# Anomalous Diffusion, Lévy Processes, and Fractional Calculus

Luisa Beghin
*Sapienza University of Rome*

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# Contents

# Fractional Brownian motion

The fractional Brownian motion (hereafter fBm) is another model of anomalous diffusion, out of the framework of Lèvy processes.

**Definition (fBm)**

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, then the (two-sided) fBm with Hurst parameter $H \in (0,1)$ is a Gaussian process $B^H := \{B_t^H\}_{t\in\mathbb{R}}$ on it, having the following properties

(i) $B_0^H = 0$ a.s.

(ii) $\mathbb{E}B_t^H = 0, \forall t \in \mathbb{R}$

(iii) $\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, $s, t \in \mathbb{R}$.

Since $B^H$ is Gaussian, it has a continuous modification by the Kolmogorov criterion and considering that $\forall n \in \mathbb{N}$,

$$\mathbb{E}|B_t^H - B_s^H|^n = \frac{2^{n/2}}{\sqrt{\pi}\Gamma((n+1)/2)}|t-s|^{2H}$$

.

**Special cases:**

- for $H = 1 \implies B_t^H = tZ$, for any $t \in \mathbb{R}$, where $Z \sim N(0,1)$
- for $H = 1/2 \implies B_t^H = W_t$, for any $t \in \mathbb{R}$, where $\{W_t\}_{t\in\mathbb{R}}$ is the two-sided Brownian motion (or Wiener process)

The $n$-times c.f. of the fBm reads

$$\mathbb{E}\exp\left\{i\kappa_1 B_{t_1}^H + ... + i\kappa_n B_{t_n}^H\right\} = \exp\left\{-\frac{1}{2}\langle \boldsymbol{\kappa} C_{\boldsymbol{t}}, \boldsymbol{\kappa}\rangle\right\},$$

where $\boldsymbol{\kappa} := (\kappa_1, ...\kappa_n)$, $\boldsymbol{t} := (t_1, ..., t_n)$ and the covariance matrix is $C_{\boldsymbol{t}} := \left\{\mathbb{E}B_{t_i}^H B_{t_j}^H\right\}_{1\leq i,j\leq n}$

# Self-similarity of fBm

- the fBm is self-similar of index $H$: for $c > 0$,

$$
\begin{aligned}
\mathbb{E}\exp\left\{i\sum_{j=1}^{n}\kappa_j B^H_{ct_j}\right\} &= \exp\left\{-\frac{1}{2}\langle\boldsymbol{\kappa}C_{c\boldsymbol{t}},\boldsymbol{\kappa}\rangle\right\} \\
&= \exp\left\{-\frac{c^{2H}}{2}\langle\boldsymbol{\kappa}C_{\boldsymbol{t}},\boldsymbol{\kappa}\rangle\right\} \\
&= \mathbb{E}\exp\left\{i\sum_{j=1}^{n}\kappa_j c^H B^H_{t_j}\right\}.
\end{aligned}
$$

$$\Longrightarrow \{B^H_{ct}\}_{t\in\mathbb{R}} \overset{f.d.d.}{=} \{c^H B^H_t\}_{t\in\mathbb{R}}$$

# Anomalous diffusive behavior of fBm

By the $H$-self-similarity we get that $B_t^H$ so that the spreading (spatial scale growing) rate is $t^H$, slower than standard $t^{1/2}$, for $H < 1/2$, and faster, for $H > 1/2$.

Moreover, the fBm is an anomalous diffusion in the MSD sense, for $H \neq 1/2$, i.e. its variance reads by definition

$$var(B_t^H) = |t|^{2H}, \qquad t \in \mathbb{R},$$

so that it is a

- sub-diffusion for $H < 1/2$
- super-diffusion for $H > 1/2$
- diffusion for $H = 1/2$

# Other properties of fBm

- the fBm has stationary, but not independent increments (on disjoint intervals), for $H \neq 1/2$ :

$$\begin{aligned} & \mathbb{E}(B_t^H - B_s^H)(B_u^H - B_v^H) \\ = \; & \frac{1}{2}((s-u)^{2H} + (t-v)^{2H} - (t-u)^{2H} - (s-v)^{2H}), \end{aligned}$$

for $0 \leq u < v < s < t$, which is $\neq 0$, for $H \neq 1/2$.

- its increments are negatively (resp. positively) correlated for $H \in (0, 1/2)$ (resp. $H \in (1/2, 1)$:

$$\begin{aligned} & \mathbb{E}(B_t^H - B_s^H)(B_u^H - B_v^H) \\ = \; & H(2H-1) \int_u^v \int_s^t (x-y)^{2H-2} dx dy, \end{aligned}$$

# Other properties of fBm

Thus the fBm is

- anti-persistent for $H \in (0, 1/2)$, i.e. its trajectories have extremely alternating behavior, since two subsequent intervals are negatively correlated
- persistent for $H \in (1/2, 1)$, i.e. its behavior is more regular, since two subsequent intervals are positively correlated
- purely random for $H = 1/2$, since increments are independent (BM case)

# Memory properties of fBm

The fBm has

- short-memory if $H \in (0, 1/2)$
- long-memory if $H \in (1/2, 1)$
- no memory if $H = 1/2$

Indeed, let

$$r(n) := \mathbb{E}[B_1^H(B_{n+1}^H - B_n^H)] = H(2H-1)\int_0^1 \int_n^{n+1} (x-y)^{2H-2} dx dy,$$

thus $r(n) \sim |n|^{2H-2,}$ as $|n| \to \infty$ and

$$\sum_{n \in \mathbb{Z}} r(n) \sim \sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^{2H-2} \begin{cases} < \infty, & H \in (0, 1/2), \\ = \infty, & H \in (1/2, 1). \end{cases}$$

# Full characterization of the fBm

We can characterize the fBm as the unique Gaussian $H$-self-similar process with stationary increments.

**Definition ($H$-sssi processes)**

A stochastic process $X$ on $(\Omega, \mathcal{F}, P)$ adapted to the filtration $\{\mathcal{F}_t\}$ is $H$-sssi if it is self-similar with index $H$ (i.e., for $c > 0$, $\{X_{ct}\} \overset{f.d.d.}{=} \{c^H X_t\}$) and has stationary increments.

**Theorem**

If the process $X$ is $H$-sssi, with finite variance, then $X_0 = 0$ a.s., it is centered (for $H \neq 1$), symmetric, with

(i) $C_{s,t} := \mathbb{E}(X_t X_s) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, $s, t \in \mathbb{R}$, where $\sigma^2 := \mathbb{E}X_1^2$.

(ii) the Hurst parameter $H < 1$.

