Anomalous Diffusion, Lévy Processes, and Fractional Calculus

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Right-sided Hadamard integral:

$$({}^H\mathcal{J}_-^{\beta} f)(t) \quad := \quad \frac{1}{\Gamma(\beta)} \int_t^{+\infty} \left(\log \frac{z}{t}\right)^{\beta-1} \frac{f(z)}{z} dz,$$

for $t > 0$, $\beta \in \mathbb{C}$, $\mathcal{R}e(\beta) > 0$.

Right-sided Hadamard derivative:

$$({}^H\mathcal{D}_-^{\beta} f)(t) := \frac{1}{\Gamma(n-\beta)} \left(-t\frac{d}{dt}\right)^n \int_t^{+\infty} \left(\log \frac{z}{t}\right)^{n-\beta-1} \frac{f(z)}{z} dz,$$

where $n = \lfloor \beta \rfloor + 1$ and $t > 0$.

Fractional Hadamard operators

When used in the heat equation in place of the time-derivative, the Hadamard fractional derivative is usually associated to ultra-slow diffusions (with mean-squared displacement given by a logarithmic function of time) (*).

Our aim is to verify if similar effect could be produced when we use the fractional Hadamard operators in the definition of a generalized process on $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$, as:

$$B_\alpha^{\mathcal{H}}(t,\omega) := \left\langle \omega, {}^H\mathcal{M}_-^{\alpha/2} \mathbb{1}_{[0,t)} \right\rangle, \qquad t \geq 0,\ \omega \in \mathcal{S}'(\mathbb{R}),$$

where ${}^H\mathcal{M}_-^{\alpha/2}$ is either ${}^H\mathcal{D}_-^{\alpha/2}$ or ${}^H\mathcal{I}_-^{\alpha/2}$.

(*) Liang et al. (2019), Garra-Orsingher-Polito (2018)

Preliminary results

Lemma [B.-Cristofaro-Polito, 2026]

Let $x \in \mathbb{R}_+$ and $a, b \in \mathbb{R}$, such that $a < b$. Then

$$\left({}^H\mathcal{D}_-^{\alpha/2}\mathbb{1}_{[a,b)}\right)(x) = \frac{1}{\Gamma(1+\alpha/2)}\left[\left(\log\frac{b}{x}\right)_+^{(\alpha-1)/2} - \left(\log\frac{a}{x}\right)_+^{(\alpha-1)/2}\right],$$

where $(x)_+ := x\mathbb{1}_{x\geq 0}$, and ${}^H\mathcal{D}_-^{\alpha/2}\mathbb{1}_{[a,b)} \in L^2(\mathbb{R}_+)$, for $\alpha \in (0,1)$.
Analogously

$$\left({}^H\mathcal{I}_-^{\alpha/2}\mathbb{1}_{[a,b)}\right)(x) = \frac{1}{\Gamma(1+\alpha/2)}\left[\left(\log\frac{b}{x}\right)_+^{(\alpha-1)/2} - \left(\log\frac{a}{x}\right)_+^{(\alpha-1)/2}\right],$$

and ${}^H\mathcal{I}_-^{\alpha/2}\mathbb{1}_{[a,b)} \in L^2(\mathbb{R}_+)$, for $\alpha \in (1,2)$.

Hadamard fractional Brownian motion

We define, on $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$, the *Hadamard-fractional Brownian motion* $B_\alpha^{\mathcal{H}} := \left\{ B_\alpha^{\mathcal{H}}(t) \right\}_{t \geq 0}$ as:

$$B_\alpha^{\mathcal{H}}(t,\omega) := \left\langle \omega, {}^{H}\mathcal{M}_{-}^{\alpha/2} \mathbb{1}_{[0,t)} \right\rangle, \qquad t \geq 0,\ \omega \in \mathcal{S}'(\mathbb{R}),$$

where

$$\left({}^{H}\mathcal{M}_{-}^{\alpha/2} f \right)(x) := \begin{cases} K_\alpha \left({}^{H}\mathcal{D}_{-}^{\alpha/2} f \right)(x), & \alpha \in (0,1) \\ f(x), & \alpha = 1 \\ K'_\alpha \left({}^{H}\mathcal{I}_{-}^{\alpha/2} f \right)(x), & \alpha \in (1,2) \end{cases}$$

for $K_\alpha = \Gamma(1-\alpha/2)/\sqrt{\Gamma(1-\alpha)}$, $K'_\alpha = \Gamma(1+\alpha/2)/\sqrt{\Gamma(1+\alpha)}$.

