Anomalous subdiffusion and time-fractional differential equations II

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Anomalous Transport and Anomalous Diffusion
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1 Introduction

Time-fractional diffusion

Heat equation $\partial_t u = \Delta u$ describes heat propagation in homogeneous medium.

Time-fractional diffusion equation

$$\partial_t^\beta u = \Delta u$$

with $0 < \beta < 1$ has been widely used to model the anomalous diffusions exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena.

Caputo derivative:

$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \left(f(s) - f(0) \right) ds.$$

- **Probabilistic approach**

Suppose that $\mathcal{L}$ is the generator of a strong Markov process X.

Let $\{S_\beta(t)\}_{t\geq 0}$: β-stable subordinator $\mathbf{E}[\exp(-\lambda S_\beta(t))] = e^{-t\lambda^\beta}$ (indep. of X).

Theorem 1.1. (Baeumer-Meerschaert '01; Meerschaert-Scheffler '04):

$u(t,x) = \mathbb{E}_x[f(X_{L_t})]$ *solves*

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \mathcal{L}_x u(t,x), \quad u(0,x) = f(x).$$

Here $L_t = \inf\{r \geq 0 : S_\beta(r) > t\}$*: inverse of* S_β *independent of* X*.*

Tools used: Mittag-Leffler functions, and the self-similarity of the β-subordinator,

$$\{S_{\lambda t}; t \geq 0\} = \{\lambda^{1/\beta} S_t; t \geq 0\} \quad \text{in distribution.}$$

C.f. Subordinate Markov Process (for comparison)

$S = \{S_t, \mathbb{P}; t \geq 0\}$: subordinator independent of X with Laplace exponent ϕ:

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}.$$

Then $\exists 1 \kappa \geq 0$ and $\exists 1 \nu$ meas. on $(0, \infty)$, satisfying $\int_0^\infty (1 \wedge x)\, \nu(dx) < \infty$ s.t.

$$\phi(\lambda) = \kappa\lambda + \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx).$$

X_{S_t}: Markov process, called subordinate Markov process.

When X is symmetric, the generator of X_{S_t} is $\mathcal{L}_\phi := -\phi(-\mathcal{L})$.

Hence $u(t,x) := \mathbb{E}_x[f(X_{S_t})]$ solves $\frac{\partial u}{\partial t} = \mathcal{L}_\phi u$ with $u(0,x) = f(x)$.

Example: β-stable subordinator $(0 < \beta < 1)$

Let $\{S_t; t \geq 0\}$ be a β-subordinator, then $\phi(\lambda) = \lambda^\beta$, the generator $-(-\mathcal{L})^\beta$.

When X is Brownian motion on $\mathbb{R}^d$, X_{S_t} is a rot. sym. (2β)-stable process on $\mathbb{R}^d$, whose generator is $-(-\Delta)^\beta =: \Delta^\beta$. It can also be expressed as

$$\begin{aligned}
\Delta^\beta f(x) &= \lim_{\varepsilon \to 0} \int_{\{|y-x|>\varepsilon\}} (f(y) - f(x)) \frac{c(d,\beta)}{|y-x|^{d+2\beta}} dy \\
&= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x) \cdot z 1_{\{|z|\leq 1\}}) \frac{c(d,\beta)}{|z|^{d+2\beta}} dz.
\end{aligned}$$

Why do we care?

Motivation:

- Question from industry.

  The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.).

  Predict the progress of soil contamination.

- The third and fourth slides: Nature (2006, Jan.)

  The scaling laws of human travel

  by D. Brockmann, L. Hufnagel and T. Geisel.