**Proof:**
By self-similarity $X_0 = X_{c0} \stackrel{d}{=} c^H X_0$, $\forall c > 0$, so that $X_0 = 0$ a.s.
Applying again self-similarity, we get $\mathbb{E}X_{2t} = 2^H \mathbb{E}X_t$, but, by stationarity of increments, it also holds that

$$\mathbb{E}X_{2t} = \mathbb{E}(X_{2t} - X_t) + \mathbb{E}X_t = 2\mathbb{E}X_t \qquad t \in \mathbb{R},$$

so that $\mathbb{E}X_t = 0$, for any $t \in \mathbb{R}$ and $H \neq 1$.
Symmetry follows by s.i. and by $X_0 = 0$ a.s.:

$$X_{-t} \stackrel{a.s.}{=} X_{-t} - X_0 \stackrel{d}{=} X_0 - X_t \stackrel{a.s.}{=} -X_t.$$

**Proof (cont'd):**
By self-similarity and symmetry,

$$\mathbb{E}X_t^2 = \mathbb{E}X^2_{|t|\mathrm{sign}(t)} = |t|^{2H}\mathbb{E}X^2_{\mathrm{sign}(t)} = |t|^{2H}\mathbb{E}X_1^2 = |t|^{2H}\sigma^2,$$

where $\mathrm{sign}(x) = \pm 1$, for $x > 0$ and $x < 0$, resp.
(i) the covariance follows by the previous expression and by s.i.:

$$\begin{aligned} C_{s,t} := \mathbb{E}(X_t X_s) &= \frac{1}{2}(\mathbb{E}X_s^2 + \mathbb{E}X_s^2 - \mathbb{E}[X_t - X_s]^2) \\ &= \frac{\sigma^2}{2}\left(|t|^{2H} + |s|^{2H} - \frac{1}{\sigma^2}\mathbb{E}X^2_{t-s}\right) \\ &= \frac{\sigma^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \end{aligned}$$

(ii) it can be checked that the previous function is positive-semidefinite (i.e. $\sum_{i=1}^n \sum_{j=1}^n C_{t_i,t_j} u_i u_j \geq 0$, $\forall t_i, t_j \in \mathbb{R}$ and $u_i, u_j \in \mathbb{R}$) only for $0 < H < 1$.

# Full characterization of the fBm

**Theorem (sufficient conditions for fBm)**

Let $X := \{X_t\}_{t\geq 0}$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$, such that

(i) $X_0 = 0$ a.s.,

(ii) $X$ is a centered Gaussian process with $\mathbb{E}X_t^2 = \sigma^2|t|^{2H}$, for $H \in (0,1)$ and $t \in \mathbb{R}$,

(iii) $X$ is a s.i. process,

then $X$ is a fBm with Hurst parameter $H$.

**Proof:** since $X$ is centered Gaussian, its f.d.d's are completely determined by its autocovariance function: it follows from (ii) and (iii) that the latter is given by $C_{s,t}$ (see previous theorem), so that the process coincides with $\left\{B_t^H\right\}_{t\geq 0}$.

# Full characterization of the fBm

**Corollary (full characterization for fBm)**

Let $X := \{X_t\}_{t \geq 0}$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$, and let $H \in (0, 1)$ then the following statements are equivalent:

(i) $X$ is an $H$-sssi Gaussian process

(ii) $X$ is a fBm with Hurst parameter $H$

(iii) $X$ is Gaussian with zero mean and autocovariance function $C_{s,t}$, $s, t \in \mathbb{R}$

$\Longrightarrow$ thus the fBm is the unique Gaussian $H$-sssi process.

# Stochastic integral representation of the fBm

The fBm can be defined as a stochastic integral w.r.t. the (two-sided) BM with a deterministic integrand.

**Theorem (Mandelbrot-Van Ness representation of fBm)**

The process $\overline{B}^H := \left\{\overline{B}_t^H\right\}_{t\geq 0}$ defined as

$$\overline{B}_t^H := C_H \int_{\mathbb{R}} [(t-u)_+^{H-1/2} - (-u)_+^{H-1/2}] dW_u, \quad H \in (0,1/2)\cup(1/2,1),$$

where $C_H := \sqrt{2H \sin(\pi H)\Gamma(2H)}/\Gamma(H+1/2)$ and $(x)_+ := x\mathbb{1}_{x>0}$, has a continuous modification which is the fBm $B^H := \{B_t^H\}_{t\geq 0}$

**Proof (sketch):**
The M-VN representation holds, as a consequence of the previous corollary, since

- $\overline{B}^H$ is Gaussian, with $\mathbb{E}\overline{B}_t^H = 0$, $\forall t$ (by the linearity of the stochastic integral), and $\overline{B}_0^H = 0$ a.s.
- for $t > 0$ and for $k_H(t,u) := (t-u)_+^{H-1/2} - (-u)_+^{H-1/2}$,

$$\mathbb{E}(\overline{B}_t^H)^2 = C_H^2 \left( \int_{-\infty}^0 k_H^2(t,u)du + \int_0^t (t-u)^{2H-1} du \right) = t^{2H},$$

by Ito isometry, while, for $t < 0$,

$$\mathbb{E}(\overline{B}_t^H)^2 = C_H^2 \left( \int_{-\infty}^t k_H^2(t,u)du + \int_t^0 (-u)^{2H-1} du \right) = (-t)^{2H}$$

**Proof (cont'd):**

- its autocovariance function is

$$\mathbb{E}(\overline{B}_t^H \overline{B}_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \qquad s, t \in \mathbb{R}.$$

Indeed the increment is, for $h > 0$,

$$\begin{aligned} \overline{B}_{t+h}^H - \overline{B}_t^H &= C_H \int_{-\infty}^t (k_H(t+h,u) - k_H(t,u))dW_u \\ &+ C_H \int_t^{t+h} k_H(t+h,u)dW_u =: I_1 + I_2, \end{aligned}$$

since, for $u \in (t, t+h)$, $k_H(t,u) = 0$.
$I_1$ and $I_2$ are independent, since $W$ has independent increments over disjoint intervals.

**Proof (cont'd):**

Moreover, by stationarity of increments of $W$, we can shift $t$ to being equal to zero, i.e.

$$I_1 = \int_{-\infty}^{0} (k_H(h,u) - k_H(0,u))dW_u$$
$$I_2 = \int_{0}^{h} k_H(h,u)dW_u,$$

so that, for any $t \in \mathbb{R}$, $h > 0$, $\overline{B}^H_{t+h} - \overline{B}^H_t \stackrel{d}{=} \overline{B}^H_h$ and

$$\mathbb{E}(\overline{B}^H_{t+h} - \overline{B}^H_t)^2 = \mathbb{E}(\overline{B}^H_h)^2 = h^{2H}.$$

Finally, the auto-covariance function follows by recalling that

$$\mathbb{E}(\overline{B}^H_t \overline{B}^H_s) = \frac{1}{2}[\mathbb{E}(\overline{B}^H_s)^2 + \mathbb{E}(\overline{B}^H_t)^2 - \mathbb{E}(\overline{B}^H_t - \overline{B}^H_s)^2]$$

Let $(\Omega, \mathcal{F}, P) = (\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu)$ (where $\mathcal{S}' := \mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space, $\mathcal{C}_\sigma(\mathcal{S}')$ is the cylinders' $\sigma$-field, and $\mu(\cdot)$ is the Gaussian measure) be the white noise space.

