Anomalous subdiffusion and time-fractional differential equations IV

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1 Recap

Classical case: $0 < \beta < 1$.

Riemann-Liouville fractional integral operator:

$$I^{\beta}\psi(t) = \Gamma(\beta)^{-1}\int_0^t (t-s)^{\beta-1}\psi(s)ds.$$

Caputo derivative:

$$\partial_t^{\beta}\psi(t) := \frac{d}{dt}I^{1-\beta}(\psi-\psi(0))(t) = \frac{d}{dt}I^{1-\beta}\psi(t) - \frac{\psi(0)}{t^{\beta}\Gamma(1-\beta)}.$$

Fractional diffusion equation in $\mathbb{R}^d$:

$$\partial_t^{\beta}p(t,x) = \Delta p(t,x) \qquad t>0, x\in\mathbb{R}^d.$$

Estimates of $p(t,x)$ were obtained (e.g. **Eidelman-Kochubei ('04)**) by

$E_{\beta}(z) = \sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}$: Mittag-Leffler function

$p(t,x) = \mathcal{F}^{-1}(E_{\beta}(|\xi|^2 t^{\beta}))$ and using Fourier analysis.

- **Probabilistic approach**

Suppose that $\mathcal{L}$ is the generator of a strong Markov process X.

Let $\{S_\beta(t)\}_{t\geq 0}$: β-stable subordinator $\mathbf{E}[\exp(-\lambda S_\beta(t))] = e^{-t\lambda^\beta}$ (indep. of X).

Theorem 1.1 (Baeumer-Meerschaert '01; Meerschaert-Scheffler '04):

$u(t,x) = \mathbb{E}_x[f(X_{L_t})]$ solves

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \mathcal{L}_x u(t,x), \quad u(0,x) = f(x).$$

Here $L_t = \inf\{r \geq 0 : S_\beta(r) > t\}$: inverse of S_β independent of X.

Tools used: Mittag-Leffler functions, and the self-similarity of the β-subordinator,

$$\{S_{\lambda t}; t \geq 0\} = \{\lambda^{1/\beta} S_t; t \geq 0\} \quad \text{in distribution.}$$

Why do we care?

Motivation:

- Question from industry.

  The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.).

  Predict the progress of soil contamination.

- The third and fourth slides: Nature (2006, Jan.)

  The scaling laws of human travel

  by D. Brockmann, L. Hufnagel and T. Geisel.

⇒ Want to consider more general $\mathcal{L}$ and more general ∂^w!

- (F, d, μ) metric meas. space

$(\mathcal{E}, \mathcal{F})$: reg. Dirichlet form on $L^2(F, \mu)$. $\{X_t\}$: process, $\mathcal{L}$: generator

$S = \{S_t, \mathbb{P}; t \geq 0\}$: driftless subordinator ($S_t \geq 0$) with $S_0 = 0$.

ϕ is the Laplace exponent of S (called Bernstein function). That is,

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \lambda > 0, t \geq 0.$$

ν on $(0, \infty)$:Lévy measure s.t. $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx)$. (Assume ν is infinite.)

Let $w(x) := \nu(x, \infty)$ and define generalized Caputo derivative with weight w by

$$\partial_t^w f(t) := \frac{d}{dt} \int_0^t w(t-s)(f(s) - f(0))\, ds,$$

Define

$$v(t, x) = \mathbb{E}_x\left[f(X_{L_t})\right]$$

where $L_t = \inf\{s > 0 : S_s > t\}$. Then $\cdots$

Theorem 1.2 (Strong sol. Chen '17, '24, Weak sol. Chen-Kim-K-Wang '18)

For $\forall f \in L^2(F;\mu)$, $v(t,x) = \mathbb{E}\left[T_{L_t} f(x)\right]$ *is a weak solution to*

$$\partial_t^w v(t,x) = \mathcal{L} v(t,x) \quad \textit{with } v(0,x) = f(x) \tag{1.1}$$

in the following sense:

(i) $t \mapsto v(t,x)$ *is continuous in* $L^2(F;\mu)$ *with* $\|v(t,x)\|_2 \le \|f\|_2$.

