Anomalous subdiffusion and time-fractional differential equations V

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Coupled space and time movements

In many circumstances, the spatial motions and holding times may depend on each other. How to model and address it?

- (Z_n, T_n): coupled random walks on $\mathbb{R}^d \times [0, \infty)$.
- CTRW: Z_{N_t}, where $N(t) = \max\{n : T_n \le t\}$.

Becker-Kern, Meerschaert and Scheffler: Limit theorems for coupled continuous time random walks. *Ann. Probab.* **2004**.

Assumption: $\exists A_n \in GL(\mathbb{R}^d)$ and $b_n > 0$ so that $(A_n Z_n, b_n T_n) \Rightarrow (X_1, S_1)$ (operator stable distribution), and assume X_t and S_t do not jump at the same time. [In particular, S_t is a β-stable subordinator.]

Then after suitable scaling, $\{Z_{N_t}; t \ge 0\} \Rightarrow \{X_{L_t}; t \ge 0\}$, where $L_t = \inf\{r : S_r > t\}$. Moreover, X_{L_t} has a density $h(t, x)$ satisfying $\mathcal{L}_{S,X} h(t, x) = \delta_{\{x\}} \frac{t^{-\beta}}{\Gamma(1-\beta)}$ in Fourier multiplier sense. [Fokker-Planck equation]

Questions

- (Z_n, T_n): from coupled random walks to Markov chains
- (X_t, S_t): operator-stable Lévy processes to Markov processes

How to solve

$$\partial_t^\beta u = \mathcal{L}_t u \quad \text{with } u(0,x) = f(x)?$$

More generally,

$$\partial_t^w u(t,x) = \mathcal{L}_t u(t,x) + h(t,x) \quad \text{with } u(0,x) = f(x)?$$

- How about $(\partial_t^w, \mathcal{L}_t)$ are further coupled?

Coupled time-fractional equation

Let $w \in L^1_{loc}([0,\infty))$ be an unbounded left-continuous decreasing function with $\lim_{x\to\infty} w(x) = 0$.

$$w \longleftrightarrow \text{Lévy measure } \nu \longleftrightarrow \text{subordinator } S_t$$

We first consider

$$\begin{aligned} \partial_t^w u &= \mathcal{L}_t u(t,x) + h(t,x) \\ &= \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x)\,\partial^2_{x_i x_j} u(t,x) + \sum_{i=1}^d b_i(t,x)\,\partial_{x_i} u(t,x) + h(t,x) \end{aligned}$$

with $u(0,x) = f(x)$.

Another version of time fractional derivative

Let S be a driftless subordinator with infinite Lévy measure ν on $(0,\infty)$. For a function φ defined on $[0,\infty)$, we define $\varphi(t)=\varphi(0)$ for $t<0$. When φ is Lipschitz on $[0,\infty)$, we knew

$$\partial_t^w \varphi(t) = -\int_0^t (\varphi(t-s)-\varphi(t))\nu(ds) = -(\mathcal{A}^*\varphi)(t),$$

where $\mathcal{A}$ is the generator of S. In fact, this holds more generally.

Lemma

Suppose $\int_0^t |\varphi(t)-\varphi(t-s)|\nu(ds)<\infty$ *for* $t\in(0,T]$ *and* $\int_0^T |\mathcal{A}^*\varphi(t)|dt<\infty$. *Then for every* $t\in(0,T]$, $\int_0^t \mathcal{A}^*\varphi(s)ds = I_t^w\varphi$ *and so* $\partial_t^w\varphi(t)=-\mathcal{A}^*\varphi(t)$ *for a.e.* $t\in(0,T]$.

Note $\mathcal{A}^*\varphi(t)$ is the generator of $Z_t=Z_0-S_t$.

Formulation

Thus we can write $\partial_t^w u = \mathcal{L}_t u + h(t,x)$ as
$(\mathcal{A}^* + \mathcal{L}_t)u = -h(t,x)$.

Question. Is there a Markov process associated with
$\mathcal{L}_0 := \mathcal{A}^* + \mathcal{L}_t$ on $\mathbb{R} \times \mathbb{R}^d$?

From now on, suppose $\sigma(t,x) = a(t,x)^{1/2}$ and $b(t,x)$ are bounded and Lipschitz.

Theorem (C.-Choi '26+)

For any given initial value (Z_0, X_0), the following SDE has a unique strong solution

$$\begin{cases} dZ_t = -dS_t, \\ dX_t = \sigma(Z_s, X_s)dB_s + b(Z_s, X_s)ds. \end{cases}$$

The solution (Z, X) forms a strong Markov process having infinitesimal generator $\mathcal{L}_0$.

Regularity

Recall $Z_t = Z_0 - S_t$. Let $L_t := \inf\{r : S_r = Z_0 - Z_r > t\}$.