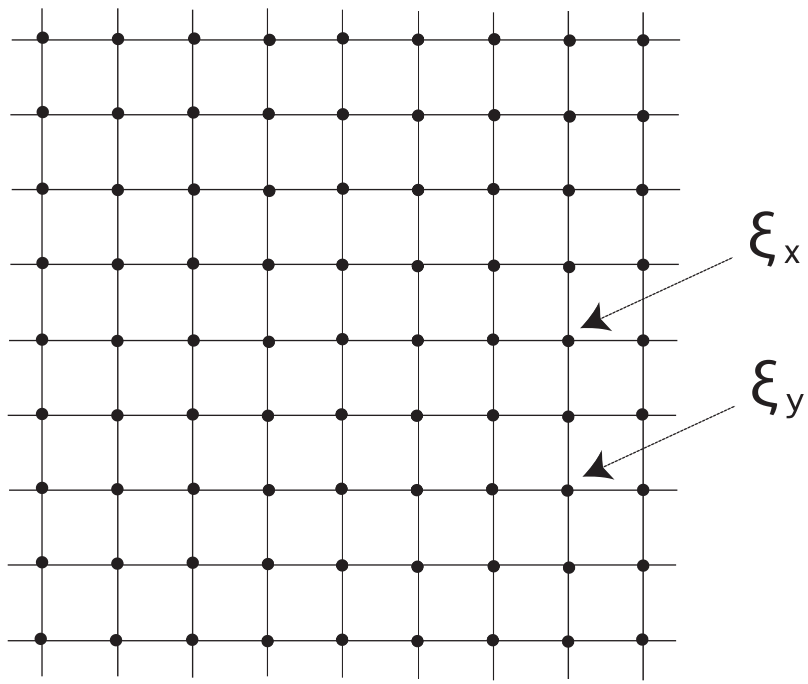
2 Trap models and their scaling limits

Example 1: Symmetric Bouchaud's trap model (BTM)

A trapping landscape $\{\xi_x\}_{x\in\mathbb{Z}^d}$: pos. i.i.d. on $\mathbf{P}$.

Random Hopping Times dynamics $(X_t^{\xi})_{t\geq 0}$: cont. time MC, trans. prob. $1/(2d)$.

Jump rate at x being $1/\xi_x$. $\exists\beta\in(0,1)$ s.t. $\mathbf{P}\left(\xi_x > u\right) = u^{-\beta}$, $\forall u \geq 1$.



High dimensional case

Theorem 2.1. $d \geq 2$ (Ben Arous-Černý '07) *For* $d \geq 3$,

$$\varepsilon X_{ct/\varepsilon^{2/\beta}} \xrightarrow{d} \mathbf{FK}_{d,\beta}(t) := BM_d(S_\beta^{-1}(t)) \quad \mathbf{P}\text{-}a.s. \ on\ D([0,\infty),\mathbb{R}^d),$$

where $\{S_\beta(t)\}_{t\geq 0}$*:* β*-stable subord. (indep. of* $\{BM_d(t)\}$*).*

For $d = 2$*, same result by replacing* $\varepsilon^{-2/\beta}$ *to* $\varepsilon^{-2/\beta}(\log \varepsilon^{-1})^{1-1/\beta}$.

$\mathbf{FK}_{d,\beta}$: Fractional-kinetics process — It is no longer a Markov process!

Density of its fixed time distribution $p(t,x)$ satisfies the fractional-kinetics equation:

$$\frac{\partial^\beta}{\partial t^\beta} p(t,x) = \frac{1}{2}\Delta p(t,x).$$

- **Limit is very different when $d = 1$. (Details on Friday.)**

Theorem 2.2. $d = 1$ (Fontes-Isopi-Newman '02)

$$\varepsilon X_{c_* t/\varepsilon^{1+1/\beta}} \xrightarrow{d} Z(t) \quad \textit{under} \quad \mathbf{P} \times P_0^{\xi}.$$

Definition 2.3. *FIN diffusion is defined by* $Z(s) = BM(\phi_\rho^{-1}(s)),\ s \in [0, \infty)$,

where $\phi_\rho(t) := \int_{\mathbb{R}} \ell(t, y)\rho(dy)$ *where* $\ell(\cdot, \cdot)$ *is the local time of BM,*

$\rho := \sum_i \nu_i \delta_{x_i}$ *where* $(x_i, \nu_i) \in \mathbb{R} \times \mathbb{R}_+$ *is distributed by PPP with intensity* $dx\beta\nu^{-1-\beta}d\nu$.