Hence $I_t^w(v(\cdot,x))$ *is absolutely convergent in* $L^2(F,\mu)$ *for* $\forall t > 0$.

(ii) *For* $\forall g \in D(\mathcal{L})$ *and* $t > 0$

$$\frac{d}{dt}\int_F g(x) I_t^w(v(\cdot,x))\,\mu(dx) = \int_F v(t,x)\mathcal{L}g(x)\,\mu(dx). \tag{1.2}$$

Conversely, if $v(t,x)$ *is a weak solution to* (1.1) *in the sense of* (i) *and* (ii) *above with* $f \in L^2(F,\mu)$, *then* $v(t,x) = \mathbb{E}\left[T_{L_t} f(x)\right]$ μ*-a.e. on* F *for every* $t \ge 0$.

(F, d, μ) metric meas. space d: geodesic metric

$(\mathcal{E}, \mathcal{F})$: reg. Dirichlet form on $L^2(F, \mu)$, conservative. $\{X_t\}$: process, $\mathcal{L}$: generator

$p_0(t, x, y)$ the corresponding heat kernel

(Q) What is the HK of the generalized FK processes?

(F, d, μ) metric meas. space d: geodesic metric

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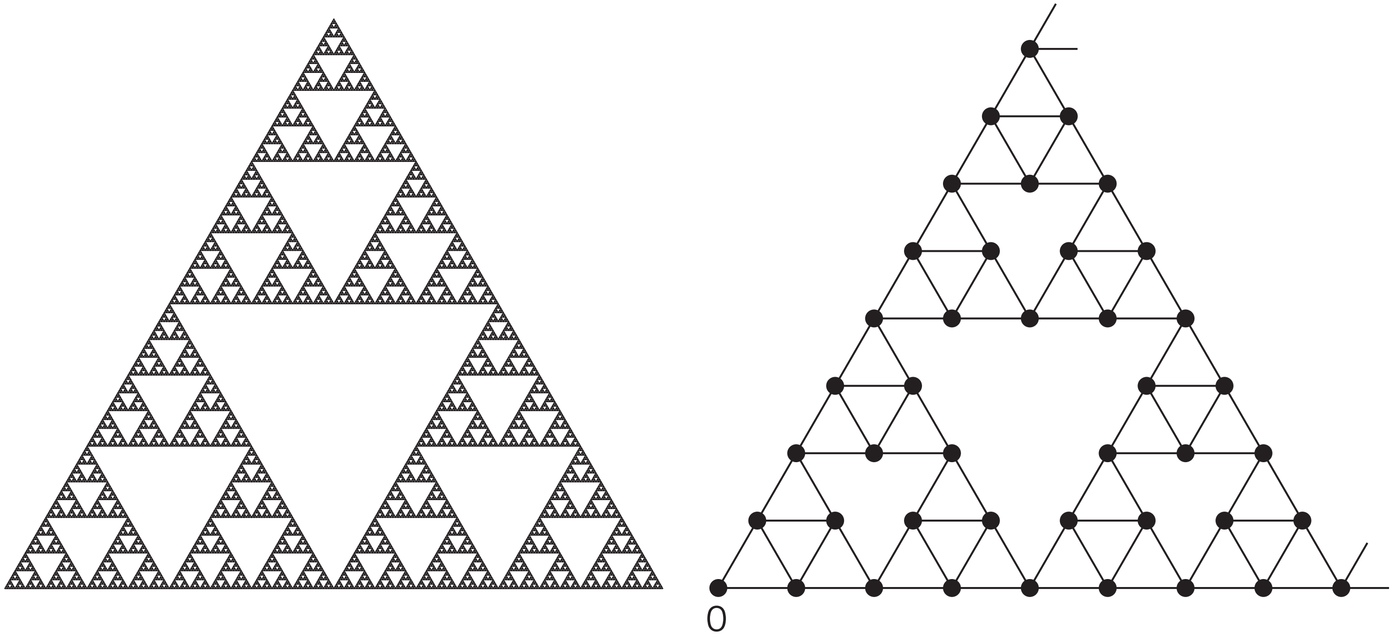
$p_0(t, x, y)$ the corresponding heat kernel