Hadamard fractional Brownian motion

Theorem [B.-Cristofaro-Polito, 2026]

For any $\alpha \in (0,1) \cup (1,2)$, the Hadamard-fBm is a Gaussian process, with zero mean,

$$var(B_\alpha^{\mathcal{H}}(t)) = t, \qquad t \geq 0,$$

and

$$cov(B_\alpha^{\mathcal{H}}(t), B_\alpha^{\mathcal{H}}(s)) = C_\alpha (s \wedge t) \Psi \left(\frac{1-\alpha}{2}, 1-\alpha; \log \left(\frac{s \vee t}{s \wedge t} \right) \right),$$

where $s, t \in \mathbb{R}_+$, $s \neq t$, $C_\alpha = 2^{1-\alpha}\sqrt{\pi}/\Gamma(\alpha/2)$ and $\Psi(a, b; z)$, $a, b, z \in \mathbb{C}$, $\Re(b) \neq 0, \pm 1, \pm 2, \ldots$, is the Tricomi's confluent hypergeometric function.

Tricomi's confluent hypergeometric function

Recall:

- *Tricomi's confluent hypergeometric function* defined as

$$\Psi\left(a,b;z\right) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)}\Phi(a,b;z)+\frac{\Gamma(b-1)}{\Gamma(a)}\Phi(1+a-b,2-b;z),$$

for $a,b,z\in\mathbb{C}$, $\Re(b)\neq 0,\pm1,\pm2,\ldots$, where

$$\Phi(a,b;z) := \sum_{k=0}^{+\infty}\frac{(a)_k}{(b)_k}\frac{z^k}{k!}$$

and $(c)_k := \frac{\Gamma(c+k)}{\Gamma(c)}$.

Hadamard fractional Brownian motion

Theorem [B.-Cristofaro-Polito, 2026]

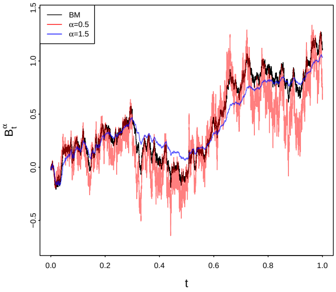
The H-fBm is self-similar with index 1/2 and has non-stationary increments, with characteristic function

$$\begin{aligned}&\mathbb{E}e^{ik\left(B_\alpha^{\mathcal{H}}(t)-B_\alpha^{\mathcal{H}}(s)\right)}\\ =\;& \exp\left\{-\frac{k^2}{2}\left[t+s-C_\alpha(t\wedge s)\Psi\left(\frac{1-\alpha}{2},1-\alpha;\log\left(\frac{t\vee s}{t\wedge s}\right)\right)\right]\right\},\end{aligned}$$

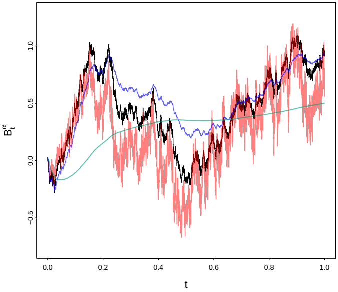
for $k\in\mathbb{R}$ and $s,t\geq 0$.

- $\left\{B_\alpha^{\mathcal{H}}(at)\right\}_{t\geq 0} \overset{f.d.d.}{=} \left\{a^{1/2}B_\alpha^{\mathcal{H}}(t)\right\}_{t\geq 0}$, where $\overset{f.d.d.}{=}$ denotes equality of the finite-dimensional distributions

Plots of sample paths



(a) $\mathcal{H}$-fBm with $\alpha = 0.5$ (red), $\alpha = 1.5$ (blue) vs Brownian motion (black)



(b) $\mathcal{H}$-fBm with $\alpha = 0.5$ (red), $\alpha = 1.5$ (blue), $\alpha = 3$ (green) vs Brownian motion (black)

Hadamard fractional Brownian motion

- no effect of the Hadamard operator on the one-d.d.
- linear variance (not anomalous diffusion)
- analogous results for any operator whose Mellin transform is equal to the Mellin transform of the indicator function multiplied by a quantity depending on α (i.e. Erdélyi-Kober type operators)

Smoothness of trajectories

Theorem [B.-Mishura-De Gregorio, 2025+]

Let $T > 0$ and $0 \le s \le t \le T$. Then the following statements are true.