Then we define there the variable $X_f(\omega) := \langle \omega, f \rangle$ (for $f \in \mathcal{S}$ and $\omega \in \mathcal{S}'$) that assigns a random value to a function $f$ which coincides with the Wiener integral $\int f(s)\mathrm{d}W_s$.

We model these generalized random variables in time, by choosing $f \in L^2(\mathbb{R})$ as follows:

If $f = \mathbb{1}_{[0,t)}$, then

$$\langle \omega, \mathbb{1}_{[0,t)} \rangle = B_t(\omega), \quad t > 0;$$

If $f = \mathbb{1}_{[s,t)}$ with $s < t$, then

$$\langle \omega, \mathbb{1}_{[s,t)} \rangle = \langle \omega, \mathbb{1}_{[0,t)} \rangle - \langle \omega, \mathbb{1}_{[0,s)} \rangle = B_t(\omega) - B_s(\omega), \quad t > 0;$$

**Theorem (Bochner-Minlos)**

On the nuclear triple $\mathcal{S} := \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})' =: \mathcal{S}'$, the white noise measure is defined as the unique probability measure $\mu$ on $\big(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}')\big)$ such that

$$\mathrm{e}^{-\frac{\langle \xi, \xi \rangle}{2}} = \int_{\mathcal{S}'} \mathrm{e}^{\mathrm{i}\langle \omega, \xi \rangle} \, \mu(\mathrm{d}\omega), \quad \xi \in \mathcal{S}.$$

Thus, we have constructed a probability space denoted by

$$\big(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu\big).$$

The corresponding Hilbert space is denoted by

$$L^2(\mu) := L^2(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu).$$

# Classical white noise analysis

- the nuclear triple is $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}'$;
- the characteristic functional is $C(\varphi) = \mathrm{e}^{-\frac{1}{2}\langle \varphi, \varphi \rangle}$, $\varphi \in \mathcal{S}(\mathbb{R})$;
- the white noise space is $\big(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu\big)$
- For $\varphi \in \mathcal{S}$, let $X_\varphi$ be the random variable defined as:

$$X_\varphi : \big(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}')\big) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})),\ \omega \mapsto \langle \omega, \varphi \rangle,\ \forall \omega \in \mathcal{S}';$$

Hida et al. (1993)

- The r.v. $X_f$ may be defined for elements $f \in L^2(\mathbb{R})$. In that case, $X_f(\omega)$ is defined only for $\mu$-a.e. $\omega \in \mathcal{S}'$ as an element of $L^2(\mu)$;
- the moments and covariance of $X_f = \langle \omega, f \rangle$, $f, g \in L^2(\mathbb{R})$ are:

$$\begin{cases} \mathbb{E}_\mu[X_f^{2n}] = \int_{\mathcal{S}'} \langle \omega, f \rangle^{2n} d\mu(\omega) = \frac{(2n)!}{2^n n!}(f, f)^n, \\ \mathbb{E}_\mu[X_f^{2n+1}] = \int_{\mathcal{S}'} \langle \omega, f \rangle^{2n+1} d\mu(\omega) = 0 \\ \mathbb{E}_\mu[X_f X_g] = \int_{\mathcal{S}'} \langle \omega, f \rangle \langle \omega, g \rangle d\mu(\omega) = (f, g) \end{cases}$$

- Hence $X_f(\omega) = \langle \omega, f \rangle \sim N(0, \|f\|_{L^2}^2)$.

# FBm as generalized process in the white noise space

Thank to the M-VN representation, we can write the fractional Brownian motion with Hurst parameter $H \in (0,1)$ as:

$$\overline{B}_t^H = C_H \int_{\mathbb{R}} \left[ (t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}W_u = \langle \omega, \mathcal{M}_-^H \mathbb{1}_{[0,t)} \rangle,$$

where $\mathcal{M}_-^H$ is a fractional operator of Riemann-Liouville type, i.e.

$$\mathcal{M}_-^H f := \begin{cases} K_H \mathcal{D}_-^{1/2-H} f, & H \in (0,1/2) \\ f, & H = 1/2 \\ K_H \mathcal{I}_-^{H-1/2} f, & H \in (1/2,1), \end{cases}$$

for $K_H := C_H \Gamma(H+1/2) = \sqrt{2H \sin(\pi H) \Gamma(2H)}$, and we denoted by $\mathbb{1}_{(a,b)}(\cdot)$, $a, b \in \mathbb{R}$, the general indicator function, i.e.

$$\mathbb{1}_{(a,b)}(t) = \begin{cases} 1, & a \le t < b, \\ -1, & b \le t < a, \\ 0, & \text{otherwise.} \end{cases}$$

# Riemann-Liouville Fractional Operators

For $\beta \in (0,1)$ and $f \in \mathcal{S}$, the (right-sided) Riemann-Liouville fractional derivative is defined as

$$\mathcal{D}_-^\beta f(x) := -\frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^\infty f(t)(t-x)^{-\beta} \mathrm{d}t, \quad x \in \mathbb{R},$$

and the (right-sided) Riemann-Liouville fractional integral is defined as

$$\mathcal{I}_-^\beta f(x) := \frac{1}{\Gamma(\beta)} \int_x^\infty f(t)(t-x)^{\beta-1} \mathrm{d}t, \quad x \in \mathbb{R}.$$

Thus it is easy to check that

$$\mathcal{M}_-^H \mathbb{1}_{[0,t)}(u) = C_H \left[ (t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right],$$

for $t \in \mathbb{R}$, and that $\mathcal{M}_-^H \mathbb{1}_{[0,t)}(\cdot) \in L^2(\mathbb{R})$.

Thus, for $t > 0$ and $H \in (0, 1/2) \cup (1/2, 1)$, we have that

$$\begin{aligned}\overline{B}_t^H &= \langle \omega, \mathcal{M}_-^H 1\!\!1_{[0,t)} \rangle = \int_{\mathbb{R}} \mathcal{M}_-^H 1\!\!1_{[0,t)} \mathrm{d}W_u \\ &= C_H \int_{\mathbb{R}} \left[ (t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right] \mathrm{d}W_u.\end{aligned}$$

Indeed, the autocovariance function is given by

$$\begin{aligned}&\mathbb{E}_\mu[\overline{B}_t^H \overline{B}_s^H] = \int_{\mathcal{S}'} \langle \omega, \mathcal{M}_-^H 1\!\!1_{[0,t)} \rangle \langle \omega, \mathcal{M}_-^H 1\!\!1_{[0,s)} \rangle d\mu(\omega) \\ &= (\mathcal{M}_-^H 1\!\!1_{[0,t)}, \mathcal{M}_-^H 1\!\!1_{[0,s)}) \\ &= c \int_{\mathbb{R}} [(t-x)_+^{H-1/2} - (-x)_+^{H-1/2}][(s-x)_+^{H-1/2} - (-x)_+^{H-1/2}] dx \\ &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).\end{aligned}$$

where $c := 1/\Gamma(H + 1/2)^2$

Thus we have the following effect by the fractional operators:

$$\begin{cases} H \in (0,1/2),\ \overline{B}_t^H = \langle \omega, \mathcal{D}_-^{1/2-H} \mathbb{1}_{[0,t)} \rangle & \to \text{short-memory} \\ H = 1/2,\ \overline{B}_t = \langle \omega, \mathbb{1}_{[0,t)} \rangle & \to \text{no memory} \\ H \in (1/2,1),\ \overline{B}_t^H = \langle \omega, \mathcal{I}_-^{H-1/2} \mathbb{1}_{[0,t)} \rangle & \to \text{long-memory} \end{cases}$$

# Grey Brownian motion

By means of the previous theory, we can define an extension of fBm as the so-called grey noise space.

