left(-\frac{\kappa^{2}}{2}\left\|\mathcal{M}_{-}^{H}\mathbb{1}_{[0,t)}\right\|_{L_{2}(\mathbb{R})}^{2}\right),\end{aligned}$$

for $H \in (0,1)$ and $\beta \in (0,1]$.

Special cases:

- for $H = \beta/2$: grey Brownian motion
- for $\beta = 1$: fractional Brownian motion
- for $\beta = 1$ and $H = 1/2$: standard Brownian motion

- it is a model of anomalous diffusion with characteristic function

$$\mathbb{E}e^{i\kappa X_\beta^H(t)} = E_\beta\left(-\frac{1}{2}\kappa^2 t^{2H}\right), \qquad \kappa \in \mathbb{R},\ t \geq 0$$

and covariance

$$cov\left(X_\beta^H(t), B_\beta^H(s)\right) = \frac{1}{\Gamma(\beta+1)}(s^{2H} + t^{2H} - |t-s|^{2H})$$

- it is a H-sssi process (but not Gaussian)
- the following one-dimensional relationship holds

$$X_\beta^H(t) \stackrel{d}{=} \sqrt{Y_\beta} B^H(t), \qquad t \geq 0,$$

where $B^H := \{B^H(t)\}_{t\geq 0}$ is a fractional Brownian and Y_β is an independent r.v. with density expressed by M-Wright function with parameter β.

Generalized grey Brownian motion

- Its marginal density function $p(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, is the fundamental solution of the stretched time-fractional master equation

$$u(x,t) = u_0(x) + \frac{2H}{\Gamma(\beta+1)} \int_0^t s^{\frac{2H}{\beta}-1} (t^{\frac{2H}{\beta}} - s^{\frac{2H}{\beta}})^{\beta-1} \frac{\partial^2}{\partial x^2} u(x,s) ds$$

This can be checked by recalling that the c.f. is $\hat{p}(\kappa,t) = E_\beta(-\kappa^2 t^{2H})$, which satisfies

$$\begin{aligned}
\hat{p}(\kappa,t) &= E_\beta(-\kappa^2 (t^{2H/\beta})^\beta) \\
&= 1 - \frac{\kappa^2}{\Gamma(\beta)} \int_0^{t^{2H/\beta}} (t^{2H/\beta} - u)^{\beta-1} E_\beta(-\kappa^2 u^\beta) du \\
&= [u = s^{2H/\beta}] \\
&= 1 - \frac{\kappa^2 2H}{\Gamma(\beta+1)} \int_0^t s^{2H/\beta-1} (t^{2H/\beta} - s^{2H/\beta})^{\beta-1} E_\beta(-\kappa^2 s^{2H}) ds
\end{aligned}$$

Generalized grey Bm $\Longleftrightarrow$ fBm

	fBm	**GgBm**
Distribution	Gaussian	M-Wright of par. β
Self-similarity	$B^H(ct) \stackrel{d}{=} c^H B_H(t)$	$X_\beta^H(ct) \stackrel{d}{=} c^H X_\beta^H(t)$
Increments	s.i.	s.i.
MSD	$\sim t^{2H}$	$\sim \frac{t^{2H}}{\Gamma(1+\beta)}$
Governing eq.		Stretched frac. master eq.

In the GgBm

- H controls self-similarity and memory
- β controls non-Gaussianity

Grey dffusions: incomplete Gamma case

An alternative model of grey diffusion can be obtained by exploiting the properties of the upper-incomplete gamma function

$$\Gamma(\rho; x) = \int_x^{+\infty} e^{-w} w^{\rho-1} dw,$$

for $x \in \mathbb{R}$ and $\rho \in \mathbb{C}$, such that $Re(\rho) > 0$.

- The upper incomplete gamma function is completely monotone and

  $$\frac{\Gamma(\rho, \lambda\eta)}{\Gamma(\rho)} = \int_0^{\infty} e^{-\eta y} \nu_\rho(y) dy, \qquad \eta \geq 0,$$

  where

  $$\nu_\rho(y) = \frac{\lambda^\rho (y-\lambda)^\rho \mathbb{1}_{y>\lambda}}{y\Gamma(\rho)\Gamma(1-\rho)}, \qquad \lambda > 0.$$

Moreover, let, for $u : \mathbb{R}^+ \to \mathbb{R}^+$ absolutely continuous function,

$$\mathcal{D}_t^{\lambda,\rho} u(t) := \frac{\rho \lambda^\rho}{\Gamma(1-\rho)} \int_0^t \frac{d}{dt} u(t-s) \Gamma(-\rho; \lambda s) ds,$$

for $\rho \in (0,1)$ and $\lambda > 0$ and $\mathcal{D}_t^{\lambda,\rho} = d/dt$, for $\rho = 1$.