1. For $\alpha \in (0,1)$

$$\mathbb{E}|B_\alpha^{\mathcal{H}}(t) - B_\alpha^{\mathcal{H}}(s)|^2 \le C_{T,\alpha}(t-s)^\alpha,$$

   where $C_{T,\alpha} > 0$ depends on T and α. Therefore $B_\alpha^{\mathcal{H}}$ belongs to $\mathcal{C}^{\alpha/2-}([0,T])$, for any $T \in (0,\infty)$.

2. For $\alpha \in (1,2)$,

$$\mathbb{E}|B_\alpha^{\mathcal{H}}(t) - B_\alpha^{\mathcal{H}}(s)|^2 \le C_\alpha(t-s),$$

   where $C_\alpha > 0$ depends only on α and does not depend on $T > 0$. Therefore $B_\alpha^{\mathcal{H}}$ belongs to $\mathcal{C}^{1/2-}([0,T])$, for any $T \in (0,\infty)$.

Hölder continuity

- the H-fBm is Hölder continuous on $[0, T]$ up to order $\alpha/2$ for $\alpha \in (0,1)$ and $1/2$ for $\alpha \in (1,2)$
- the order of Hölder continuity can not be improved (previous upper bounds are optimal - as it was proved by lower bounds)
- the Hölder exponent on $[\delta, T]$, for $0 < \delta < T$, is the same as before (i.e. $\alpha/2$) for $\alpha \in (0,1)$, while it is $\alpha/2 < 1/2$, for $\alpha \in (1,2)$
- increasing smoothness with departure from 0 (on the contrary to the case of fBm)

Further properties: p-variations

Let us define the p-variation on $[0, T]$ of a stochastic process $X := \{X(t)\}_{t\geq 0}$ as

$$V^p(X,[0,T]) = \lim_{n\to\infty} \mathbb{E}\sum_{k=1}^{n}\left|X\left(\frac{Tk}{n}\right) - X\left(\frac{T(k-1)}{n}\right)\right|^p.$$

Theorem [B.-Mishura-De Gregorio, 2025+]

Let $T > 0$ and let $B_\alpha^{\mathcal{H}}$ be the $\mathcal{H}$-fBm with $\alpha \in (0,1)$. Then the p-variation $V^p(B_\alpha^{\mathcal{H}}, [0, T])$ is finite and nonzero for $p = \frac{2}{\alpha}$, zero for $p > \frac{2}{\alpha}$ and infinite for $p < \frac{2}{\alpha}$.

- In particular, the quadratic variation is infinite despite the analogies with the Wiener process (1/2 self-similarity, linear in t variance)

Further properties: p-variations

Theorem [B.-Mishura-De Gregorio, 2025+]

The p-variation $B_\alpha^{\mathcal{H}}$ of the the $\mathcal{H}$-fBm with $\alpha \in (1,2)$. $V^p(B_\alpha^{\mathcal{H}}, [0, T])$ is finite and non-zero for $p = \frac{2}{\alpha}$, zero for $p > \frac{2}{\alpha}$ and infinite for $p < \frac{2}{\alpha}$.

- In particular, quadratic variation $V^2(B_\alpha^{\mathcal{H}}, [0, T])$ is zero, contrary to the well-known $V^2(B, [0, T]) = T$ for the Wiener process $B := \{B(t)\}_{t \geq 0}$

Further properties: local non-determinism

Recall that, according to Berman definition, a zero mean Gaussian process $\{X(t)\}_{t\in\mathbb{R}}$ with strictly positive incremental variance on some interval $\mathbb{T} = (a, b)$ is called locally nondeterministic (LND) if, for any $m \geq 2$,

$$\liminf_{\epsilon\downarrow 0} V_m = \frac{\mathrm{Var}(X(t_m) - X(t_{m-1})|X(t_1), \ldots, X(t_{m-1}))}{\mathrm{Var}(X(t_m) - X(t_{m-1}))} > 0,$$

for $t_1 < t_2 < \ldots < t_m \in (a, b)$ with $|t_1 - t_m| < \epsilon$.