— Atoms of ρ are dense in $\mathbb{R}$ a.e.

$$\frac{\partial}{\partial t} p(t, x) = \frac{\partial^2}{\partial \rho \partial x} p(t, x)$$

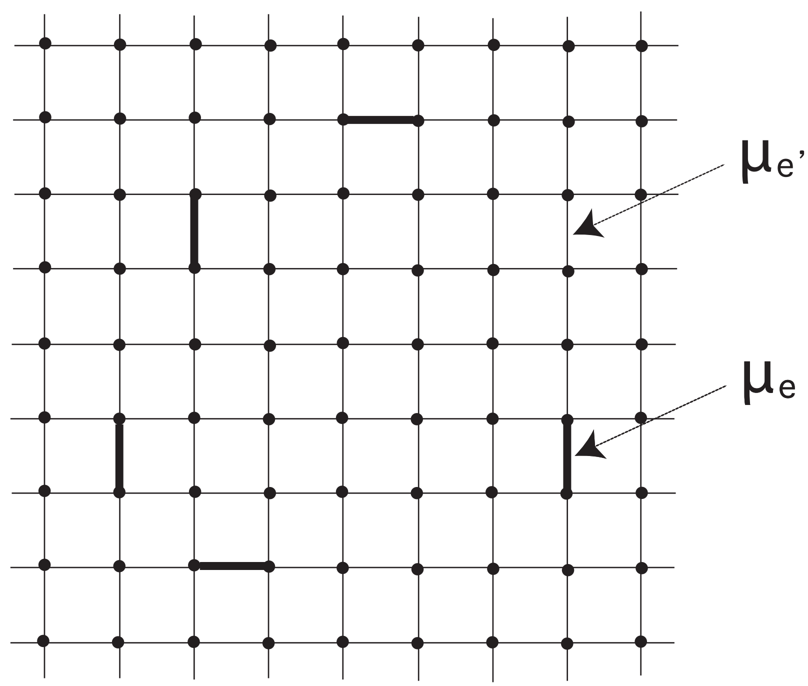
ρ plays the role of speed measure.

Example 2: Random conductance model (RCM)

$\{\mu_e\}$: random conductance, *i.i.d.* on each edge e of $\mathbb{Z}^d$ s.t. $\exists \beta \in (0,1)$

$$\mathbb{P}(\mu_e \geq c_1) = 1, \quad \mathbb{P}(\mu_e \geq u) = c_2 u^{-\beta}(1+o(1)) \text{ as } u \to \infty. \tag{2.1}$$

(Note that $\mathbb{E}\mu_e = \infty$.) $\{X_t\}_{t\geq 0}$: cont. time MC on $\mathbb{Z}^d$ (holding time exp(1)).



Theorem 2.1. (Barlow-Černý '11) *For* $d \geq 3$,

$$\varepsilon X_{ct/\varepsilon^{2/\beta}} \xrightarrow{d} \mathbf{FK}_{d,\beta}(t) := BM_d(S_\beta^{-1}(t)) \quad \mathbf{P}\text{-}a.s. \ on \ D([0,\infty),\mathbb{R}^d),$$

where $\{S_\beta(t)\}_{t\geq 0}$*:* *β-stable subordinator* $\mathbf{E}[\exp(-\lambda S_\beta(t))] = e^{-t\lambda^\beta}$ (indep. of $\{BM_d(t)\}$).

For $d = 2$ *(Černý '11), same result by replacing* $\varepsilon^{-2/\beta}$ *to* $\varepsilon^{-2/\beta}(\log \varepsilon^{-1})^{1-1/\beta}$.

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Rem 1: The result contains the scaling limit of non-symmetric Bouchaud's trap model (BTM), namely $a \in [0,1]$, $\mu_{xy} = \xi_x^a \xi_y^a$ if $x \sim y$ and measure $\mu_x = \xi_x$.

Rem 2: As before, for $d = 1$: $Z(s) = BM(\phi_\rho^{-1}(s))$, FIN diffusion.

$\phi_\rho(t) := \int_{\mathbb{R}} \ell(t,y)\rho(dy)$, $\rho := \sum_i \nu_i \delta_{x_i}$: PPP with intensity $dx\beta\nu^{-1-\beta}d\nu$.

To summarize, **Theorem** Let $\{X_t\}_{t\geq 0}$ be the CSRW of RCM that satisfies (2.1).