(Q) What is the HK of the generalized FK processes?

$$\begin{aligned} v(t,x) &= \mathbb{E}[f(X(S_t^{-1}))] = \int_0^\infty T_r f(x)\, d_r \mathbb{P}(S_t^{-1} \le r) = \int_0^\infty T_r f(x)\, d_r \mathbb{P}(S_r \ge t) \\ &= \int_0^\infty \int_F f(y) p_0(r, x, y)\, \mu(dy)\, d_r \mathbb{P}(S_r \ge t) \\ &= \int_F f(y) \left(\int_0^\infty p_0(r, x, y)\, d_r \mathbb{P}(S_r \ge t) \right) \mu(dy). \end{aligned}$$

So the HK of the FK process $p_t(x, y)$ is $p_t(x,y) := \int_0^\infty p_0(r, x, y)\, d_r \mathbb{P}(S_r \ge t)$.



2 Heat kernel estimates for diffusions and jump processes on metric measure spaces

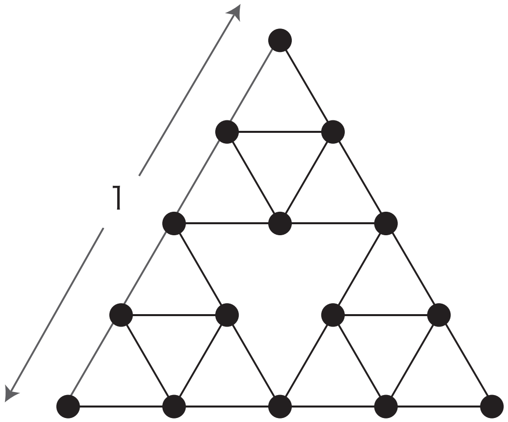
2.1 Diffusions

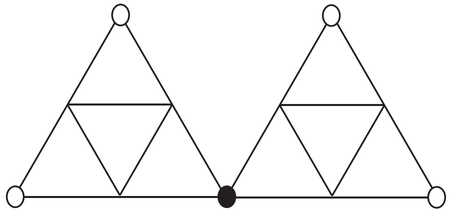
RW and Brownian motion on fractals

G: Sierpinski gasket graph (RHS, extended to infinity)

$\{S_n : n = 0, 1, 2, \cdots\}$: simple RW on G

(In each step, the particle moves to one of the nearest neighbors with equal probability.)





Brownian motion on fractals

Consider simple RW $\{2^{-m}S_n\}$ on $2^{-m}G$.

Want to construct random motion of the particle on the gasket by taking $m \to \infty$.

What is the average time for $2^{-1}S_n$ to reach $\circ$? → (Answer) $5\,(=2^{\log 5/\log 2})$

Set $2^{-m}S_{[5^m t]}$. Then the average time to reach $\{-1, 1\}$ is 1.

$$\boxed{2^{-m}S_{[2^{m\log 5/\log 2}t]} \xrightarrow{m\to\infty} B(t) \quad \text{(Brownian motion on the gasket)}}$$ (Goldstein, Kusuoka '87)

The corresponding Laplacian $5^m\{\frac{1}{4}\sum_{i=1}^{4} f(x_m^i) - f(x)\} \xrightarrow{m\to\infty} \mathcal{L}f(x)$ (Kigami '89)

Anomalous diffusion: $d_w = \log 5/\log 2 > 2$ Diffusive speed of particles (for long time) is much slower than that in the Euclidean space. $E^c[|B(t)|] \asymp t^{1/d_w}$.