Theorem [B.-Mishura-De Gregorio, 2025+]

The $\mathcal{H}$-fBm $B^{\mathcal{H}}_{\alpha}$ is locally nondeterministic on any interval $(0, T)$, $T > 0$, for $\alpha \in (0, 1) \cup (1, 2)$.

- level of randomness at all scales prevented from collapsing into a deterministic process

Memory properties

Let $X_\alpha^{\mathcal{H}} := \left\{X_\alpha^{\mathcal{H}}(n)\right\}_{n\geq 0}$ be the discrete-time increment process defined as $X_\alpha^{\mathcal{H}}(n) := B_\alpha^{\mathcal{H}}(n) - B_\alpha^{\mathcal{H}}(n-1)$, for $n \in \mathbb{N}$.

Theorem [B.-Mishura, De Gregorio, 2025+]

The process $X_\alpha^{\mathcal{H}}(n)$, is short-range dependent, for $\alpha \in (0,1)$.

- For $\alpha \in (0,1)$: $\sum_{n=1}^{\infty} \left|\mathbb{E}X_\alpha^{\mathcal{H}}(1)X_\alpha^{\mathcal{H}}(n)\right| < \infty$

Theorem [B.-Cristofaro-Polito, 2026]

The process $X_\alpha^{\mathcal{H}}$ is long-range dependent, for $\alpha \in (1,2)$.

- For $\alpha \in (1,2)$ and for any $t \in \mathbb{N}$, $m \to +\infty$:

$$\Delta_t^{(m)} := \frac{var\left[\sum_{j=tm-m+1}^{tm} X_\alpha^{\mathcal{H}}(j)\right]}{\sum_{j=tm-m+1}^{tm} var\left[X_\alpha^{\mathcal{H}}(j)\right]} \to +\infty.$$

Mandelbrot-Van Ness representation

Recall: M-VN finite-dimensional representation of fBm as

$$B_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} (t-s)_+^{(\alpha-1)/2} dB(s),$$

where $\{B(t)\}_{t\geq 0}$ is a standard Brownian motion.

For H-fBm we have that

$$B_\alpha^{\mathcal{H}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \left(\log \frac{t}{s}\right)_+^{(\alpha-1)/2} dB(s),$$

for $\alpha \in (0,1) \cup (1,2)$.

Wiener integral w.r.t. H-fBm

By the previous Mandelbrot-Van Ness representation of the H-fBm we give

Definition [B.- De Gregorio, Mishura (2025+)]

Let the inner product on the space $L_{2,\alpha}(\mathbb{R}^+) := \left\{ f : {}_{\mathcal{H}}\mathcal{M}^{\alpha}_{-} f \in L_2(\mathbb{R}^+) \right\}$ be

$$\langle f, g \rangle_{L_{2,\alpha}} := \int_0^\infty {}_{\mathcal{H}}\mathcal{M}^{\alpha}_{-} f(x) {}_{\mathcal{H}}\mathcal{M}^{\alpha}_{-} g(x) dx = \langle {}_{\mathcal{H}}\mathcal{M}^{\alpha}_{-} f, {}_{\mathcal{H}}\mathcal{M}^{\alpha}_{-} g \rangle_{L_2},$$

then we define the following stochastic integral w.r.t. the $\mathcal{H}$-fBm

$$I_\alpha(f) := \int_{\mathbb{R}^+} f(s) dB^{\mathcal{H}}_{\alpha}(s) = \int_{\mathbb{R}^+} \left({}_{\mathcal{H}}\mathcal{M}^{\alpha}_{-} f \right)(s) dB(s).$$

Steps of the integral's construction

- integral of a step function $f(t) := \sum_{k=1}^{n} a_k \mathbb{1}_{[t_{k-1}, t_k)}(t)$:

$$\mathcal{J}_\alpha(f) = \sum_{k=1}^{n} a_k [B_\alpha^{\mathcal{H}}(t_k) - B_\alpha^{\mathcal{H}}(t_{k-1})]$$