(i) (Barlow-Černý '10) Let $d \geq 3$, $\beta \in (0,1)$ in (2.1) and let $X_t^{(\varepsilon)} := \varepsilon X_{t/\varepsilon^{2/\beta}}$. Then

$$X^{(\varepsilon)} \xrightarrow{d} c \cdot \mathbf{FK}_{d,\beta} \quad \text{under } P_\omega^0, \ \mathbb{P}\text{-a.s. on } D([0,\infty), \mathbb{R}^d) \text{ with } J_1\text{-topology.} \tag{2.2}$$

(ii) (Černý '11) Let $d = 2$, $\beta \in (0,1)$ in (2.1) and let $X_t^{(\varepsilon)} := \varepsilon X_{t(\log(1/\varepsilon))^{1-1/\beta}/\varepsilon^{2/\beta}}$.

Then (2.2) holds.

(iii) (Černý '11) Let $d = 1$, $\beta \in (0,1)$ in (2.1) and let $X_t^{(\varepsilon)} := \varepsilon X_{c_* c_\varepsilon t/\varepsilon}$, where $c_* = \mathbb{E}[\mu_e^{-1}]$,

$$c_\varepsilon := \inf\{t \geq 0 : \mathbb{P}(\mu_e > t) \leq \varepsilon\} = \varepsilon^{-1/\beta}(1 + o(1)).$$

$$\text{Then} \qquad X^{(\varepsilon)} \xrightarrow{d} Z(t) \quad \text{under } \mathbb{P} \times P_0^\mu.$$

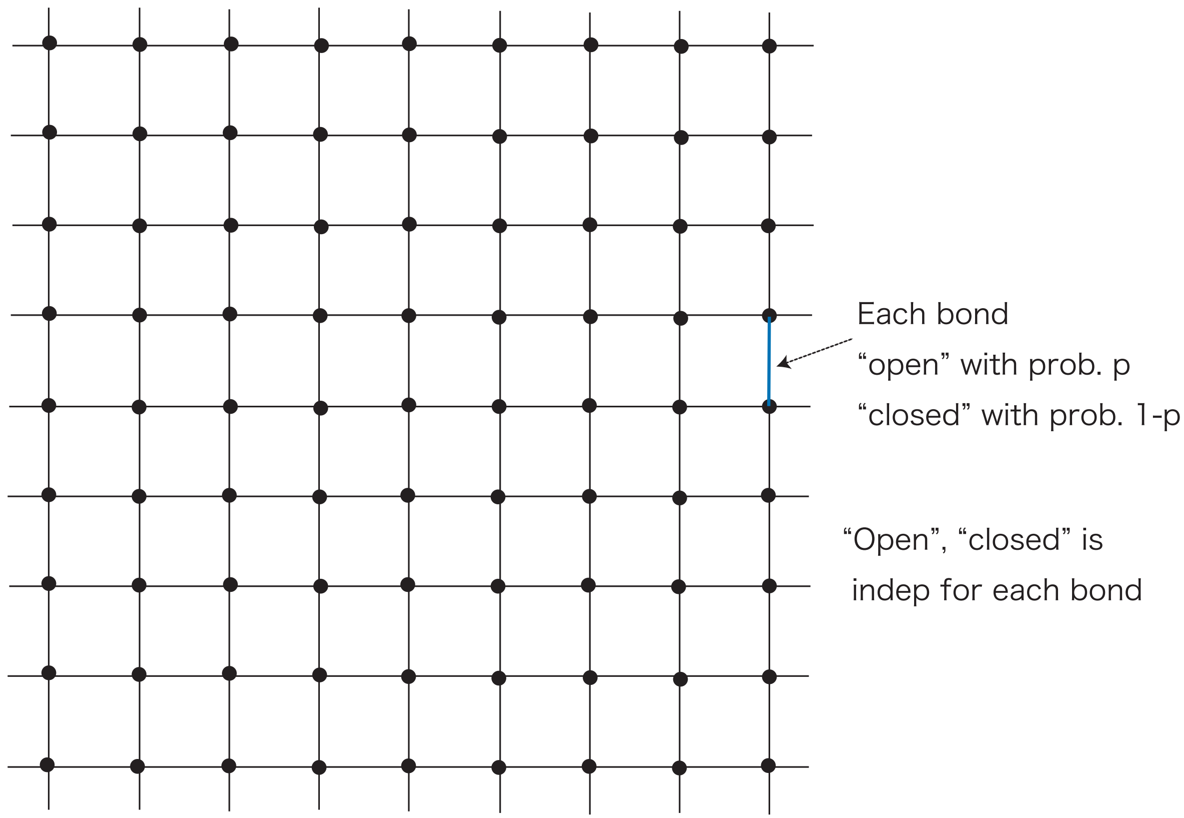
(iv) (Barlow-Zheng '10) Let $d \geq 3$, $\beta = 1$ in (2.1) with $c_1 = c_2 = 1$ and