Sub-Gaussian heat kernel estimates:(Barlow-Perkins '88) (HK(d_w))

$$c_1 t^{-d_f/d_w} \exp\Big(-c_2\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\Big) \le p_t(x,y) \le c_3 t^{-d_f/d_w} \exp\Big(-c_4\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\Big)$$

$d_f = \log 3/\log 2$: Hausdorff dimension, $d_w = \log 5/\log 2$: Walk dimension

$t^{d_f/d_w} = \mu(B(x, t^{1/d_w}))$

Asymptotic of eigenvalues: (Fukushima-Shima '92) Consider compact gasket.

Spectrum of $\mathcal{L}$ consists of eigenvalues. $\rho(x) := \sharp\{\lambda_i : \lambda_i \text{ is an e.v. of } \mathcal{L}, \lambda_i \le x\}$, then

$$0 < \liminf_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \limsup_{x\to\infty} \frac{\rho(x)}{x^{d_s/2}} < \infty,$$

where $d_s := 2d_f/d_w$ is called the Spectral dimension of SG(2).

Stability of HKE

Thm [Barlow-Bass '03, Barlow-Bass-K '06, Andres-Barlow '15, Grigor'yan-Hu-Lau '15]

$$(\mathbf{VD}) + (\mathbf{PI}(\mathbf{d_w})) + (\mathbf{CS}(\mathbf{d_w})) \Leftrightarrow (\mathbf{HK}(\mathbf{d_w})).$$

- (VD): volume doubling condition

$$\mu(B(x,2R)) \le c_1 \mu(B(x,R)) \qquad \forall \;\; x \in M, R > 0.$$

- (PI(d_w)): scaled Poincaré inequality $\quad \forall B_R = B(x_0,R)$, $R > 0$

$$\int_{B_R} (f(x) - \bar{f}_{B_R})^2 \mu(dx) \le c_1 R^{d_w} \, \mathcal{E}_{B_R}(f,f), \quad \forall f$$

where $\bar{f}_B = \mu(B)^{-1} \int_B f(x)\mu(dx)$.

- (CS (β)): energy ineq. for cut-off funct. **Remark.** (CS(2)) always holds.

2.2 Jumps

- α-stable-like process [Chen-K('03)]: $M \subset \mathbb{R}^n$; a d-set.

$$\mathcal{E}(f,f) = \int\int_{M\times M} (u(x)-u(y))^2 J(x,y)\mu(dx)\mu(dy),$$

where $J(x,y) = \frac{c(x,y)}{d(x,y)^{d+\alpha}}$, $c(x,y) \in [M^{-1}, M]$ $0 < \alpha < 2$. Then

$$c_1\left\{t^{-d/\alpha} \wedge \frac{t}{d(x,y)^{d+\alpha}}\right\} \le p_t(x,y) \le c_2\left\{t^{-d/\alpha} \wedge \frac{t}{d(x,y)^{d+\alpha}}\right\}. \tag{2.1}$$

- More generally, (M,d,μ): Metric meas. space ($diam\, M = \infty$)

$(\mathcal{E},\mathcal{F})$: reg. DF on $L^2(M,\mu)$ (pure jump), $\{X_t\}_{t\ge 0}$: corresp. jump proc.

$$\mathcal{E}(f,g) = \int\int_{M\times M\backslash d} (f(x)-f(y)(g(x)-g(y))J(x,y)\mu(dx)\mu(dy).$$

Volume and time scale (doubling conditions) $\exists c_1 > 1$ s.t.

$$c_1^{-1}\mu(B(x,r)) \le \mu(B(x,2r)) \le c_1\mu(B(x,r)) \qquad \forall x \in M, r > 0, \tag{2.2}$$

and $T : \mathbb{R}_+ \to \mathbb{R}_+$ str. increasing, $T(0) = 0$ and $\exists c_2 > 1$ s.t.

$$c_2^{-1}T(r) \le T(2r) \le c_1 T(r) \qquad \forall r > 0. \tag{2.3}$$

- We consider the following HKE (HK (T)).

$$p(t,x,y) \asymp \frac{1}{\mu(B(x,T^{-1}(t)))} \wedge \frac{t}{\mu(B(x,d(x,y)))T(d(x,y))},$$

for all $x, y \in M, t > 0$.