- proof that the linear span of the set $\left\{ {}_{\mathcal{H}}\mathcal{M}_{-}^{\alpha} \mathbb{1}_{(a,b)}; a, b \in \mathbb{R}^+, a < b \right\}$ is dense in $L_2(\mathbb{R}^+)$

by resorting to the Plancherel and Parseval formulas of Mellin transform

Riemann-Stieltjes integration of smooth integrands w.r.t. H-fBm

By Hölder properties of $\mathcal{H}$-fBm we get

Theorem [B.- Mishura, De Gregorio (2025+)]

Let $T > 0$; then the integral

$$\int_0^T f(s) dB_\alpha^{\mathcal{H}}(s)$$

exists as the a.s. limit of Riemann-Stieltjes sums if $f : [0, T] \to \mathbb{R}$ is Hölder continuous up to order β such that

1. if $\alpha \in (0, 1)$, then $\beta > 1 - \frac{\alpha}{2}$,
2. if $\alpha \in (1, 2)$, then $\beta > \frac{1}{2}$.

Riemann-Stieltjes integration of smooth integrands w.r.t. H-fBm

Theorem [B.- Mishura, De Gregorio (2025+)]

For any $\alpha \in (0,1) \cup (1,2)$ and for any function $f \in \mathcal{C}^{(1)}([0, T])$, the following integration-by-parts formula holds true

$$\int_0^T f(s) dB_\alpha^{\mathcal{H}}(s) = f(T) B_\alpha^{\mathcal{H}}(T) - \int_0^T B_\alpha^{\mathcal{H}}(s) f'(s) ds.$$

Definition of non-Gaussian measure: Le Roy measure

Definition.

The Le Roy function is defined as

$$\mathcal{R}_\beta(x) := \sum_{j=0}^{\infty} \frac{x^j}{(j!)\beta}$$

for $x \in \mathbb{R}$ and $\beta \in (0, 1]$.

We use the complete monotonicity of the function $\mathcal{R}_\beta(-x)$, for $x > 0$, in order to define a measure on the space $(\mathcal{S}', \sigma*)$. The choice is motivated by the fact that it satisfies the following equation

$$^H\mathcal{D}_{0+}^{\beta} f(t) = tf(t), \qquad t \geq 0.$$

Simon (2022)

Definition of non-Gaussian measure: Le Roy measure

Definition.

Let $\beta \in (0,1]$, we define the *n-dimensional Le Roy measure* $\nu_\beta^n(\cdot)$ as the unique probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ that satisfies:

$$\mathcal{R}_\beta\left(-\frac{\langle \xi, \xi\rangle}{2}\right) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} d\nu_\beta^n(x), \qquad \xi \in \mathbb{R}^n. \qquad (-1)$$

Lemma [B.-Cristofaro-Polito, 2026]

The mixed moments of orders $r_1, ..., r_n \in \mathbb{N}$ of $(X_{\beta,1}, ..., X_{\beta,n})$ with characteristic function given in (1) are

$$\begin{aligned} M_{r_1,\dots,r_n} \quad : \quad &= \mathbb{E}\left[{X_{\beta,1}}^{r_1} \cdots {X_{\beta,n}}^{r_n}\right] \\ &= \begin{cases} 0, & \text{for at least one } r_j = 2m_j + 1 \\ 2^{-m}(m!)^{1-\beta} \prod_{j=1}^{n} \frac{(2m_j)!}{m_j!}, & \text{for } r_j = 2m_j, \end{cases} \end{aligned}$$

for $j = 1, ..., n$, $m_j = 1, 2, ...$ and $m = \sum_{j=1}^n m_j$.

Le Roy grey noise space

Definition.

Let $\beta \in (0,1]$, we define the *Le Roy grey noise space* $(\mathcal{S}', \sigma^*, \nu_\beta)$ where $\nu_\beta(\cdot)$ is the unique probability measure such that

$$\int_{\mathcal{S}'} e^{i\langle x,\xi\rangle} d\nu_\beta(x) = \mathcal{R}_\beta\left(-\frac{\langle \xi,\xi\rangle}{2}\right), \qquad \xi \in \mathcal{S}. \qquad (0)$$

Corollary [B.-Cristofaro-Polito, 2026]

The even-order moments of the measure ν_β are given by

$$\int_{\mathcal{S}'(\mathbb{R})} \langle u,\xi\rangle^{2m} d\nu_\beta(u) = \frac{(2m)!}{2^m (m!)^\beta} \langle \xi,\xi\rangle^m,$$

$$\int_{\mathcal{S}'(\mathbb{R})} \prod_{j=1}^{2m} \langle u,\xi_j\rangle d\nu_\beta(u) = (m!)^{1-\beta} \sum_{SP} \prod_{k=1}^{m} \langle \xi_{r_k}, \xi_{s_k}\rangle,$$

for $m \in \mathbb{N}$, $\xi, \xi_i \in \mathcal{S}(\mathbb{R})$, $i \in \mathbb{N}$.