Recall that, by the Bochner-Minlos theorem, for any CM * function $F(\cdot)$ on $\mathcal{S}(\mathbb{R})$, there exists a unique functional $\Phi(\cdot)$ such that $\Phi(\xi) := F(\|\xi\|^2) \in L^2(\mathbb{R})$ and

$$\Phi(\xi) = \int_{\mathcal{S}'} e^{i\langle x,\xi\rangle} d\mu(x), \qquad \xi \in \mathcal{S},$$

for a unique probability measure $\mu$.

As we have seen, for $F(x) = e^{-x}$, the measure given by

$$\Phi(\xi) = \mathrm{e}^{-\frac{(\xi,\xi)}{2}} = \int_{\mathcal{S}'} \mathrm{e}^{\mathrm{i}\langle \omega,\xi\rangle} \, \mu(\mathrm{d}\omega), \quad \xi \in \mathcal{S},$$

is the white noise measure (Gaussian case).

(*) i.e. $C^\infty$ and $F(x) = a + bx + \int_0^{+\infty} e^{-xt}\mu(dt), \qquad a, b \geq 0$

# Grey diffusions: Mittag-Leffler case

If, instead, we choose the CM function $F(t) = E_\beta(-t)$, for $\beta \in (0,1)$, where $E_\beta(x) := \sum_{j=0}^{\infty} x^j/\Gamma(\beta j + 1)$ is the Mittag-Leffler function, then

$$\Phi(\xi) = E_\beta(-(\xi,\xi)^2) = \int_{\mathcal{S}'} \mathrm{e}^{\mathrm{i}\langle \omega, \xi \rangle} \, \mu_\beta(\mathrm{d}\omega), \quad \xi \in \mathcal{S},$$

where $\mu_\beta$ is the grey noise measure and $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_\beta)$ grey noise space.

For $\beta = 1 \implies E_\beta(-x) = e^{-x} \implies \mu_\beta$ reduces to the Gaussian measure.

We define on the grey noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu_\beta)$ the generalized process

$$X(\cdot, \varphi) := \langle \cdot, \varphi \rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing. Then

- if $\varphi = 1\!\!1_{[0,t)} \Longrightarrow$ grey Brownian motion

$$X_\beta(\omega, t) := \left\langle \omega, 1\!\!1_{[0,t)} \right\rangle, \qquad \omega \in \mathcal{S}'(\mathbb{R}),$$

- if $\varphi = \mathcal{M}_-^H 1\!\!1_{[0,t)} \Longrightarrow$ generalized grey Brownian motion

$$X_\beta^H(\omega, t) := \left\langle \omega, 1\!\!1_{[0,t)} \right\rangle, \qquad \omega \in \mathcal{S}'(\mathbb{R}).$$

Schneider (1988), Grothaus-Jahnert (2015), Mura-Taqqu (2008)

# Generalized grey Brownian motion

Then, for any $\kappa \in \mathbb{R}$, the c.f. of the generalized grey Brownian motion is given by

$$\begin{aligned}\mathbb{E}e^{i\kappa\left\langle\omega,\mathcal{M}_{-}^{H}1\!\!1_{[0,t)}\right\rangle} &= \int_{\mathcal{S}'(\mathbb{R})} e^{i\kappa\left\langle\omega,\mathcal{M}_{-}^{H}1\!\!1_{[0,t)}\right\rangle} d\mu_\beta(\omega) \\ &= E_\beta\left(-\frac{\kappa^2}{2}\left\|\mathcal{M}_{-}^{H}1\!\!1_{[0,t)}\right\|_{L_2(\mathbb{R})}^2\right),\end{aligned}$$

for $H \in (0,1)$ and $\beta \in (0,1]$.

**Special cases:**

- for $H = \beta/2$: grey Brownian motion
- for $\beta = 1$: fractional Brownian motion
- for $\beta = 1$ and $H = 1/2$: standard Brownian motion

# Generalized grey Brownian motion

- it is a model of anomalous diffusion with characteristic function

$$\mathbb{E}e^{i\kappa X_\beta^H(t)} = E_\beta\left(-\frac{1}{2}\kappa^2 t^{2H}\right), \qquad \kappa \in \mathbb{R},\ t \geq 0$$

and covariance

$$cov\left(X_\beta^H(t), B_\beta^H(s)\right) = \frac{1}{\Gamma(\beta+1)}(s^{2H} + t^{2H} - |t-s|^{2H})$$

- it is a $H$-sssi process (but not Gaussian)
- the following one-dimensional relationship holds

$$X_\beta^H(t) \stackrel{d}{=} \sqrt{Y_\beta} B^H(t), \qquad t \geq 0,$$

where $B^H := \{B^H(t)\}_{t\geq 0}$ is a fractional Brownian and $Y_\beta$ is an independent r.v. with density expressed by $M$-Wright function with parameter $\beta$.

# Generalized grey Brownian motion

- Its marginal density function $p(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, is the fundamental solution of the stretched time-fractional master equation

$$u(x,t) = u_0(x) + \frac{2H}{\Gamma(\beta+1)} \int_0^t s^{\frac{2H}{\beta}-1} (t^{\frac{2H}{\beta}} - s^{\frac{2H}{\beta}})^{\beta-1} \frac{\partial^2}{\partial x^2} u(x,s) ds$$

This can be checked by recalling that the c.f. is $\hat{p}(\kappa,t) = E_\beta(-\kappa^2 t^{2H})$, which satisfies

$$\begin{aligned}
\hat{p}(\kappa,t) &= E_\beta(-\kappa^2 (t^{2H/\beta})^\beta) \\
&= 1 - \frac{\kappa^2}{\Gamma(\beta)} \int_0^{t^{2H/\beta}} (t^{2H/\beta} - u)^{\beta-1} E_\beta(-\kappa^2 u^\beta) du \\
&= [u = s^{2H/\beta}] \\
&= 1 - \frac{\kappa^2 2H}{\Gamma(\beta+1)} \int_0^t s^{2H/\beta-1} (t^{2H/\beta} - s^{2H/\beta})^{\beta-1} E_\beta(-\kappa^2 s^{2H}) ds
\end{aligned}$$