let $X_t^{(\varepsilon)} := \varepsilon X_{t\log(1/\varepsilon)/\varepsilon^2}$. Then

$$X^{(\varepsilon)} \xrightarrow{d} BM \quad \text{with cov. } \tfrac{1}{2}\sigma_V^2 I \quad \text{under } P_\omega^0, \ \mathbb{P}\text{-a.s. on } D([0,\infty), \mathbb{R}^d).$$

3 Quenched invariance principle for random conductance model (Toward the proof of Theorem 2.1)

- Bond percolation on $\mathbb{Z}^d$ $(d \geq 2)$



$\exists p_c = p_c(d) \in (0,1)$ s.t. $p > p_c \Rightarrow \exists 1$ infinite open cluster $\mathcal{G}(\omega)$ (random media!)

$(\Omega, \mathcal{F}, \mathbb{P})$: prob. space for the randomness of the space, $\quad p_n^\omega(x,y) := \mathbb{P}^x(Y_n = y)/\mu_y$.

$p > p_c$: No anomalous behavior for long time.

(a) [Gaussian heat kernel estimates] (Barlow '04)

$$\frac{c_1}{t^{d/2}} \exp(-c_2 \frac{d(x,y)^2}{t}) \le p_t^\omega(x,y) \le \frac{c_3}{t^{d/2}} \exp(-c_4 \frac{d(x,y)^2}{t}), \tag{3.1}$$

$\mathbb{P}$-a.s. ω for $t \ge d(x,y) \vee \exists U_x$, $x, y \in \mathcal{G}$.

(b) [Quenched invariance principle]

(Sidoravicius-Sznitman '04, Berger-Biskup '07, Mathieu-Piatnitski. '07)

$$n^{-1} Y^\omega_{n^2 t} \to B_{\sigma t} \ \ \mathbb{P}\text{-a.s. } \omega \text{ for some } \ \sigma > 0$$

Cf. "Annealed" invariance principle: known since 80's

• **Random conductance model** (symmetric (reversible) RWRE)

Consider $(\mathbb{Z}^d, E_d)$, $d \geq 2$ where E_d is the set of non-oriented n.n. bonds.

Let the conductance $\{\mu_e : e \in E_d\}$ be i.i.d. (more generally stat. ergo.) on $(\Omega, \mathcal{F}, \mathbb{P})$.

Two natural MCs Trans. prob. $P(x, y) = \mu_{xy}/\mu_x$.

1. Constant speed random walk (CSRW): holding time is $exp(1)$ for each point
2. Variable speed random walk (VSRW): holding time at x is exp. distri. with mean μ_x^{-1}

The corresponding discrete Laplace operators are

$$\mathcal{L}_C f(x) = \frac{1}{\mu_x} \sum_y (f(y) - f(x))\mu_{xy}, \quad \mathcal{L}_V f(x) = \sum_y (f(y) - f(x))\mu_{xy}.$$

Let ν be s.t. $\nu(x) = 1$, $\forall x \in \mathbb{Z}^d$. Then, for each finite supported f, g,

$$\mathcal{E}(f, g) = -(\mathcal{L}_V f, g)_\nu = -(\mathcal{L}_C f, g)_\mu.$$

RW on supercrit. perco. is a special case ($\mathbb{P}(\mu_e = 1) = p$, $\mathbb{P}(\mu_e = 0) = 1 - p$)

Assume $\mathbb{P}(\mu_e > 0) > p_c(\mathbb{Z}^d)$. Then $\exists 1 \mathcal{C}$ infinite cluster. We consider $\mathbb{P}(\cdot | 0 \in \mathcal{C})$.