Thm [Chen-K-Wang '21] (**Stability of HKE**) Under (2.2),(2.3)

$$\mathbf{HK(T)} \Leftrightarrow \mathbf{J_T} + \mathbf{CSJ(T)}, \qquad \text{where}$$

- J_T : $\quad J(x,y) \asymp \frac{1}{\mu(B(x,d(x,y)))T(d(x,y))}$.
- CSJ(T) : energy ineq. for cut-off funct.

To be precise,

- CSJ(T): $\exists C_0 \in (0,1], C_1, C_2 > 0$ s.t. $\forall R \geq r > 0$, $\forall x \in M$, $\forall f \in \mathcal{F}$,

$\exists$ cutoff fu. φ: $\varphi|_{B(x,R)} = 1$, $\varphi|_{B(x,R+r)^c} = 0$ s.t.

$$\int_{B(x,R+(1+C_0)r)} f^2\, d\Gamma(\varphi,\varphi) \leq C_1 \int_{U\times U^*} (f(x)-f(y))^2 J(dx,dy) + \frac{C_2}{T(r)} \int_{B(x,R+(1+C_0)r)} f^2\, d\mu,$$

where $d\Gamma(\varphi,\varphi)(x) = \int_M (\varphi(x)-\varphi(y))^2 J(x,dy)$,

and $U = B(x,R+r)\setminus B(x,R)$, $U^* = B(x,R+(1+C_0)r)\setminus B(x,R-C_0 r)$.

3 Heat kernel estimates for generalized FK processes

HK of the FK process $p_t(x,y)$ is $p_t(x,y) := \int_0^\infty p_0(r,x,y)\, d_r\mathbb{P}(S_r \geq t)$.

Suppose $p_0(t,x,y) \asymp t^{-d/2}\exp(-d(x,y)^2/t)$, and suppose $0 < \exists\beta_1 \leq \beta_2 < 1$ s.t.

$$c_1\kappa^{\beta_1} \leq \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \leq c_2\kappa^{\beta_2}, \quad \forall\lambda > 0, \kappa \geq 1. \tag{3.1}$$

Theorem 3.1 *(i) If* $d(x,y)\phi(t^{-1})^{1/2} \leq 1$, *then*

$$p_t(x,y) \asymp \begin{cases} \phi(t^{-1})^{d/2} & \text{if } d < 2, \\ \phi(t^{-1}) \log\left(\dfrac{2}{d(x,y)\phi(t^{-1})^{1/2}}\right) & \text{if } d = 2, \\ \phi(t^{-1})^{d/2}\left(d(x,y)\phi(t^{-1})^{1/2}\right)^{-d+2} = \phi(t^{-1})/d(x,y)^{d-2} & \text{if } d > 2. \end{cases}$$

(ii) Suppose $d(x,y)\phi(t^{-1})^{1/2} \geq 1$, *then*

$$p_t(x,y) \asymp \phi(t^{-1})^{d/2}\exp\Big(-t\bar{\phi}_2^{-1}((d(x,y)/t)^2)\Big), \tag{3.2}$$

where $\bar{\phi}_2(\lambda) = \lambda^2/\phi(\lambda)$, *and* $\bar{\phi}_2^{-1}(\lambda)$ *is the inverse function of* $\bar{\phi}_2(\lambda)$

Special case: $\phi(s) = s^\beta$, $0 < \beta < 1$ β-stable subordinator

In that case (3.2) is

$$p_t(x,y) \asymp t^{-\beta d/2} \exp\Big(- (d(x,y)t^{-\beta/2})^{2/(2-\beta)}\Big).$$

Rem : Here we assume $\mu(B(x,r)) \asymp r^d$.

We have more general version: under vol. doubling, more general shape of HK.

Proposition 3.2 (Key Proposition)

$$\mathbb{P}(S_r \geq t) \asymp r\phi(t^{-1}) \qquad \textit{if} \quad r\phi(t^{-1}) << 1,$$

$$\mathbb{P}(S_r \leq t) \asymp \exp(-t(\phi')^{-1}(t/r)) \qquad \textit{if} \quad r\phi(t^{-1}) >> 1.$$

Rem 2: Roughly, Theorem 3.1 is obtained from Gaussian HK by taking $t \to 1/\phi(t^{-1})$.