# Generalized grey Bm $\Longleftrightarrow$ fBm

| | **fBm** | **GgBm** |
|---|---|---|
| **Distribution** | Gaussian | M-Wright of par. $\beta$ |
| **Self-similarity** | $B^H(ct) \stackrel{d}{=} c^H B_H(t)$ | $X_\beta^H(ct) \stackrel{d}{=} c^H X_\beta^H(t)$ |
| **Increments** | s.i. | s.i. |
| **MSD** | $\sim t^{2H}$ | $\sim \frac{t^{2H}}{\Gamma(1+\beta)}$ |
| **Governing eq.** | | Stretched frac. master eq. |

In the GgBm

- $H$ controls self-similarity and memory
- $\beta$ controls non-Gaussianity

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# Contents

# Motivation

Recall that, for the fBm with Hurst parameter $H = \alpha/2$,

$$\begin{cases} \alpha \in (0,1),\ \langle \omega, \mathcal{D}_-^{(1-\alpha)/2} 1\!\!1_{[0,t)} \rangle & \rightarrow \text{short-memory / sub-diffusion} \\ \alpha = 1,\ \langle \omega, 1\!\!1_{[0,t)} \rangle & \rightarrow \text{no memory/standard diffusion} \\ \alpha \in (1,2),\ \langle \omega, \mathcal{I}_-^{(\alpha-1)/2} 1\!\!1_{[0,t)} \rangle & \rightarrow \text{long-memory/super-diffusion} \end{cases}$$

We define an extension of fBm in the white noise space which preserves this relation between the operators and the memory behaviour by generalizing the fractional derivatives and integrals.

We are aimed to describe under which conditions on the generalized fractional operators the anomalous diffusing behavior is preserved, at least in some relevant special cases.

# White noise space

Recall:

- the nuclear triple is $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}'$;
- the characteristic functional is $C(\varphi) = \mathrm{e}^{-\frac{1}{2}\langle \varphi, \varphi \rangle}$, $\varphi \in \mathcal{S}(\mathbb{R})$;
- the white noise space is $\left(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu\right)$
- For $\varphi \in \mathcal{S}$, let $X_\varphi$ be the random variable defined as:

$$X_\varphi : \left(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}')\right) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})),\ \omega \mapsto \langle \omega, \varphi \rangle,\ \forall \omega \in \mathcal{S}';$$

Hida et al. (1993)

# FBm in the white noise analysis

By recalling that $X_f = \langle \omega, f \rangle$ and we are under Gaussian measure $\mu$, we represent the fractional Brownian motion with $f(\cdot) := \mathcal{M}_-^{\alpha/2} 1\!\!1_{[0,t)}(\cdot) \in L^2(\mathbb{R})$, where

$$\mathcal{M}_-^{\alpha/2} 1\!\!1_{[0,t)}(u) = \frac{\sqrt{K_\alpha}}{\Gamma((1+\alpha)/2)} \left[ (t-u)_+^{\frac{\alpha-1}{2}} - (-u)_+^{\frac{\alpha-1}{2}} \right],$$

for $K_\alpha = \Gamma(\alpha+1)\sin(\pi\alpha/2)$. Thus, for $t > 0$, we have that

$$\begin{aligned} B_t^{\alpha/2} &= \frac{\sqrt{K_\alpha}}{\Gamma((1+\alpha)/2)} \int_{\mathbb{R}} \left[ (t-u)_+^{\frac{\alpha-1}{2}} - (-u)_+^{\frac{\alpha-1}{2}} \right] \mathrm{d}W_u \\ &= \langle \omega, \mathcal{M}_-^{\alpha/2} 1\!\!1_{[0,t)} \rangle. \end{aligned}$$

### Definition 1

A function $\psi : (0, \infty) \overset{C^\infty}{\to} [0, \infty)$ is completely monotone if

$$\forall n \in \mathbb{N}_0, \quad (-1)^n \psi^{(n)}(\lambda) \geq 0, \quad \forall \lambda > 0.$$

### Definition 2

A function $\varphi(\cdot)$ is said to be complete Bernstein (cB) if it admits the representation:

$$\varphi(x) = a + bx + \int_0^{+\infty} (1 - \mathrm{e}^{-sx})\nu(s)\mathrm{d}s, \qquad x > 0,$$

where $\nu(\cdot)$ is its completely monotone density with the property $\int_0^{+\infty} (s \wedge 1)\nu(\mathrm{d}s) < \infty$.
We define $\overline{\nu}(y) := \int_y^\infty \nu(s)\mathrm{d}s$, $y > 0$, as its tail Lévy density.

The terms "derivative" and "integral" of generalized fractional operators make sense if they satisfy the fundamental theorem of calculus.

**Definition 3 (Sonine pair)**

The functions $\overline{\nu}(s)$ and $\kappa(s)$, $s > 0$ are a Sonine pair if:

$$\int_0^t \overline{\nu}(z)\kappa(t-z)\mathrm{d}z \equiv 1, \quad \text{for any } t > 0$$

If we choose the cB function $\varphi(x) = x^{\beta}$, $\beta \in (0,1)$, the corresponding Sonine pair is given by $\overline{\nu}(s) = s^{-\beta}/\Gamma(1-\beta)$ and $\kappa(s) = s^{\beta-1}/\Gamma(\beta)$.

# Generalized fractional operators

Let $\varphi(\cdot)$ be a complete Bernstein function (with some additional hypotheses), $\overline{\nu}(\cdot)$ be its tail Lévy density and $\kappa(\cdot)$ the corresponding associate Sonine kernel.

We will define the generalized fractional operators on $L^1(\mathbb{R})$ by means of the convolution with the above Sonine pair.

For $\beta \in (0,1)$, we recover the classical Riemann-Liouville fractional operators by using the Sonine pair

$$\overline{\nu}(s) = \frac{s^{-\beta}}{\Gamma(1-\beta)} \qquad\qquad \kappa(s) = \frac{s^{\beta-1}}{\Gamma(\beta)}.$$

We will restrict ourselves to the case where the Bernstein function $\varphi$ satisfies the following conditions:

**(C1)** $\lim_{\theta\to 0}\varphi(\theta) = 0$

**(C2)** $\lim_{\theta\to +\infty}\varphi(\theta) = +\infty$.

Conditions **(C1)-(C2)** imply that $\overline{\nu}$ is the tail of a Lévy density with infinite mass on $\mathbb{R}^+$ (no compound Poisson case): indeed, by the initial and finite value theorems, respectively, we have that

$$\begin{aligned} \lim_{x\to 0}\overline{\nu}(x) &= \lim_{\theta\to +\infty}\theta\tilde{\overline{\nu}}(\theta) = \lim_{\theta\to +\infty}\varphi(\theta) = +\infty, \\ \lim_{x\to +\infty}\overline{\nu}(x) &= \lim_{\theta\to 0}\theta\tilde{\overline{\nu}}(\theta) = \lim_{\theta\to 0}\varphi(\theta) = 0, \end{aligned}$$

where we denote by $\tilde{f}(\cdot)$ the Laplace transform of a function $f$.