Let $(\{Y_t\}_{t\geq 0}, \{P_\omega^x\}_{x\in\mathbb{Z}^d})$ be either the CSRW or VSRW and define

$$q_t^\omega(x,y) = P_\omega^x(Y_t = y)/\theta_y$$

be the heat kernel of $\{Y_t\}_{t\geq 0}$ where θ is either ν or μ.

(Q1) Heat kernel estimates? (Q2) Invariance principle?

Let $T > 0$, F: bdd. cont. on $D([0,T], \mathbb{R}^d)$. Set $\Psi_\varepsilon := E_\omega^0 F(\varepsilon Y_{\cdot/\varepsilon^2})$, $\Psi_0 := E_{BM} F(\sigma W_\cdot)$.

Weak FCLT ("Annealed IP") $\Psi_\varepsilon \to \Psi_0$ in $\mathbb{P}$-prob.

QFCLT (Quenched IP) $\Psi_\varepsilon \to \Psi_0$ in $\mathbb{P}$-a.s.

Note: Our interest is QFCLT. When $\mathbb{E}\mu_e < \infty$, weak FCLT was obtained in 80's

(Kipnis-Varadhan '86, De Masi-Ferrari-Goldstein-Wick '89 ($\sigma > 0$))

For (Q1) Heat kernel estimates:

Theorem 3.1. (Barlow-Deuschel '10) *(3.1) holds for VSRW if* $\mathbb{P}(1 \leq \mu_e) = 1$.

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Theorem 3.1. (Barlow-Deuschel '10) *(3.1) holds for VSRW if* $\mathbb{P}(1 \le \mu_e) = 1$.

For (Q2): Quenched inv. princ. For $t \ge 0$, let $\{Y_t\}_{t\ge 0}$ be either CSRW or VSRW and

$$Y_t^{(\varepsilon)} := \varepsilon Y_{t/\varepsilon^2}. \tag{3.2}$$

Theorem 3.2. ($\mu_e \le 1$ case: Biskup-Prescott '07, Mathieu '08, $\mu_e \ge 1$ case: Barlow-Deuschel '10, unified: Andres-Barlow-Deuschel-Hambly '11)

(i) Let $\{Y_t\}_{t\ge 0}$ *be the VSRW. Then* $\mathbb{P}$*-a.s.* $Y_t^{(\varepsilon)} \to B_{\sigma_V^2 t}$ *where* $\sigma_V > 0$.

(ii) Let $\{Y_t\}_{t\ge 0}$ *be the CSRW. Then* $\mathbb{P}$*-a.s.* $Y^{(\varepsilon)} \to B_{\sigma_C^2 t}$ *where*

$\sigma_C^2 = \sigma_V^2/(2d\mathbb{E}\mu_e)$ *if* $\mathbb{E}\mu_e < \infty$ *and* $\sigma_C^2 = 0$ *if* $\mathbb{E}\mu_e = \infty$.

- Local CLT also holds.

4 Strategy of the proof of Theorem 3.2

*** Abuse of notation: We sometimes write $\mathbb{P}$ for $\mathbb{P}(\cdot|0 \in \mathcal{C})$. ***

Time changed process (surgery of the cluster)

$\mathcal{C}$: ∞-cluter of $\{e \in E_d : \mu_e > 0\}$. Choose $K > 0$ large enough so that

$$q(K) := \mathbb{P}(0 < \mu_e < K^{-1}) + \mathbb{P}(\mu_e > K) < p - p_c(\mathbb{Z}^d),$$

and let $\mathcal{C}_2$ be the ∞-cluter of $\{e \in E_d : \mu_e \in [K^{-1}, K]\}$ (to be precise, remove also bonds that connect to "bad" bonds).

Let Y be VSRW on $\mathcal{C}$ and define Z as a trace of Y on $\mathcal{C}_2$. Namely, let $A_t := \int_0^t 1_{\{Y_s \in \mathcal{C}_2\}} ds$,

$$Z_t := Y_{A_t^{-1}}, \quad t \geq 0, \qquad \text{where } A_t^{-1} := \inf\{s : A_s > t\}.$$

Note $\{Z_t\}$ is a jump process in general.

We can then "induce" conductance μ'_{xy} which gives trans. prob. of $\{Z_t\}$.