Lemma (C.-Choi '26+)

Suppose $f \in C_b^{2,\alpha}(\mathbb{R}^d)$ *for some* $0 < \alpha < 1$. *Let*

$$u(t,x) = \mathbb{E}_{(t,x)}[f(X_{L_t})] \quad \text{for } t > 0.$$

Then $u(t,\cdot) \in C_b^2(\mathbb{R}^d)$ *for each* $t > 0$ *and* $u(\cdot,x) \in C_b(\mathbb{R})$ *for each* $x \in \mathbb{R}^d$. *Moreover,* $\exists c > 0$ *such that for any* $0 < t_1 < t_2$ *and* $x \in \mathbb{R}^d$,

$$|u(t_2,x) - u(t_1,x)| \le c\left(U((t_1,t_2]) + t_2 - t_1\right),$$

where $U(A) := \int_0^\infty \mathbb{P}(S_r \in A)\,dr$.

$$\mathbb{E}L_t = \mathbb{E}\int_0^\infty \mathbf{1}_{\{S_r \le t\}}\,dr = U((0,t]).$$

Classical solution

We say $u = u(t,x)$ is a classical solution to $\partial_t^w u = \mathcal{L}_t u + h(t,x)$ if $u(\cdot,t) \in C_b^2$, and $\exists c > 0$ so that for any $0 \le t_1 < t_2$ and $x \in \mathbb{R}^d$,

$$|u(t_2,x) - u(t_1,x)| \le c\left(U((t_1,t_2]) + t_2 - t_1\right), \qquad (1)$$

and that $\partial_t^w u = \mathcal{L}_t u + h(t,x)$ pointwise.

Recall [C, '24] for every $t > 0$,

$$\int_0^t U((t-s,t])\nu(ds) \le \int_0^t w(t-s)U(ds) = 1.$$

So under (1), $\partial_t^w u = -\int_0^\infty (u(t-s,x) - u(t,x))\nu(ds)$ is well defined for a.e. $t > 0$.

Time-fractional equation

Theorem (C.-Choi '26+)

Suppose that $f \in C_b^{2,\alpha}(\mathbb{R}^d)$ *for some* $0 < \alpha < 1$. *The function*

$$u(t,x) := \mathbb{E}_{(t,x)}[f(X_{L_t})], \quad t \geq 0, x \in \mathbb{R}^d,$$

is the unique classical solution to the equation

$$\begin{cases} \partial_t^w u(t,x) = \mathcal{L}_t u(t,x) & \text{for each } (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = f(x). \end{cases}$$

Key proposition

Let $L_t := \inf\{r : S_t := Z_0 - Z_r > t\}$.

Proposition (C.-Choi '26+)

For any $t_0, t > 0$, $x \in \mathbb{R}^d$ *and* $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}_{(t_0,x)}(X_{L_t} \in A) = \int_0^\infty \mathbb{E}_{(t_0,x)}\left[\mathbf{1}_{\{X_s \in A\}} w(t - S_s) \mathbf{1}_{\{t > S_s\}}\right] ds.$$

Theorem (C.-Choi '26+)

Suppose $h(t,x) \in Lip(\mathbb{R}_+ \times \mathbb{R}^d)$ *be such that* $h(\cdot,t) \in C_b^{2,\alpha}(\mathbb{R}^d)$ *for some* $0 < \alpha < 1$. *Define*

$$u(t,x) = \mathbb{E}_{(t,x)}\left[\int_0^{L_t} h(Z_r, X_r)dr\right], \quad t \geq 0, x \in \mathbb{R}^d.$$

1. *Then on each bounded time interval,* $u(t,\cdot)$ *is* C_b^2 *on* $\mathbb{R}^d$*, and* $u(\cdot,x) \in C_b([0,T])$*. Moreover,* $\exists c > 0$ *s.t. for any* $t_2 > t_1 \geq 0$ *and* $x \in \mathbb{R}^d$*,*

$$|u(t_2,x) - u(t_1,x)| \leq c\left(U((t_1,t_2]) + t_2 - t_1\right).$$

2. u *is the unique classical solution to* $\partial_t^w u = \mathcal{L}_t u + h(t,x)$ *with* $u(0,x) = 0$.