Kochubei (2011), Chen (2017)

**Definition 4**

Let $\varphi$ be a Bernstein function and $\overline{\nu}$ be the tail Lévy density associated to $\varphi$; if $\overline{\nu}:\mathbb{R}\to\mathbb{R}^+$ is such that $\overline{\nu}(x)=0$, for $x\leq 0$, and $\overline{\nu}\in L^{p_1}(0,\varepsilon)\cap L^{p_2}(\varepsilon,\infty)$, for any $\varepsilon>0$ and $p_1,p_2\geq 1$, then, for $f\in L^1(\mathbb{R})$, we define the generalized right-sided Riemann-Liouville fractional derivative as follows

$$(\mathcal{D}_-^{(\overline{\nu})}f)(x):=-\frac{d}{dx}\int_x^{+\infty}f(t)\overline{\nu}(t-x)dt, \quad \text{a.e. } x\in\mathbb{R},$$

where the derivative is given in distributional sense (*).

(*) i.e., for any $\phi\in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}}(\mathcal{D}_-^{(\overline{\nu})}f)(x)\,\phi(x)\,dx=\int_{\mathbb{R}}\left(\int_x^{+\infty}f(t)\,\overline{\nu}(t-x)\,dt\right)\phi'(x)\,dx.$$

B.-Cristofaro-Mishura (2024)

**Lemma**

Let $p_1 \geq 1$ and $p_2 \geq 1$ and let $\overline{\nu} : \mathbb{R} \to \mathbb{R}^+$ be such that $\overline{\nu}(x) = 0$ for $x \leq 0$, and $\overline{\nu} \in L^{p_1}(0, \varepsilon) \cap L^{p_2}(\varepsilon, \infty)$, for any $\varepsilon > 0$.
Then the Fourier transform of $\mathcal{D}_{-}^{(\overline{\nu})} q$ is given by

$$\mathcal{F}\left(\mathcal{D}_{-}^{(\overline{\nu})} q\right)(\xi) := \int_{\mathbb{R}} e^{i\xi x} \left(\mathcal{D}_{-}^{(\overline{\nu})} q\right)(x) dx = \varphi(i\xi)\widehat{q}(\xi), \qquad \xi \in \mathbb{R},$$

where $q \in \mathcal{S}(\mathbb{R})$ and $\widehat{q}(\xi) := \mathcal{F}(q)(\xi)$.

If we choose $\varphi(x) = x^{\beta}$, $\beta \in (0,1) \Longrightarrow$ the tail Lévy density is given by $\overline{\nu}(s) = s^{-\beta}/\Gamma(1-\beta)$ and $\mathcal{F}\left(\mathcal{D}_{-}^{(\overline{\nu})} f\right)(\xi) = (i\xi)^{\beta} \widehat{f}(\xi)$ the generalized Riemann-Liouville derivative coincides with the fractional R.L. derivative of order $\beta$.

We extend the definition of $\mathcal{D}_{-}^{(\overline{\nu})}$ for $f \in \mathcal{S}(\mathbb{R})$ to the indicator function in distributional sense.

**Theorem 5 (B.-Cristofaro-Mishura (2024))**

*Let $a, b \in \mathbb{R}$ such that $a < b$, then the generalized fractional derivative of the indicator function of the interval $(a, b)$ is given by*

$$(\mathcal{D}_{-}^{(\overline{\nu})}\mathbb{1}_{(a,b)})(x) = \overline{\nu}((b-x)_{+}) - \overline{\nu}((a-x)_{+}), \quad a.e.\, x \in \mathbb{R}.$$

*Moreover, if $\nu(\cdot)$ and $\overline{\nu}(\cdot)$ satisfy the following assumptions:*

**(A1)** $\nu \in L^2(1, +\infty)$.

**(A2)** $\overline{\nu} \in L^2(0,1) \cap L^p(1, +\infty)$, *for some* $p \geq 1$.

*Then* $\mathcal{D}_{-}^{(\overline{\nu})}\mathbb{1}_{(a,b)} \in L^2(\mathbb{R})$.

# Generalized Fractional Integral: Definition

## Definition 6

Let $\kappa(\cdot)$ be the associate Sonine kernel of $\overline{\nu}$, for a certain Bernstein function $\varphi$. If $\kappa \in L^{p_1}(0,\varepsilon) \cap L^{p_2}(\varepsilon,+\infty)$, for any $\varepsilon > 0$ and for some $p_1, p_2 \geq 1$, then, for $f \in L^1(\mathbb{R})$, we define the generalized right-sided Riemann-Liouville fractional integral as follows

$$(\mathcal{I}_{-}^{(\kappa)} f)(x) := -\int_x^{+\infty} f(t)\kappa(t-x)dt, \quad \text{a.e. } x \in \mathbb{R}.$$

**Lemma**

The operator $\mathcal{I}_-^{(\kappa)}$ is well-defined and its Fourier transform is given by

$$\mathcal{F}\left(\mathcal{I}_-^{(\kappa)}\eta\right)(\xi) := \int_{\mathbb{R}} e^{i\xi x}\left(\mathcal{I}_-^{(\kappa)}\eta\right)(x)dx = \varphi(i\xi)^{-1}\widehat{\eta}(\xi), \qquad \xi \in \mathbb{R},$$

for $\eta \in \mathcal{S}(\mathbb{R})$ and $\widehat{\eta}(\xi) := \mathcal{F}(\eta)(\xi)$.

If we choose $\kappa(s) = s^{\beta-1}/\Gamma(\beta)$, for $\beta \in (0,1)$, which corresponds to $\overline{\nu}(s) = s^{-\beta}/\Gamma(1-\beta) \Longrightarrow$ R.L. integral

$$(\mathcal{I}_-^{\beta}\eta)(x) = \frac{1}{\Gamma(\beta)}\int_x^{+\infty}\eta(t)(t-x)^{\beta-1}dt,$$

with Fourier transform $\mathcal{F}\left(\mathcal{I}_-^{\beta}\eta\right)(\xi) = (i\xi)^{-\beta}\widehat{\eta}(\xi)$.

# Assumptions and first results

**Lemma (B.-Cristofaro-Mishura, [1])**

Let $a, b \in \mathbb{R}$, such that $a < b$, and let $\varphi$ be a complete Bernstein function satisfying **(C1)-(C2)** and let the corresponding Sonine kernel $\kappa(\cdot)$ and the function $\chi(x) := \int_0^x \kappa(z)dz$ satisfy the following conditions, resp., for any $\varepsilon > 0$,

**(B1)** $\kappa \in L^1(0, \varepsilon) \cap L^2(\varepsilon, +\infty)$,

**(B2)** $\chi \in L^2(0, \varepsilon)$.

Then

$$(\mathcal{I}_-^{(\kappa)} \mathbb{1}_{(a,b)})(x) = \chi((b-x)_+) - \chi((a-x)_+).$$

Moreover, $\mathcal{I}_-^{(\kappa)} \mathbb{1}_{(a,b)} \in L^2(\mathbb{R})$.