Le Roy grey noise space

The Laplace transform of ν_β is well-defined and holomorphic in $\mathcal{S}_{\mathbb{C}} := \mathcal{S} \oplus i\mathcal{S} = \{\xi_1 + i\xi_2 | \xi_1, \xi_2 \in \mathcal{S}\}$:

Lemma [B.-Cristofaro-Polito, 2026]

Let $\beta \in (0,1)$ and $\lambda \in \mathbb{R}/\{0\}$, then the exponential function $\mathcal{S}' \ni \omega \mapsto e^{|\lambda\langle x,\phi\rangle|}$ is integrable w.r.t. $\nu_\beta(\cdot)$, and

$$\ell_\beta(\phi) := \int_{\mathcal{S}'} e^{\lambda\langle x,\phi\rangle} d\nu_\beta(x) = \mathcal{R}_\beta\left(\frac{\lambda^2||\phi||^2}{2}\right),$$

for ϕ s.t. $||\phi|| < \infty$, is holomorphic in $\mathcal{S}_{\mathbb{C}}$.

$\Longrightarrow$ ν_β has analytic Laplace transform in a neighborhood of 0
$\Longrightarrow$ ν_β belongs to the class of measures for which the Appell systems exist

Hadamard-fBm in the Le Roy space

Definition.

Let $\mathbb{1}_{[a,b)}$ be the indicator function of $[a,b)$, then we define on the probability space $(\mathcal{S}'(\mathbb{R}), \sigma^*, \nu_\beta)$ the following process

$$B^H_{\alpha,\beta}(t,\omega) := \left\langle \omega, {}^H\mathcal{M}^{\alpha/2}_{-}\mathbb{1}_{[0,t)} \right\rangle, \qquad t \geq 0,\ \omega \in \mathcal{S}'(\mathbb{R}).$$

One-dimensional characterization:

- characteristic function, for $\theta \in \mathbb{R}$, $t \geq 0$,

$$\mathbb{E}e^{i\theta B^H_{\alpha,\beta}(t)} = \mathcal{R}_\beta\left(-\frac{\theta^2}{2}\left\| {}^H\mathcal{M}^{\alpha/2}_{-}\mathbb{1}_{[0,t)} \right\|^2\right) = \mathcal{R}_\beta\left(-\frac{\theta^2 \overline{K}^2_\alpha t}{2}\right),$$

  where

$$\overline{K}_\alpha = \begin{cases} \Gamma(1-\alpha/2)/\sqrt{\Gamma(1-\alpha)}, & \text{for } \alpha \in (0,1) \\ \Gamma(1+\alpha/2)/\sqrt{\Gamma(1+\alpha)}, & \text{for } \alpha \in (1,2). \end{cases}$$

Hadamard-fBm in the Le Roy space

- one-dimensional representation:

$$B^{H}_{\alpha,\beta}(t) \overset{d}{=} B(T_{\alpha,\beta}(t)), \qquad t \geq 0,$$

where $\{T_{\alpha,\beta}(t)\}_{t\geq 0}$, independent of $\{B(t)\}_{t\geq 0}$, has density $g_{\alpha,\beta}(x,t) = m_{\beta}(x/\overline{K}^{2}_{\alpha}t)/\overline{K}^{2}_{\alpha}t$, for $x, t \in \mathbb{R}_{+}$.

