Anomalous subdiffusion and time-fractional differential equations VI

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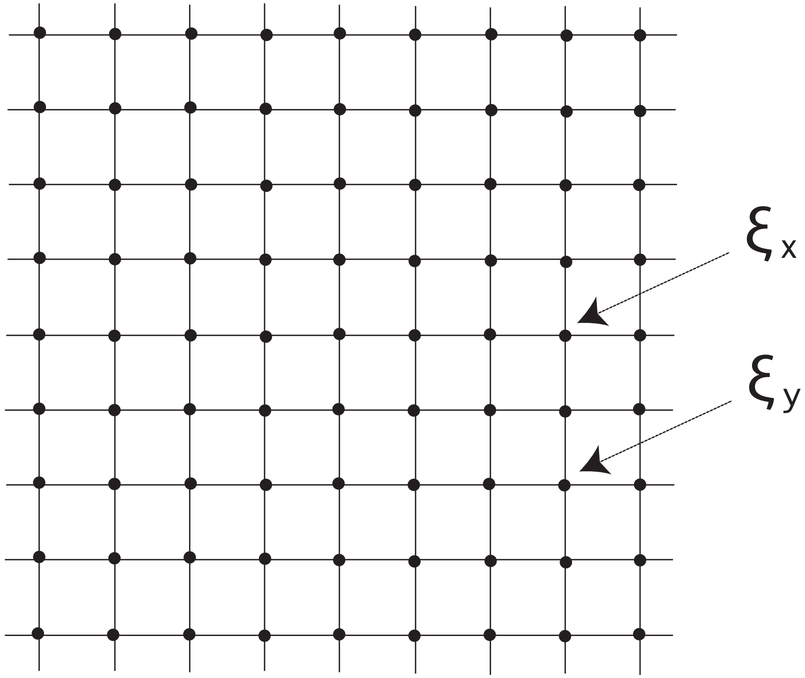
1 Introduction

Example 1: Symmetric Bouchaud's trap model (BTM)

A trapping landscape $\{\xi_x\}_{x\in\mathbb{Z}^d}$: pos. i.i.d. on $\mathbf{P}$.

Random Hopping Times dynamics $(X_t^\xi)_{t\geq 0}$: cont. time MC, trans. prob. $1/(2d)$.

Jump rate at x being $1/\xi_x$. $\exists\alpha\in(0,1)$ s.t. $\mathbf{P}(\xi_x>u)=u^{-\alpha}$, $\forall u\geq 1$.



Theorem 1.1 $d = 1$ (Fontes-Isopi-Newman '02)

$$\varepsilon X_{c_* t/\varepsilon^{1+1/\alpha}} \xrightarrow{d} Z(t) \quad \textit{under} \quad \mathbf{P} \times P_0^{\xi}.$$

Definition 1.2 *FIN diffusion is defined by* $Z(s) = BM(\phi_\rho^{-1}(s))$, $s \in [0, \infty)$,

where $\phi_\rho(t) := \int_{\mathbb{R}} \ell(t, y)\rho(dy)$ *where* $\ell(\cdot, \cdot)$ *is the local time of BM,*

$\rho := \sum_i \nu_i \delta_{x_i}$ *where* $(x_i, \nu_i) \in \mathbb{R}\times\mathbb{R}_+$ *is distributed by PPP with intensity* $dx\alpha\nu^{-1-\alpha}d\nu$.

— Atoms of ρ are dense in $\mathbb{R}$ a.e.

$$\boxed{\frac{\partial}{\partial t} p(t, x) = \frac{\partial^2}{\partial \rho \partial x} p(t, x)}$$

ρ plays the role of speed measure.

Limit is very different for $d \geq 2$.

Theorem 1.3 $d \geq 2$ (Ben Arous-Černý '07) *For* $d \geq 3$,

$$\varepsilon X_{ct/\varepsilon^{2/\alpha}} \xrightarrow{d} \mathbf{FK}_{d,\alpha}(t) := BM_d(S_\alpha^{-1}(t)) \quad \mathbf{P}\text{-}a.s. \ \ on \ D([0,\infty),\mathbb{R}^d),$$

where $\{S_\alpha(t)\}_{t\geq 0}$*:* α*-stable subord.* *(indep. of* $\{BM_d(t)\}$*).*

For $d = 2$, *same result by replacing* $\varepsilon^{-2/\alpha}$ *to* $\varepsilon^{-2/\alpha}(\log \varepsilon)^{-2/(1-\alpha)}$.

$\mathbf{FK}_{d,\alpha}$: Fractional-kinetics process — It is no longer a Markov process!