Part 1, Proof of FCLT for $\{Z_t\}$:

This involves classical results and we discuss later except one thing.

Environment seen from the particle (Kipnis-Varadhan)

$\Omega = [K^{-1}, K]^{E_d}$. $\{\mu'_e : e \in E_d\}$ are defined on $(\Omega, \mathbb{P})$ and we write $\mu'_{\{x,y\}}(\omega) = \omega_{x,y}$.

Let $T_x \colon \Omega \to \Omega$ denote the shift by x, namely $(T_z\omega)_{xy} := \omega_{x+z,y+z}$. Define

$$V_t = V_t(\omega) = T_{Z_t}(\omega), \qquad \forall t \in [0, \infty), \tag{4.1}$$

where $\{Z_t\}_{t\geq 0}$ is the MC. Note $\mathbb{P}$ is erg. w.r.t. $V_t(\omega) := T_{Z_t}(\omega)$ (De Masi et al. '89)

Part 2, Proof of FCLT for $\{Y_t\}$:

$\mathcal{H} := \mathcal{C}\backslash\mathcal{C}_2$, $x \in \mathcal{C}$, let $\mathcal{H}(x)$ be the connected component of $\mathcal{C}\backslash\mathcal{C}_2$ containing x.

Lemma 4.1. *For K chosen large enough, the following holds.*

(i) All the con. comp. of $\mathcal{H}$ are finite. Further, $\exists c_1, c_2$ s.t. $\forall x \in \mathbb{L}$,

$$\mathbb{P}\left(x \in \mathcal{C}_1 \textit{ and } \operatorname{diam}\mathcal{H}(x) \geq n\right) \leq c_1 e^{-c_2 n}.$$

(ii) $\mathbb{P}$-a.s., for n large, the vol. of any hole intersecting $[-n, n]^d$ is bdd by $(\log n)^{\exists\alpha}$.

- $\lim_{t\to\infty} A_t/t = \mathbb{P}(0 \in \mathcal{C}_2) =: C_0 > 0, \quad \mathbb{P} \times P_\omega^0$-a.s.

(☺$\mathbb{P}$ is erg. w.r.t. $V_t(\omega) = T_{Z_t}(\omega)$ and $A_t = \int_0^t 1_{\{0\in\mathcal{C}_2(V_s(\omega))\}} ds$)

Since

$$Y_t^{(\varepsilon)} = \varepsilon(Y_{t/\varepsilon^2} - Z_{A_t/\varepsilon^2}) + \varepsilon(Z_{A_t/\varepsilon^2} - Z_{C_0 t/\varepsilon^2}) + Z_{C_0 t}^{(\varepsilon)},$$

Lemma 4.1 and tightness of Y (which can be proved by HK upper bound) imply that the first two terms go to 0. We thus obtain FCLT for $Y_t^{(\varepsilon)}$ with $\sigma_V^2 = C_0 \sigma_Z^2$.

Part 3, Proof of FCLT for CSRW:

Note VSRW and CSRW are time changes of each others. Set

$$\tilde{A}_t := \int_0^t \mu_{Y_s} ds = \int_0^t \mu_0(T_{Y_s}\omega) ds.$$

Then CSRW $\{X_t\}$ can be written as

$$X_t = Y_{\tilde{A}_s^{-1}}, \qquad \text{where } \tilde{A}_t^{-1} := \inf\{s : \tilde{A}_s > t\}.$$

Again by the ergo. thm, $\lim_{t\to\infty} \tilde{A}_t/t = \mathbb{E}\mu_0 = 2d\mathbb{E}\mu_e \quad \mathbb{P} \times P_\omega^0$-a.s.

So $X^{(\varepsilon)}$ converges to $\sigma_C B_t$ where B_t is BM and $\sigma_C^2 = \sigma_V^2/(2d\mathbb{E}\mu_e)$ if $\mathbb{E}\mu_e < \infty$

and $\sigma_C^2 = 0$ if $\mathbb{E}\mu_e = \infty$.