Lemma [B.-Cristofaro-Polito, 2026]

Let ${}^{H}\overline{\mathcal{D}}^{\beta}_{t}$ the (left-sided) Hadamard derivative, in the Caputo form, of order $\beta \in (0,1)$, then the transition density of $B^{H}_{\alpha,\beta}$ satisfies the following differential equation

$${}^{H}\overline{\mathcal{D}}^{\beta}_{t} u(x,t) = \frac{\overline{K}^{2}_{\alpha} t}{2} \frac{\partial^{2}}{\partial x^{2}} u(x,t),$$

with $u(x,0) = \delta(x)$, where $\delta(\cdot)$ is the Dirac's delta function.

Hadamard-fBm in the Le Roy space

Therefore

- the process $B^H_{\alpha,\beta}$ has zero mean and

$$var\left(B^H_{\alpha,\beta}(t)\right) = \mathbb{E}T_{\alpha,\beta}(t) = t,$$

  regardless of the parameter β

- $$cov(B^H_{\alpha,\beta}(t), B^H_{\alpha,\beta}(s)) = cov(B^H_\alpha(t), B^H_\alpha(s)),$$

- the persistence and memory properties of $B^H_{\alpha,\beta}$ coincide with those of the Hadamard-fBm and are independent from the parameter β.

Hadamard-fBm in the Le Roy space

Finite-dimensional representation

- n-times characteristic function of $B^H_{\alpha,\beta}$ as

$$\mathbb{E}e^{i\sum_{j=1}^{n}\theta_j B^H_{\alpha,\beta}(t_j)} = \mathcal{R}_\beta\left\{-\frac{1}{2}\left\|\sum_{j=1}^{n}\theta_j \mathcal{M}_{-}^{\alpha/2}\mathbb{1}_{[0,t_j)]}\right\|^2\right\},$$

  for $0 \leq t_1 < t_2 < ... < t_n$ and $\theta_j \in \mathbb{R}$, for $j = 1, ..., n$

- $$\left\{B^H_{\alpha,\beta}(t)\right\}_{t\geq 0} \overset{f.d.d.}{=} \left\{\sqrt{Y_\beta}B^H_\alpha(t)\right\}_{t\geq 0} \overset{f.d.d.}{=} \left\{B^H_\alpha\left(\sqrt{Y_\beta}t\right)\right\}_{t\geq 0},$$

  where Y_β, independent of the H-fBm B^H_α, has distribution $P(Y_\beta \in B) = \mu_\beta(B)$, for any $B \in \mathcal{B}(\mathbb{R}_+)$.

Shot-noise process weakly converging to the H-fBm

Let

$$S(t) := \sum_{j=1}^{+\infty} g(t/T_j) R_j(T_j), \qquad t \geq 0,$$

where

- $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the response function (depending on ratios instead of the distances between current times and instants T_j of last Poissonian occurrences of shots, with rate λ)
- $R_j(T_j)$ is the noise caused by the j-th shot at time T_j and is assumed to depend on T_j, but conditionally independent of each other and i.d. with

$$F_u(x) := P(R_j \leq x | T_j = u), \qquad x \in \mathbb{R}, u \in \mathbb{R}^+.$$

Definition of the shot-noise process

Further assumptions:

- The conditional moments of R_j, i.e. $K_r(u) := \int_{\mathbb{R}} x^r dF_u(x)$, for any $u \in \mathbb{R}$ and $j \in \mathbb{Z}$, are assumed to be finite at least for $r = 1, 2, 3, 4$
- $K_1(u) = 0$, for any u, without loss of generality.

Then, using the relation $\sum_{j=1}^{N(t)} g(t/T_j) = \int_0^{+\infty} g(t/u)N(du)$ (where $\{N(t)\}_{t \geq 0}$ is a Poisson process with intensity λ), we can write, for any $t \geq 0$,

$$S(t) = \int_0^{+\infty} \int_{\mathbb{R}} g(t/u) r N(du, dr),$$

where $N(\cdot, \cdot)$ is a Poisson measure with intensity $\lambda F_u(dr)du$.

Logarithmic shot-noise

Let now

$$g(t) = \log^{\beta}(t)\mathbb{1}_{[1,+\infty)}(t), \tag{1}$$

for $\beta \in (0, 1/2)$ and $t \geq 0$, so that $S(t)$ reduces to

$$S_{\beta}(t) = \sum_{j=1}^{+\infty} \log^{\beta}(t/T_j)R_j(T_j)\mathbb{1}_{t \geq T_j} = \sum_{j=1}^{+\infty}(\log t - \log T_j)_{+}^{\beta} R_j(T_j),$$

where $x_{+} := x\mathbb{1}_{x \geq 0}$.