**Definition 7**

For a complete Bernstein function $\varphi(\cdot)$ satisfying (**C1**)-(**C2**), let

$$\mathcal{M}_{-}^{(\varphi)} := \begin{cases} \mathcal{D}_{-}^{(\overline{\nu})}, & \text{under } \textbf{(A1)-(A2)} \\ \mathcal{I}_{-}^{(\kappa)}, & \text{under } \textbf{(B1)-(B2)} \end{cases}$$

**Definition 8 (B., Cristofaro, Mishura [1])**

Under the previous assumptions, we define on $(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'), \mu)$, where $\mu(\cdot)$ is the Gaussian measure, the Bernstein Gaussian process as

$$X_t^{\varphi}(\omega) := \langle \omega, C_\varphi \mathcal{M}_{-}^{(\varphi)} \mathbb{1}_{[0,t)} \rangle, \qquad t > 0, \ \mu\text{-a.e.}\, \omega,$$

for a normalizing constant $C_\varphi > 0$.

# Bernstein Gaussian Processes: Properties I

**Lemma (B., Cristofaro, Mishura [1])**

Under the previous assumptions $X_t^\varphi$ is centered, Gaussian, with stationary increments and covariance

$$\mathrm{Cov}(X_t^\varphi, X_s^\varphi) = \int_{\mathbb{R}} A_\varphi(x) \frac{1-\cos(tx)-\cos(sx)+\cos((t-s)x)}{x^2}\mathrm{d}x$$

for

$$A_\varphi(x) := \begin{cases} 2C_\varphi|\varphi(\mathrm{i}x)|^2, & \text{if } \mathcal{M}_-^{(\varphi)} = \mathcal{M}_-^{(\overline{\nu})} \\ 2C_\varphi/|\varphi(\mathrm{i}x)|^2, & \text{if } \mathcal{M}_-^{(\varphi)} = \mathcal{M}_-^{(\kappa)}. \end{cases}$$

Its characteristic function reads

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^k \theta_j X_{t_j}^\varphi} = \exp\left(-\frac{1}{2}\langle \mathcal{M}_-^{(\varphi)} \sum_{j=1}^k \theta_j \mathbb{1}_{[0,t_j)}, \mathcal{M}_-^{(\varphi)} \sum_{i=1}^k \theta_i \mathbb{1}_{[0,t_i)}\rangle\right),$$

for $0 < t_1 ... < t_k$, $k \in \mathbb{N}$, and $\theta_j \in \mathbb{R}$, $j = 1, 2, , ...k,$.

Let now define the following inner product $\langle \cdot, \cdot \rangle_\varphi$ on $\mathcal{S}(\mathbb{R})$:

$$\langle \phi, \xi \rangle_\varphi := C \int_{\mathbb{R}} |\varphi(ix)|^2 \overline{\hat{\xi}(x)} \hat{\phi}(x) dx,$$

for $\phi, \xi \in \mathcal{S}(\mathbb{R})$, where $\varphi(\cdot)$ is the Bernstein function.
Then the following relationship holds with the inner product in $L^2(\mathbb{R}, dx)$:

$$\langle \phi, \xi \rangle_\varphi = \langle \mathcal{M}_-^{(\varphi)} \phi, \mathcal{M}_-^{(\varphi)} \xi \rangle_{L^2},$$

for $\phi, \xi \in \mathcal{S}(\mathbb{R})$ and, by the previous lemmas, we have that

$$\begin{aligned} \langle \mathcal{M}_-^{(\varphi)} \mathbb{1}_{[0,t)}, \mathcal{M}_-^{(\varphi)} \mathbb{1}_{[0,s)} \rangle_{L^2} &= \langle \mathbb{1}_{[0,t)}, \mathbb{1}_{[0,s)} \rangle_\varphi \\ &= \int_{\mathbb{R}} \frac{|\varphi(ix)|^2}{x^2} \left[ (e^{-itx} - 1)(e^{isx} - 1) \right] dx. \end{aligned}$$

# Bernstein Gaussian Processes: Properties II

**Theorem (B., Cristofaro, Mishura [1])**

Let the function $\varphi(\cdot)$ satisfy the following condition, for any $a > 0$, for $C, K, a, p, q > 0$ and $\beta \in (0, 1)$

$$Ca^{\beta} \overset{*}{<} \int_{\mathbb{R}} \frac{|\varphi(iax)|^2}{x^2}(1 - \cos x)dx \overset{**}{<} Ka^{1-\frac{q+1}{p}}.$$

Then there exists a version of the process with $\gamma$-Holder continuous sample paths, for $0 < \gamma < q/p$, and with local time $L([0, t], x)$ continuous in $t$, a.e. $x \in \mathbb{R}$, and square-integrable w.r.t. $x$.

Continuity from (**) - local time from (*)

**Theorem 9 (B., Cristofaro, Mishura [1] )**

*Under the assumptions on $\overline{\nu}(\cdot)$ and $\kappa(\cdot)$, then the generalized process $(0,\infty) \ni t \mapsto X_t^{\varphi}$ is differentiable, i.e. the following limit*

$$\lim_{h\to 0} \frac{X_{t+h}^{\varphi} - X_t^{\varphi}}{h}, \quad t > 0$$

*exists and coincides with $\mathscr{N}_t^{\varphi}$ s.t.*

$$(\mathcal{S}\mathscr{N}_t^{\varphi})(\xi) = (\mathcal{M}_{+}^{(\varphi)}\xi)(t),$$

*for every $\xi$ in a suitable neighborhood of zero in $\mathcal{S}_{\mathbb{C}}$, where*

$$(\mathcal{S}\Psi)(\xi) := e^{-\frac{1}{2}\langle\xi,\xi\rangle} \int_{S'} e^{\langle\omega,\xi\rangle}\Psi(\omega)\mu(d\omega),$$

*and $\mathcal{M}_{+}^{(\varphi)}$*

- Define and study the Ornstein-Uhlenbeck Bernstein process as a solution of

$$U_t^\varphi = u_0 - \theta \int_0^t U_s^\varphi + \sigma X_t^\varphi, \quad t \geq 0$$

for $\theta > 0$ and $\sigma \in \mathbb{R}$, or as a solution of the ODE

$$\begin{cases} \frac{d}{dt}u(t) = -\theta u(t) + \sigma(\mathcal{D}_+^{(\overline{\nu})}\xi)(t) & t \geq 0, \\ u(0) = u_0 & t = 0. \end{cases}$$