- **More general version** Suppose the corresponding heat kernel enjoys

$$p_0(t,x,y) \asymp t^{-d/\alpha}\Psi(d(x,y)/t^{1/\alpha})$$

for some Ψ monotone decreasing

$\Rightarrow$ (Grigor'yan-K '08) Either $\mathcal{E}$ is local, $\alpha \geq 2$ and $\Psi(s) \asymp \exp(-s^{\alpha/(\alpha-1)})$: sub-Gaussian

or $\mathcal{E}$ is non-local, $\alpha > 0$ and $\Psi(s) \asymp (1+s)^{-(d+\alpha)}$: α-stable-like.

Note: $\alpha = 2$ and $\Psi(s) = \exp(-s^2)$ is the classical case mentioned before.

Suppose (3.1) holds. (We may assume ϕ is a complete Bernstein function.)

Then $\cdots$

Theorem 3.3 *(i) If* $d(x,y)\phi(t^{-1})^{1/\alpha} \le 1$, *then*

$$p_t(x,y) \asymp \begin{cases} \phi(t^{-1})^{d/\alpha} & \text{if } d<\alpha, \\ \phi(t^{-1}) \log\left(\dfrac{2}{d(x,y)\phi(t^{-1})^{1/\alpha}}\right) & \text{if } d=\alpha, \\ \phi(t^{-1})^{d/\alpha}\left(d(x,y)\phi(t^{-1})^{1/\alpha}\right)^{-d+\alpha} = \phi(t^{-1})/d(x,y)^{d-\alpha} & \text{if } d>\alpha. \end{cases}$$

(ii) Suppose $d(x,y)\phi(t^{-1})^{1/\alpha} \ge 1$. • *When the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *is local,*

$$p_t(x,y) \asymp \phi(t^{-1})^{d/\alpha} \exp\Big(-t\bar{\phi}_\alpha^{-1}((d(x,y)/t)^\alpha)\Big), \tag{3.3}$$

where $\bar{\phi}_\alpha(\lambda) = \lambda^\alpha/\phi(\lambda)$, *and* $\bar{\phi}_\alpha^{-1}(\lambda)$ *is the inverse function of* $\bar{\phi}_\alpha(\lambda)$

- *When* $(\mathcal{E},\mathcal{F})$ *is non-local,*

$$p_t(x,y) \asymp \phi(t^{-1})^{d/\alpha}(d(x,y)\phi(t^{-1})^{1/\alpha})^{-d-\alpha} = \frac{1}{\phi(t^{-1})d(x,y)^{d+\alpha}}.$$

Part of the proof • **[Lower]** For $x, y \in M$, $t > 0$ with $d(x,y)^\alpha \phi(t^{-1}) \le 1$, then

$$p(t,x,y) \ge c_1 \left(\phi(t^{-1})^{d/\alpha} \vee \frac{\phi(t^{-1})}{d(x,y)^{d-\alpha}} \right).$$

Proof:

$$\begin{aligned} p(t,x,y) &\ge c \int_{\kappa_2/\phi(t^{-1})}^{\kappa_1/\phi(t^{-1})} p_0(r,x,y)\, d_r \mathbb{P}(S_r \ge t) \\ &\ge c \left(\min_{\kappa_2/\phi(t^{-1}) \le r \le \kappa_1/\phi(t^{-1})} p_0(r,x,y) \right) \left(\mathbb{P}(S_{\kappa_1/\phi(t^{-1})} \ge t) - \mathbb{P}(S_{\kappa_2/\phi(t^{-1})} \ge t) \right) \qquad (3.4) \\ &\ge c\phi(t^{-1})^{d/\alpha}. \end{aligned}$$