- $\log^{\beta}(x)$ is slowly varying, for any $\beta \in \mathbb{R}$
  $\Longrightarrow$ the lingering effect of the noises, represented by $g(\cdot)$, is slowly varying over time
- we will assume that the conditional variance of the noises, given the Poisson times, i.e. $K_2(t)$ is asymptotically constant, when the latter tend to infinity

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Recall: $L(\cdot)$ be a positive slowly varying function at $+\infty$ if $\lim_{x \to +\infty} L(ax)/L(x) = 1$, for any $a \in \mathbb{R}^{+}$.

Convergence of f.d.d.'s

Theorem.

Let $K_{\alpha,\lambda} := K\lambda\Gamma(\alpha)$. *If* $K_2(\cdot)$ *and* $K_4(\cdot)$ *are positive, bounded functions such that* $\lim_{t\to\infty} K_2(t) = K > 0$ *and* $K_4(u)/K_2(u) \le \kappa$, *for any* $u \in \mathbb{R}$ *and* $\kappa > 0$, *then the scaled shot-noise process defined as* $\hat{S}_{\alpha,c} := \left\{\hat{S}_{\alpha,c}(t)\right\}_{t\ge 0}$, *where*

$$\hat{S}_{\alpha,c}(t) := \frac{S(ct)}{\sqrt{cK_{\alpha,\lambda}}} = \frac{1}{\sqrt{cK_{\alpha,\lambda}}} \sum_{j=1}^{+\infty} (\log(ct) - \log(T_j))_+^{\frac{\alpha-1}{2}} R_j(T_j),$$

$t \ge 0$, $\alpha \in (1,2)$, *weakly converges, in the sense of finite-dimensional distributions, to the H-fBm* B_α^H, *as* $c \to +\infty$.

Preliminary result

Lemma.

The incremental variance of the limiting process B_α^H, i.e.

$$\begin{aligned} \rho_H(s,t) &:= \mathbb{E}\left[B_\alpha^H(t) - B_\alpha^H(s)\right]^2 \\ &= t + s - C_\alpha s \Psi\left(\frac{1-\alpha}{2}, 1-\alpha; \log\left(\frac{t}{s}\right)\right) \end{aligned}$$

for $0 \leq s < t$, satisfies the following properties, for $\alpha \in (1,2)$:

1. $\rho(0,0) = 0$ *and* $\rho(s,t) \geq 0$, *for any* $0 \leq s \leq t$
2. *"super-additive":* $\rho(r,s) + \rho(s,t) \leq \rho(r,t)$, *for any* $0 \leq r \leq s \leq t$
3. *"non-decreasing":* $\rho(s,t) \leq \rho(s,t')$, *for any* $0 \leq s \leq t \leq t'$
4. $\lim_{h\to 0} \rho(t, t+h) = 0$
5. $\rho(0,t)$ *is continuous at* t.

Weak convergence

Theorem.

Under the previous assumptions, the scaled shot-noise process $\left\{\hat{S}_{\alpha,c}(t)\right\}_{t\geq 0}$ *weakly converges to the H-fBm, i.e.*

$$\hat{S}_{\alpha,c} \Rightarrow B_\alpha^H, \qquad \textit{in } (\mathbb{D}[0,T], J_1).$$

Sketch of the proof:

- check that $\hat{S}_{\alpha,c}$ has sample paths in $\mathbb{D}$
- $\lim_{\delta\to 0} P(|B_\alpha^H(T) - B_\alpha^H(T-\delta)| \geq \epsilon) = 0$ holds by stochastic continuity of B_α^H
- check that for $C > 0$ such that, for any $0 \leq r \leq s \leq t \leq T$, with $t - r < 2\delta$, for some $\delta > 0$, $\beta \geq 0$,

$$P(|\hat{S}_{a,c}(r) - \hat{S}_{a,c}(s)| \wedge |\hat{S}_{a,c}(s) - \hat{S}_{a,c}(t)| \geq \epsilon) \leq \frac{C}{\epsilon^{4\beta}} \rho_H^2(r,t],$$

- $\rho_H(0,t]$ is continuous in t.

Some references

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Thank you for your attention!