Then we have that $\mathbb{E}(U_t^\varphi) = u_0 e^{-\theta t}$ and $cov(U_t^\varphi, U_s^\varphi) = \sigma^2 \langle h_{\overline{\nu},t}, h_{\overline{\nu},s} \rangle$, where, for $x \in \mathbb{R}$,

$$\begin{aligned} & h_{\overline{\nu},t}(x) \\ = \ & \overline{\nu}(t-x)_+ - \overline{\nu}(-x)_+ - \theta \int_0^t e^{\theta(s-t)} \left[\overline{\nu}(s-x)_+ - \overline{\nu}(-x)_+\right] ds. \end{aligned}$$

**Theorem [B.-Cristofaro-Mishura, 2024]**

If we define $\rho_n := \mathbb{E}[X_1^{(\overline{\nu})}(X_{n+1}^{(\overline{\nu})} - X_n^{(\overline{\nu})})]$, for $n \in \mathbb{N}$, then

- $\{X_t^{(\overline{\nu})}\}_{t\geq 0}$ is not persistent, at least for $n < N_0$ (for some $N_0$), and it is short-range dependent (i.e. $\sum_{n=1}^{+\infty} |\rho_n| < +\infty$), for any $\overline{\nu}(\cdot)$, satisfying **(A1)-(A2)**.
- $\{X_t^{(\kappa)}\}_{t\geq 0}$ is persistent and long-range dependent (i.e. $\sum_{n=1}^{+\infty} |\rho_n| < +\infty$), for any $\kappa$ satisfying **(B1)-(B2)**.

# Special cases

- **Fractional BM** with $H = \alpha/2$:
  for $\overline{\nu}(s) = s^{(\alpha-1)/2}/\Gamma((1+\alpha)/2)$ and $\varphi(x) = x^{(1-\alpha)/2}$, $\alpha \in (0,1)$, we have

$$X_t^{(\overline{\nu})} = K_\alpha \int_{\mathbb{R}} \left[ (t-u)_+^{(\alpha-1)/2} - (-u)_+^{(\alpha-1)/2} \right] dW_u$$

- **Tempered FBM**:
  for $\overline{\nu}(s) = e^{-\theta s} s^{(\alpha-1)/2}/\Gamma((1+\alpha)/2)$, $\alpha \in (0,1)$, and $\varphi(x) = x(x+\theta)^{-(\alpha+1)/2}$, we get

$$X_t^{(\overline{\nu})} = K_\alpha^\theta \int_{\mathbb{R}} \left[ e^{-\theta(t-u)_+} (t-u)_+^{(\alpha-1)/2} - e^{-\theta(-u)_+} (-u)_+^{(\alpha-1)/2} \right] dW$$

Meerschaert-Sabzikar (2013)

- **Alternative tempered FBM**:
  for $\overline{\nu}(s) = \alpha\theta^\alpha\Gamma(-\alpha, \theta s)/\Gamma(1-\alpha)$ and
  $\varphi(x) = (\theta + x)^\alpha - \theta^\alpha$, $\theta \geq 0$, $\alpha \in (0,1)$, we have

$$X_t^{(\overline{\nu})} = \frac{C\alpha\theta^\alpha}{\Gamma(1-\alpha)} \int_{\mathbb{R}} [\Gamma(-\alpha, \theta(t-u)_+) - \Gamma(-\alpha, \theta(-u)_+)]\, dW_u$$

- **Ornstein-Ulhenbeck process**
  for $\overline{\nu}(s) = e^{-s} = \overline{\overline{\nu}}(s)$ and $\varphi(x) = x/(1+x)$ we have

$$X_t^{(\overline{\nu})} = C \int_{\mathbb{R}} \left[ e^{-(t-u)_+} - e^{-(-u)_+} \right] dW_u.$$

  Note: the Lévy measure is finite on $[0, +\infty)$ and thus $\mathcal{D}_-^{(\overline{\nu})}$ has a non-singular kernel.

# Diffusing behaviour

**Theorem [B.-Cristofaro-Mishura, 2024]**

In the case $\mathcal{M}_{-}^{(\varphi)} = \mathcal{D}_{-}^{(\overline{\nu})}$ we have the following alternative situations:

(i) If $\nu \in L^2(\mathbb{R}^+)$, then $\lim_{t\to+\infty} \mathrm{Var}\left(X_t^{(\nu)}\right) = C > 0$

(ii) If $\nu \notin L^2(\mathbb{R}^+)$, then $\lim_{t\to+\infty} \mathrm{Var}\left(X_t^{(\nu)}\right) = +\infty$ and $\lim_{t\to+\infty} \frac{\mathrm{Var}\left(X_t^{(\nu)}\right)}{t} = 0.$

# Diffusing behavior

We don't have general results on behavior of the MSD, w.r.t. $t$, i.e.

$$Var(X_t^\varphi) = \int_{\mathbb{R}} A_\varphi(x) \frac{1 - 2\cos(tx)}{x^2} \mathrm{d}x$$

for

$$A_\varphi(x) := \begin{cases} 2C_\varphi |\varphi(\mathrm{i}x)|^2, & \text{if } \mathcal{M}_-^{(\varphi)} = \mathcal{M}_-^{(\overline{\nu})} \\ 2C_\varphi / |\varphi(\mathrm{i}x)|^2, & \text{if } \mathcal{M}_-^{(\varphi)} = \mathcal{M}_-^{(\kappa)}, \end{cases}$$

for any Bernstein function $\varphi(\cdot)$ (i.e. for any GFO) satisfying the assumptions.

# Diffusing behavior

We can describe the diffusing behavior in the special cases:

- **fBm with $H = \alpha/2$:** for $\varphi(x) = x^{(1-\alpha)/2}$

$$Var(X_t^{\varphi}) = t^{\alpha}, \qquad \alpha \in (0,2)$$

  - sub-diffusion for $\alpha < 1$
  - super-diffusion for $\alpha > 1$
  - diffusion for $\alpha = 1$

- **Tempered fBm:** for $\varphi(x) = x(x+\theta)^{-(1+\alpha)/2}$, $\alpha \in (0,1)$,

$$Var(X_t^{\varphi}) \begin{cases} \simeq t^{\alpha}, & t \to 0^+ \\ \to const. & t \to +\infty \end{cases}$$

  Thus it behaves as the fBm for small $t$, while it is a mean-reverting process as the stationary OU.

- **Alternative tempered fBm:**
for $\varphi(x) = (x+\theta)^\alpha - \theta^\alpha$, $\alpha \in (0,1)$,

$$Var(X_t^\varphi) \begin{cases} \simeq t, & \alpha \in (0,1/2), \\ \text{diverges} & \alpha \in (1/2,1). \end{cases}$$

Thus it behaves as the Bm and as the tempered $\alpha$-stable process (standard diffusions), for $\alpha \in (0,1/2)$, while it is a super-diffusion, for $\alpha \in (1/2,1)$, as the $\alpha$-stable process.

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