$$\begin{aligned} p(t,x,y) &\ge c \int_{\kappa_4 d(x,y)^\alpha}^{\kappa_3 d(x,y)^\alpha} p_0(r,x,y)\, d_r \mathbb{P}(S_r \ge t) \\ &\ge c \left(\min_{\kappa_4 d(x,y)^\alpha \le r \le \kappa_3 d(x,y)^\alpha} p_0(r,x,y) \right) \left(\mathbb{P}(S_{\kappa_3 d(x,y)^\alpha} \ge t) - \mathbb{P}(S_{\kappa_4 d(x,y)^\alpha} \ge t) \right) \\ &\ge \frac{c d(x,y)^\alpha \phi(t^{-1})}{d(x,y)^d}. \end{aligned}$$

- (Diffusion case) Fix $x \in M$ and set $\bar{p}_0(t,r) := t^{-d/\alpha} \exp\left(-m(t,r)\right), \ t, r > 0$. Then

$$\left|\frac{\partial \bar{p}_0(t,r)}{\partial t}\right| \le c_0 t^{-d/\alpha-1} \exp\left(-c_1 m(t,r)\right) = \frac{c_0}{t} \bar{p}_0(t,r),$$

- (Diffusion case) [Upper] For $x, y \in M$, $t > 0$ with $d(x,y)^\alpha \phi(t^{-1}) \le 1$, then

$$p(t,x,y) \le c_1 \left(\phi(t^{-1})^{d/\alpha} \vee \phi(t^{-1}) \int_{z^\alpha}^{2/\phi(t^{-1})} r^{-d/\alpha} dr\right).$$

Proof: Let $z = d(x,y)$ and

$$\begin{aligned} p(t,x,y) &\asymp \int_0^\infty \bar{p}_0(r,z)\, d_r \mathbb{P}(S_r \ge t) \\ &= \int_0^{2/\phi(t^{-1})} \bar{p}_0(r,z)\, d_r \mathbb{P}(S_r \ge t) - \int_{2/\phi(t^{-1})}^\infty \bar{p}_0(r,z)\, d_r \mathbb{P}(S_r \le t) =: I_1 + I_2. \end{aligned}$$

Then

$$\begin{aligned} I_1 &\le c\bar{p}_0(2/\phi(t^{-1}), z) - \int_0^{2/\phi(t^{-1})} \mathbb{P}(S_r \ge t)\, d_r \bar{p}_0(r,z) \\ &\le c\bar{p}_0(2/\phi(t^{-1}), z) + c \int_0^{2/\phi(t^{-1})} r\phi(t^{-1}) \cdot \frac{1}{r} \cdot \bar{p}_0(r,z)\, dr =: c\bar{p}_0(2/\phi(t^{-1}), z) + cI_{1,1}. \end{aligned}$$

Suppose that $z^\alpha \phi(t^{-1}) \le 1$. Then,

$$I_{1,1} \le c\phi(t^{-1}) \left(\int_0^{z^\alpha} r^{-d/\alpha} \exp\left(-m(r,z)\right)\, dr + \int_{z^\alpha}^{2/\phi(t^{-1})} r^{-d/\alpha}\, dr \right)$$

$$= c\phi(t^{-1}) \left(z^{\alpha-d} + \int_{z^\alpha}^{2/\phi(t^{-1})} r^{-d/\alpha}\, dr \right) \le 2c\phi(t^{-1}) \int_{z^\alpha}^{2/\phi(t^{-1})} r^{-d/\alpha}\, dr,$$

and $$I_2 = -\,\bar{p}_0(r,z)\mathbb{P}(S_r \le t)\Big|_{2/\phi(t^{-1})}^{\infty} + \int_{2/\phi(t^{-1})}^{\infty} \mathbb{P}(S_r \le t)\, d_r\bar{p}_0(r,z)$$

$$\le c\bar{p}_0(2/\phi(t^{-1}), z) + c\int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot \frac{1}{r} \cdot \bar{p}_0(r,z)\, dr.$$