Fractional Poisson equation

Corollary (C.-Choi '26+)

Suppose $f \in C_b^{2,\alpha}(\mathbb{R}^d)$ *and* $h \in Lip_b(\mathbb{R}_+ \times \mathbb{R}^d)$ *be such that* $h(t,\cdot) \in C_b^{2,\alpha}(\mathbb{R}^d)$ *for some* $0 < \alpha < 1$. *The equation*

$$\begin{cases} \partial_t^w u(t,x) = \mathcal{L}_t u(t,x) + h(t,x) \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = f(x) \end{cases}$$

has a unique classical solution. Moreover,

$$u(t,x) := \mathbb{E}_{(t,x)} \left[f(X_{L_t}) + \int_0^{L_t} h(Z_t, X_r) dr \right], \quad t \geq 0, x \in \mathbb{R}^d.$$

When the coupled process is Lévy

Let (Z, X) be the Lévy process with generator

$$\mathcal{L}u(t,x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i}(t,x)$$
$$+ \int_{(0,\infty)\times\mathbb{R}^d} \left[u(t-s, x+y) - u(t,x) - \nabla_x u(t,x)\cdot y\mathbf{1}_{\{|y|\le 1\}}\right] J(ds,dy),$$

where $\int_{(0,\infty)\times\mathbb{R}^d} \left((s+|y|^2)\wedge 1\right) J(ds,dy) < \infty$.

Marginal Lévy measure $\nu([s,\infty)) := J([s,\infty)\times\mathbb{R}^d) < \infty$ satisfying $\int_{(0,\infty)}(s\wedge 1)\nu(ds) < \infty$.

- $S_t := Z_0 - Z_t$ is a subordinator with Lévy measure ν.

Let $L_t := \inf\{r > 0 : S_r = Z_0 - Z_r > t\}$.

Coupled time fractional equation

Theorem (C.-Choi '26+)

For $f \in C_b^2(\mathbb{R}^d)$,

$$u(t,x) := \mathbb{E}_{(t,x)}[f(X_{L_t})]\mathbf{1}_{\{t>0\}} + f(x)\mathbf{1}_{\{t\le 0\}}$$

is in $C_b^{2,0}(\mathbb{R}^d \times \mathbb{R})$. Moreover, $\mathcal{L}u(t,x)$ exists pointwise and u is the unique classical solution to

$$\mathcal{L}u(t,x) = \int_{(t,\infty)\times\mathbb{R}^d} (f(x+y) - f(x))\, J(ds, dy)$$

for each $t > 0$ and $x \in \mathbb{R}^d$.

General case

Consider an integro-differential operator of the form

$$\begin{aligned}\mathcal{L}\varphi(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\,\partial^2_{x_i x_j}\varphi(t,x) + \sum_{i=1}^{d} b_i(t,x)\,\partial_{x_i}\varphi(t,x) \\ &\quad + \int_{(0,\infty)\times\mathbb{R}^d\setminus\{(0,0)\}} \Big(\varphi(x+y,t-s)-\varphi(t,x) \\ &\qquad -\nabla_x\varphi(t,x)\cdot y\mathbf{1}_{\{|y|\le 1\}}\Big)\big)J(t,x;ds,dy),\end{aligned}$$

where $\int_{(0,\infty)\times\mathbb{R}^d}(1\wedge(s+|y|^2))J(t,x;ds,dy)<\infty$.

Suppose the martingale problem for $(\mathcal{L}, C_b^{1,2}(\mathbb{R}\times\mathbb{R}^d))$ is well posed. Denote the process by (Z,X).

When $Z_0 - Z$ is a subordinator

Proposition (C.-Choi '26+)

For a strong Markov process (Z, X) associated with $\mathcal{L}$, the process $Z_0 - Z$ is a driftless subordinator with Lévy measure ν if for every $(t, x) \in \mathbb{R}^d \times \mathbb{R}$,

$$J(t, x; [s, \infty), \mathbb{R}^d) = \nu([s, \infty)) \quad \text{for every } s > 0. \tag{2}$$

Conversely, if for each $s > 0$, the function $J(t, x; [s, \infty), \mathbb{R}^d)$ is continuous in (t, x), then (2) is the necessary condition for the process $Z_0 - Z$ to be a driftless subordinator.

In the rest of the talk, we assume (2) holds and ν is an infinite measure.

Key proposition

Let $L_t := \inf\{r : S_t := Z_0 - Z_r > t\}$ and $w(x) = \nu([x, \infty))$.

Proposition (C.-Choi '26+)

For any $t_0, t > 0$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}_{(t_0,x)}(X_{L_t} \in A) = \int_0^\infty \mathbb{E}_{(t_0,x)}\left[\mathbf{1}_{\{X_s \in A\}} w(t - S_s) \mathbf{1}_{\{t > S_s\}}\right] ds.$$

Time fractional equation

Theorem (C.-Choi '26+)

Suppose that $\mathcal{L}g$ is continuous for each $g \in C_b^2(\mathbb{R} \times \mathbb{R}^d)$. For $f \in C_b^2(\mathbb{R}^d)$, define

$$u(t,x) = \mathbb{E}_{(t,x)}[f(X_{L_t})]\mathbf{1}_{\{t>0\}} + f(x)\mathbf{1}_{\{t\le 0\}}.$$

Then the limit

$$\overline{\mathcal{L}}u(t,x) := \lim_{h\downarrow 0} \frac{\mathbb{E}_{(t,x)}[u(Z_h, X_h)] - u(t,x)}{h}$$

exists for each $t \in (0,\infty)$ and $x \in \mathbb{R}^d$. Moreover,

$$\overline{\mathcal{L}}u(t,x) = \int_{\mathbb{R}^d \times [t,\infty)} (f(x+y) - f(x))\, J(t,x;ds,dy)$$

for each $t \in (0,\infty)$ and $x \in \mathbb{R}^d$.