Lemma [B.-Gajda (2020)]

The solution of the following fractional relaxation equation

$$\mathcal{D}_t^{\lambda,\rho} u(t) = -\lambda^\rho u(t), \qquad \lambda > 0, \rho \in (0,1),$$

with $u(0) = 1$, is given by

$$u_{\lambda,\rho}(t) = \frac{\Gamma(\rho; \lambda t)}{\Gamma(\rho)}.$$

Incomplete Gamma measure and space

Steps:

- apply the Bochner-Minlos theorem (by CM) to define the infinite-dimensional measure ν_ρ on $S'(\mathbb{R})$
- use the moments of ν_ρ in order to apply the Gram-Schmidt orthogonalization to the space $\mathcal{L}^2(\mathbb{R}, \mathcal{B}, \nu_\rho)$
- construct the Γ-noise space
- define the generalized process on $(S'(\mathbb{R}), \mathcal{B}, \nu_\rho)$

Problem: the moments of ν_ρ are infinite $\Longrightarrow$ introduce a tempering factor with parameter $\theta > 0$.

Theorem [B.-Cristofaro-Gajda (2023)]

Let $\rho \in (0,1)$, $\lambda > 0$, $\theta \geq 0$, then

$$\frac{\Gamma\left(\rho, \lambda\left(\eta+\theta\right)\right)}{\Gamma(\rho, \lambda\theta)} = \int_0^{+\infty} e^{-\eta y} \nu_{\rho,\theta}(y) dy,$$

where

$$\nu_{\rho,\theta}(y) = \frac{e^{-\theta y} \lambda^{\rho} (y-\lambda)^{-\rho} \mathbb{1}_{y>\lambda}}{y \Gamma(\rho, \lambda\theta) \Gamma(1-\rho)}.$$

The moments of $\nu_{\rho,\theta}$ are equal to zero, for $k = 2n+1$, $n \in \mathbb{N}$ and

$$\int_{\mathbb{R}} x^k \nu_{\rho,\theta}(x) dx = \frac{(-1)^{n+1} 2(2n-1)! \Gamma(\rho) \theta^{\rho-n}}{\Gamma(\rho,\theta)(n-1)!} E^{\rho}_{1,\rho+1-n}\left(-\theta\right),$$

for $k = 2n$, $n \in \mathbb{N}$, where $E^{\rho}_{\alpha,\beta}\left(\cdot\right)$ is the Prabhakar function.

The first polynomials $H_n^{\rho,\theta}$, $n = 0, 1, 2, 3$, orthogonal in $L^2(\mathbb{R}, \nu_{\rho,\theta})$, are given by

$$H_0^{\rho,\theta}(x) = 1 \qquad H_1^{\rho,\theta}(x) = x$$

$$H_2^{\rho,\theta}(x) = x^2 - \frac{\theta^{\rho-1}e^{-\theta}}{\Gamma(\rho,\theta)} \qquad H_3^{\rho,\theta}(x) = x^3 - 3x(1-(1-\rho)\theta^{-1})$$

$$\dots$$

For $\rho = 1 \implies$ Hermite polynomials

Note: the dependence on θ automatically vanishes for $\rho = 1$

Γ-grey noise space

We define the infinite-dimensional Γ-grey measure $\nu_{\rho,\theta}$ on $\mathcal{S}'(\mathbb{R})$, as the unique probability measure fulfilling

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle x,\xi\rangle} \nu_{\rho,\theta}(x)dx = \frac{\Gamma(\rho, \|\xi\|^2_{\mathcal{L}_2} + \theta)}{\Gamma(\rho,\theta)}, \qquad \xi \in \mathcal{S}(\mathbb{R}),\ \rho \in (0,1].$$

- For $\rho \neq 1 \Longrightarrow$ Γ-grey noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \nu_{\rho,\theta})$
- For $\rho = 1 \Longrightarrow$ white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$

Γ-GBM as generalized process

Definition 1

The Γ-GBM (with tempering parameter θ) is defined as the generalized process

$$B^{\theta}_{\alpha,\rho}(\omega,t) := \left\langle \omega, \mathcal{M}_{-}^{\alpha/2}\mathbb{1}_{[0,t)} \right\rangle, \qquad t \geq 0,$$

for $\omega \in (\mathcal{S}'(\mathbb{R}), \mathcal{B}, \nu_{\rho,\theta})$.