Suppose that $z^\alpha \phi(t^{-1}) \le 1$. Then,

$$\int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot \bar{p}_0(r,z)\, dr$$

$$\le c\int_{2/\phi(t^{-1})}^{\infty} \exp(-c_1 t(\phi')^{-1}(t/r)) \cdot r^{-1} \cdot r^{-d/\alpha}\, dr \le \cdots \le c\phi(t^{-1})^{d/\alpha},$$

hence we get $I_2 \le c'\phi(t^{-1})^{d/\alpha}$. **Λ**

4 Heat kernel estimates for fundamental solutions of Poisson equations

Set up Assume that $\{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator with Lévy measure ν

and having bounded density $\bar{p}(t, \cdot)$ for each $t > 0$.

Suppose $\{P_t^0;\, t \geq 0\}$ is unif. bdd strongly cont. semigroup and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$: generator.

Consider $\partial_t^w u = \mathcal{L}u + f(t, x)$ on $(0, T_0] \times E$ with $u(0, x) = g(x)$.

Using $w(x) = \nu(x, \infty)$, define generalized Caputo derivative with weight w by

$$\partial_t^w f(t) := \frac{d}{dt} \int_0^t w(t-s)(f(s) - f(0))\, ds.$$

Suppose $\{P_t^0; t \geq 0\}$ has a kernel $p_0(t, x, y)$ w.r.t. some measure μ on M. Define

$$q(t, x, y) := \int_0^\infty p_0(r, x, y)\bar{p}(r, t) dr.$$

Then the unique solution of the above Poisson eq. can be expressed as

$$u(t, x) = \int_M p(t, x, y) g(y) \mu(dy) + \int_0^t \int_M q(s, x, y) f(t-s, y) \mu(dy) ds.$$

(Recall $p(t, x, y) = \mathbb{E}\left[p_0(E_t, x, y)\right]$.)

For $\beta \in (0,1)$, define

$$H_{\le 1}(t,r) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & \text{if } d < 2\alpha, \\ t^{-1-\beta}\log\left(\dfrac{2t^{\beta}}{r^{\alpha}}\right), & \text{if } d = 2\alpha, \\ t^{-1-\beta}/r^{d-2\alpha}, & \text{if } d > 2\alpha, \end{cases}$$

$$H^{(j)}_{\ge 1}(t,r) = t^{2\beta-1}/r^{d+\alpha},$$

$$H^{(c)}_{\ge 1}(t,r) = t^{\beta-1-\beta d/\alpha}\exp\Big(-(r^{\alpha}/t^{\beta})^{1/(\alpha-\beta)}\Big).$$

Theorem 4.1 *Suppose that $\mathcal{L}$ is the generator of a Markov process and its fundamental solution $p^0(t,x,y)$ admits the two-sided estimates*

$$p^0(t,x,y) \asymp t^{-d/\alpha}\Psi(d(x,y)/t^{1/\alpha}), \tag{4.1}$$

and that $\{S_t, \mathbb{P}; t \ge 0\}$ is a β-stable subordinator with $0 < \beta < 1$.

(i) *Suppose* $\Psi(r) = (1+r)^{-d-\alpha}$ *with* $\alpha > 0$. *Then* $q(t,x,y)$ *satisfies*

$$q(t,x,y) \asymp \begin{cases} H_{\leq 1}(t, d(x,y)) & \textit{if } d(x,y) \leq t^{\beta/\alpha}, \\ H_{\geq 1}^{(j)}(t, d(x,y)) & \textit{if } d(x,y) \geq t^{\beta/\alpha}. \end{cases}$$

(ii) *Suppose* $\Psi(r) = \exp(-r^{\alpha/(\alpha-1)})$ *with* $\alpha \geq 2$. *Then* $q(t,x,y)$ *satisfies*

$$q(t,x,y) \asymp H_{\leq 1}(t, d(x,y)) \qquad \textit{if } d(x,y) \leq t^{\beta/\alpha},$$

$$q(t,x,y) \asymp H_{\geq 1}^{(c)}(t, d(x,y)) \qquad \textit{if } d(x,y) \geq t^{\beta/\alpha}.$$

Thank you!