4.1 QIP for $\{Z_t\}$ 1: HK estimates

Lemma 4.2. (Perco. est. (Antal-Pisztora '96)) $\exists c_1, c_2, c_3 > 0$ *s.t.* $\forall x, y \in \mathbb{L}$,

$$\mathbb{P}\left(x, y \in \mathcal{C}_2 \text{ and } d(x,y) \leq c_1|x-y|\right) \leq c_2 e^{-c_3|x-y|},$$

$$\mathbb{P}\left(x, y \in \mathcal{C}_2 \text{ and } d(x,y) \geq c_1^{-1}|x-y|\right) \leq c_2 e^{-c_3|x-y|},$$

where $|\cdot - \cdot|$ *is the Euclidean dist. and* $d(\cdot,\cdot)$ *is the graph dist.*

- (Percolation est.) $\Rightarrow$ (HK estimates (3.1))

In principle, stability of HK est. helps. (Finding "good balls" inside which one has good control of volume and Poincaré exp. Then use the Borel-Cantelli to obtai a.s. results.

4.2 QIP for $\{Z_t\}$ 2: Correctors

Idea behind $\varphi = \varphi_\omega : \mathbb{Z}^d \to \mathbb{R}^d$ be a harmonic map (so $M_t = \varphi(Y_t)$ is a P_ω^0-mart.)

$$\text{Corrector} \quad \chi(x) := (\varphi - I)(x) = \varphi(x) - x.$$

For simplicity, let us consider CLT (instead of FCLT) for Y. We have

$$\frac{Y_t}{t^{1/2}} = \frac{M_t}{t^{1/2}} - \frac{\chi(Y_t)}{t^{1/2}}.$$

Martingale CLT gives $M_t/t^{1/2} \xrightarrow{w}$ Normal distri. So ETP $\chi(Y_t)/t^{1/2} \to 0$.

This can be done once we have

(a) $P_\omega^0(|Y_t| \geq At^{1/2})$ is small ($\Leftarrow$ Can be shown by HK upper bound)

(b) $|\chi(x)|/|x| \to 0$ as $|x| \to \infty$.

Note There maybe many global harm. fu., so we should chose one s.t. (b) holds.

5 Idea of the proof of Theorem 2.1

Theorem 2.1 (Barlow-Černý '11) Let $d \geq 3$, $\beta \in (0,1)$, and $\{X_t\}_{t\geq 0}$ be the Markov chain of RCM that satisfies (2.1). Then

$$\varepsilon X_{t/\varepsilon^{2/\beta}} \xrightarrow{d} c \cdot \mathbf{FK}_{d,\beta} \quad \text{under } P_\omega^0,\ \mathbb{P}\text{-a.s. on } D([0,\infty),\mathbb{R}^d) \text{ with } J_1\text{-topology.}$$

Idea of the proof:

VSRW $\{Y_t\}$ converges to BM (Thm 3.2), and CSRW is a time change of VSRW:

$$\text{Clock process} \quad \tilde{A}_t := \int_0^t \mu_{Y_s} ds = \int_0^t \mu_0(T_{Y_s}\omega) ds, \qquad X_t = Y_{\tilde{A}_t^{-1}}.$$

In fact, (using transience of RW)

$$(n^{-1}Y(n^2\cdot), n^{-2/\beta}\tilde{A}_{n^2\cdot}) \to (c_1 B_d, c_2 V_\beta) \ \text{ weakly under } P_\omega^0,\ \mathbb{P}\text{-a.s.}$$

$$\Rightarrow n^{-1}X(n^{2/\beta}t) = n^{-1}Y(\tilde{A}^{-1}_{n^{2/\beta}t}) = n^{-1}Y(n^2(F_t^n)^{-1}) \to c_1 B_d((c_2 V_\beta)^{-1})$$

$$(\tilde{A}^{-1}_{n^{2/\beta}t} = \inf\{s : \tilde{A}_s > n^{2/\beta}t\} = n^2 \inf\{s : F_t^n := n^{-2/\beta}\tilde{A}_{n^2 s} > t\} = n^2(F_t^n)^{-1})$$

To show $n^{-2/\beta}\tilde{A}_{n^2 s} \to c_2 V_\beta$,

• one does truncation, before the exit from a large ball $B(0, Kn)$

• prove that only finitely many clusters of visits occur and they are well separated.

(coarse-graining)

• VSRW spends to in going from one cluster to another are asymptotically indep.

and exp. distributed.

• very fine estimates of Green functions (resistances)

Thank you!