Its n-times characteristic function, for any $\xi_k \in \mathbb{R}$, $k = 1, ..., n$ and $n \in \mathbb{N}$, reads

$$\Phi_{\alpha,\rho}(\xi_1, ..., \xi_n; t_1, ...t_n) = \frac{\Gamma\left(\rho, \lambda\left(\frac{1}{2}\sum_{j,k=1}^{n} \xi_j \xi_k \gamma_\alpha(t_j, t_k) + \theta\right)\right)}{\Gamma(\rho, \lambda\theta)},$$

where $\gamma_\alpha(t_j, t_k) = t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha$ and $0 \leq t_1 \leq ... \leq t_n < \infty$.

Equivalently, we denote by $\|\cdot\|_\alpha$ the norm on $\mathcal{S}(\mathbb{R})$ defined as

$$\|f\|_\alpha^2 := C_\alpha \int_{\mathbb{R}} |x|^{1-\alpha} |\widehat{f}(x)|^2 dx, \qquad f \in \mathcal{S}(\mathbb{R}), 0 < \alpha < 2.$$

Then the measure $\nu_{\alpha,\rho}^\theta$ is the unique measure satisfying

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle \omega, \xi \rangle} d\nu_{\alpha,\rho}^\theta(\omega) = \frac{\Gamma\left(\rho, \|\xi\|_\alpha^2 + \theta\right)}{\Gamma(\rho, \theta)}, \quad \xi \in \mathcal{S}(\mathbb{R}),\ \theta \geq 0,\ 0 < \rho \leq 1$$

and we call the probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \nu_{\alpha,\rho}^\theta)$ generalized Γ-grey noise space.

Full characterization (as Normal variance mixture)

Theorem [B.-Cristofaro-Gajda (2023)]

$$B^{\theta}_{\alpha,\rho}(t) \overset{f.d.d.}{=} \sqrt{Y^{\rho}_{\theta,\lambda}} B^{\alpha/2}(t), \qquad t \geq 0,$$

where $B^{\alpha/2} := \{B^{\alpha/2}(t), t \geq 0\}$ is a fractional Brownian motion with Hurst-parameter $H = \alpha/2$, for $\alpha \in (0, 2)$ and $Y^{\rho}_{\theta,\lambda}$ is a r.v. with density

$$l^{\rho}_{\theta,\lambda}(y) = \frac{e^{-\theta y} \lambda^{\rho} (y - \lambda)^{-\rho} \mathbb{1}_{y>\lambda}}{y \Gamma(\rho, \lambda\theta) \Gamma(1 - \rho)}, \qquad \rho \in (0, 1), \lambda > 0, \theta \geq 0,$$

independent from $B^{\alpha/2}$.

Some properties

- The k-th order moment of the $\Gamma-$GBM is given by

$$\mathbb{E}\left[B^{\theta}_{\alpha,\rho}(t)^{k}\right] = \begin{cases} 0, & k = 2n+1 \\ \frac{2\lambda^{n} t^{\alpha n}}{\Gamma(\rho,\lambda\theta)} G^{2,0}_{1,2}\left[\lambda\theta \middle| \begin{matrix} 1-n & \\ 0, & \rho-n \end{matrix}\right], & k = 2n \end{cases}$$

  for $k, n \in \mathbb{N}$

- The autocovariance is

$$cov\left[B^{\theta}_{\alpha,\rho}(t), B^{\theta}_{\alpha,\rho}(s)\right] = \frac{\lambda^{\rho} e^{-\theta\lambda} \theta^{\rho-1}}{\Gamma(\rho,\lambda\theta)} \left[t^{\alpha} + s^{\alpha} - |t-s|^{\alpha}\right].$$

- Stationarity of the increments, $\xi \in \mathbb{R},\ t, s \geq 0$,

$$\mathbb{E}\exp\left\{i\xi[B^{\theta}_{\alpha,\rho}(t) - B^{\theta}_{\alpha,\rho}(s)]\right\} = \frac{\Gamma\left(\rho, \lambda\left(\frac{\xi^2}{2}|t-s|^{\alpha} + \theta\right)\right)}{\Gamma(\rho, \lambda\theta)},$$

- Hölder continuity

$$\mathbb{E}_{\nu_{\rho,\theta}}\left(\left|B^{\theta}_{\alpha,\rho}(t) - B^{\theta}_{\alpha,\rho}(s)\right|^{2n}\right) \leq K^{n}_{\theta,\rho}(t-s)^{\alpha n}.$$

- Self-similarity

$$B^{\theta}_{\alpha,\rho}(at) \overset{f.d.d.}{=} a^{\alpha/2} B^{\theta}_{\alpha,\rho}(t),$$

for $t \geq 0$ and $a > 0$.