Feynman-Kac transform for sub-diffusions

Theorem (C.-Deng-Peng, SIAM J. Math. Anal. '21)

Suppose $\kappa(x)$ is a bounded function on $\mathcal{X}$. Let $f \in \mathbb{B}$ and define

$$u(t,x) := \mathbb{E}_x\left[e^{-\int_0^t \kappa(X_{L_s})ds} f(X_{L_t})\right].$$

Then $u(t,x)$ is the unique mild solution in $(\mathbb{B}, \|\cdot\|)$ to Kolmogorov backward equation

$$\partial_t^{w,\kappa(x)} u(t,x) = \mathcal{L}u(t,x) - \kappa(x) I_t^{w,\kappa(x)} u(t,x),$$

with $u(0,x) = f(x)$. Here

$$\begin{aligned} I_t^{w,\kappa}\varphi &:= \int_0^t \varphi(t-s)e^{-\kappa s} w(s)ds, \\ \partial_t^{w,\kappa}\varphi &:= \frac{d}{dt} I_t^{w,\kappa}(\varphi - \varphi(0)). \end{aligned}$$

Dual Feynman-Kac transform

Suppose that X is in weak duality with $\widehat{X}$ w.r.t. m on $\mathcal{X}$.

Theorem (Zhang-C. SPA '22)

Suppose that $\kappa(x)$ is a bounded function on $\mathcal{X}$. For $f, g \in L^2(\mathcal{X}; m)$,

$$\mathbb{E}_{gm}\left[e^{-\int_0^t \kappa(X_{L_s})ds} f(X_{L_t})\right] = \int_{\mathcal{X}} f(y) v_g(t,y) m(dy),$$

where

$$v_g(t,y) = \widehat{\mathbb{E}}_y\left[e^{-\int_0^t \kappa(\widehat{X}_{L_t - L_r})dr} g(\widehat{X}_{L_t})\right].$$

Thus under the measure $\mathbb{P}_{gm}$, the "distribution" of $Y_t = X_{L_t}$ under the Feynman-Kac transform $e^{-\int_0^t \kappa(X_{L_s})ds}$ is $v_g(t,y)m(dy)$.

Fokker-Planck equation

Theorem (Zhang-C. SPA '22)

Suppose $\kappa(x)$ is bounded on $\mathcal{X}$. Define for $g \in L^2(\mathcal{X}; m)$

$$v_g(t,x) := \widehat{\mathbb{E}}_x \left[e^{-\int_0^t \kappa(\widehat{X}_{L_t - L_s}) ds} g(\widehat{X}_{L_t}) \right].$$

Then $v(t,x) := v_g(t,x)$ is the unique mild solution in $L^2(\mathcal{X}; m)$ to the non-local Fokker-Planck equation

$$\frac{\partial}{\partial t} v(t,y) = \mathcal{L}^* \left(\frac{\partial}{\partial t} + \kappa(y) \right) I_t^{*,w,\kappa(y)} v(\cdot, y) - \kappa(y) v(t,y)$$

with $v(0,x) = g(x)$. Here

$$I_t^{*,w,\kappa} \varphi := \int_0^t \varphi(t-s) e^{-\kappa s} U(ds).$$

Black-Scholes model for bear markets

During bear market, financial activities are less frequent. We propose to model stock price by sub-diffusions:

$$d\mathcal{S}_t = \mathcal{S}_t \left(r_t dt + \overline{\mu}_t dL_{(t-a)^+} + \sigma_t dB_{L_{(t-a)^+}} \right),$$

where $a \geq 0$ is the initial wake up time for the market, $r_t \geq 0$ is the interest rate.

How to price European call option $(\mathcal{S}_T - K)^+$ with maturity T and strike price K?

S. Zhang and C. Fractional Black-Scholes model and Girsanov transform for sub-diffusions. Preprint 2025.

Stochastic optimal control

Joint with Shuaiqi Zhang

1. Stochastic maximum principle for sub-diffusions and its applications. *SIAM J. Control Optim.* **62** (2024), 953-981.
2. Fully coupled forward-backward stochastic differential equations driven by sub-diffusions. *J. Differential Equations* **405** (2024), 337-358
3. Stochastic maximum principle for fully coupled forward-backward stochastic differential equations driven by sub-diffusion. *SIAM J. Control Optim.* **62** (2024), 2433-2455.

Thank you!