Class of H-sssi processes

$\left\{B_{\alpha,\rho}^{\theta}(t)\right\}_{t\geq 0}$, for $\rho \in (0,1]$, $\alpha \in (0,2)$ and $\theta \geq 0$, represents a class of H-sssi processes, with $H = \alpha/2$, which includes

- Fractional Brownian motion (for $\rho = 1$)
- Brownian motion (for $\rho = \alpha = 1$)

Time-change representation (for the one-dimensional law)

Let $Y^{\rho}_{\theta,\lambda}(t)$, $t \geq 0$, be the (tempered) process with Laplace transform of the n-times density equal to

$$\mathbb{E}e^{-\sum_{k=1}^{n}\eta_k Y^{\rho}_{\theta,\lambda}(t_k)} = \frac{\Gamma\left(\rho, \lambda\left(\sum_{k=1}^{n}\eta_k t_k\right)+\theta\right)}{\Gamma\left(\rho\right)}, \qquad \eta_1, ...\eta_n > 0,$$

then $Y^{\rho}_{\theta,\lambda}(1) \stackrel{d}{=} Y^{\rho}_{\theta,\lambda}$ and $Y^{\rho}_{\theta,\lambda}(t)$, $t \geq 0$

- is self-similar (with scaling parameter equal to one)
- has stationary (non-independent) increments
- has non-decreasing trajectories

Then the following equality of the one-dimensional distribution holds

$$B^{\theta}_{\alpha,\rho}(t) \overset{d}{=} B(Y^{\rho}_{\theta,\lambda}(t^{\alpha})), \qquad t \geq 0,$$

for an independent Brownian motion B.

Theorem [B.-Cristofaro-Gajda (2023)]

Let $e^{z}_{\rho} := z^{\rho-1}E_{\rho,\rho}(z^{\rho})$ be the so-called ρ-exponential function, then the characteristic function of $B^{\theta}_{\alpha,\rho}$, for $\theta = 0$, satisfies the following integral equation

$$\Phi_{\alpha,\rho}(\xi,t) = 1 - \frac{\alpha\lambda\xi^2}{2}\int_0^t e^{-\frac{\lambda\xi^2}{2}(t^{\alpha}-s^{\alpha})} e_{\rho}^{\frac{\lambda\xi^2}{2}(t^{\alpha}-s^{\alpha})} s^{\alpha-1}\Phi_{\alpha,\rho}(\xi,s)ds,$$

for $t \geq 0$ and $\xi \in \mathbb{R}$.

- For $\rho = 1$

$$u(\xi, t) = 1 - \frac{\alpha\xi^2}{2} \int_0^t s^{\alpha-1} u(\xi, s) ds,$$

satisfied by the characteristic function of the fractional Brownian motion

- For any $\rho < 1$, compare with the Fourier transform of the GGBM master equation (Mura-Mainardi (2009)):

$$u(\xi, t) = 1 - \frac{\alpha\xi^2}{2\Gamma(\beta+1)} \int_0^t s^{\frac{\alpha}{\beta}-1} (t^{\alpha/\beta} - s^{\alpha/\beta})^{\beta-1} u(\xi, s) ds.$$

Further research directions

- proof of existence of Appel polynomials in the incomplete Γ-case
- definition of a class of completely monotone generalized Wright functions
- extension to generalized Wright analysis in infinite dimensions
- definition of Feynman-Kac formula for the Γ-grey case

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References

- L. Beghin, J. Gajda, Tempered relaxation equation and generalized stable processes, *Fract. Calcul. Appl. Anal.*, 23, n.5, 2020, 1248-1273.
- L. Beghin, L. Cristofaro, J. Gajda, Non-Gaussian measures in infinite dimensional spaces: the Gamma-grey noise, *Potential Analysis*, 2023, 59 (2), 1-23.
- L. Beghin, L.Cristofaro, J.L. Da Silva Fox-H densities and completely monotone generalized Wright functions, *Journal Theoretical Probability*, 2025, 38, 18.
- L. Beghin, L.Cristofaro, J.L. Da Silva, Generalized Wright Analysis in Infinite Dimensions, *Integral Equations and Operator Theory*, 2025, 97:28.

Grothaus, M., Jahnert, F., Riemann, F., & da Silva, J. L. (2015). *Mittag-Leffler analysis I: Construction and characterization.* Journal of Functional Analysis, 268(7), 1876-1903.

Hida, T., Kuo, H. H., Potthoff, J., & Streit, L. (1993). *White Noise and Infinite Dimensional Calculus*, Springer Science & Business Media.

Schneider, W. R. (1988). Grey noise. Stochastic processes, physics and geometry (Ascona and Locarno, 1988), 676-681.