Density of its fixed time distribution $p(t,x)$ satisfies the fractional-kinetics equation:

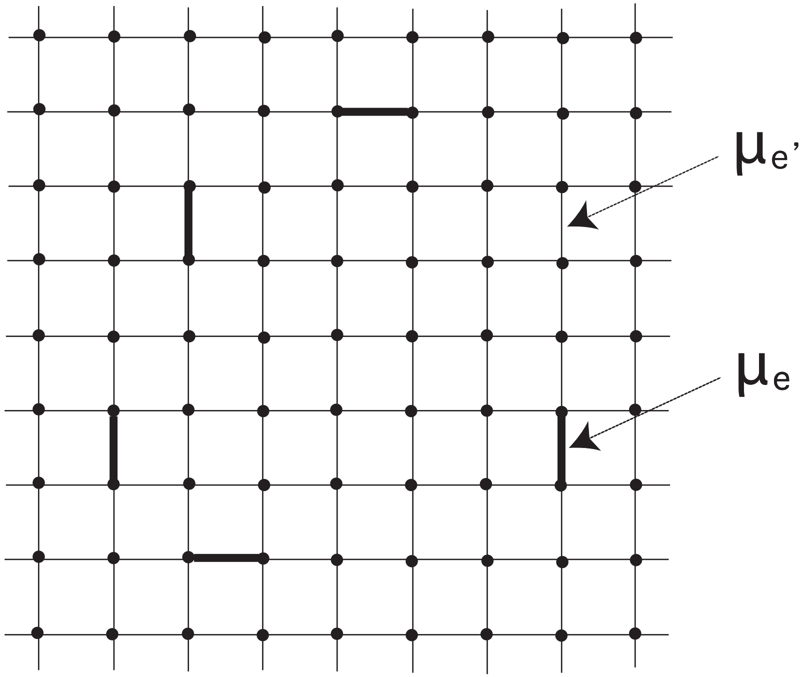
$$\boxed{\frac{\partial^\alpha}{\partial t^\alpha} p(t,x) = \frac{1}{2}\Delta p(t,x).}$$

Example 2: Random conductance model (RCM)

$\{\mu_e\}$: random conductance, *i.i.d.* on each edge e of $\mathbb{Z}^d$ s.t. $\exists \alpha \in (0,1)$

$$\mathbb{P}(\mu_e \geq c_1) = 1, \quad \mathbb{P}(\mu_e \geq u) = c_2 u^{-\alpha}(1 + o(1)) \ \text{ as } u \to \infty. \tag{1.1}$$

(Note that $\mathbb{E}\mu_e = \infty$.) $\{X_t\}_{t\geq 0}$: cont. time MC on $\mathbb{Z}^d$ (holding time exp(1)).



(More involved, but same results hold.) $d = 1, 2$: Černý '11, $d \geq 3$: Barlow-Černý '11

AIM To consider such a time-changed process as a scaling limit of (strongly recurrent) Markov chains in a unified manner on general metric spaces (including fractals), and obtain its heat kernel estimates.

To do so, we discuss;

(i) General convergence theorem on strongly recurrent processes.

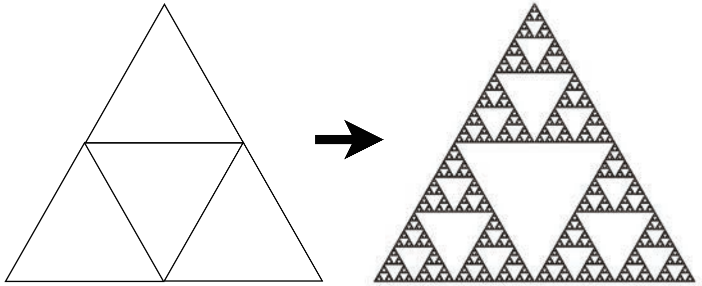
(ii) General convergence theorem on time-changed strongly recurrent processes.

Note: Our results generalize FIN case (include more time-changed processes).

Here is is a typical example (before time change) we have in mind.

Example 3: Fractal graphs (for simplicity pre-Sierpinski gasket)

G^N: pre-SG, $\{\mu^N_{xy}\}$ random (i.i.d.) conductance $\mu^N_{xy} \in [c_1, c_2]$, X^N: corresponding MC



$$(2^{-N} X^N_{[5^N t]})_{t\geq 0} \longrightarrow (B^F_t)_{t\geq 0} \quad \text{in prob.,}$$

where F is the gasket, B^F is BM on F (K-Kusuoka ’96, K ’04).

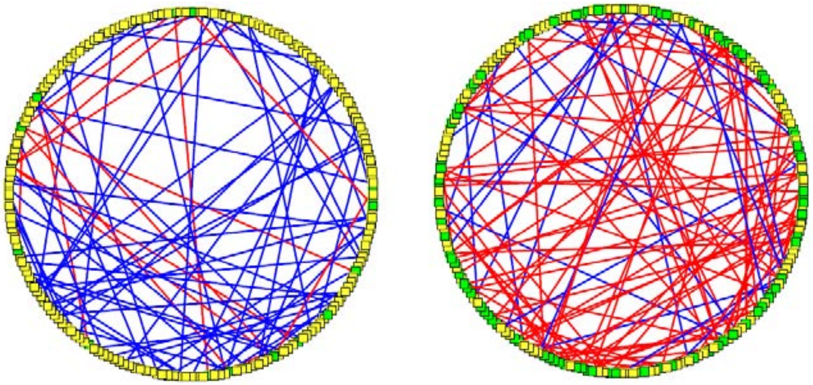
(Q) What happens for the ”heavy tailed” conductance?

Example 4: Erdős-Rényi random graph in critical window

$G(N,p)$: Erdős-Rényi random graph I.e. $V_N := \{1, 2, \cdots, N\}$ vertices

Percolation on the complete graph: each bond open w.p. $p \sim c/N$.

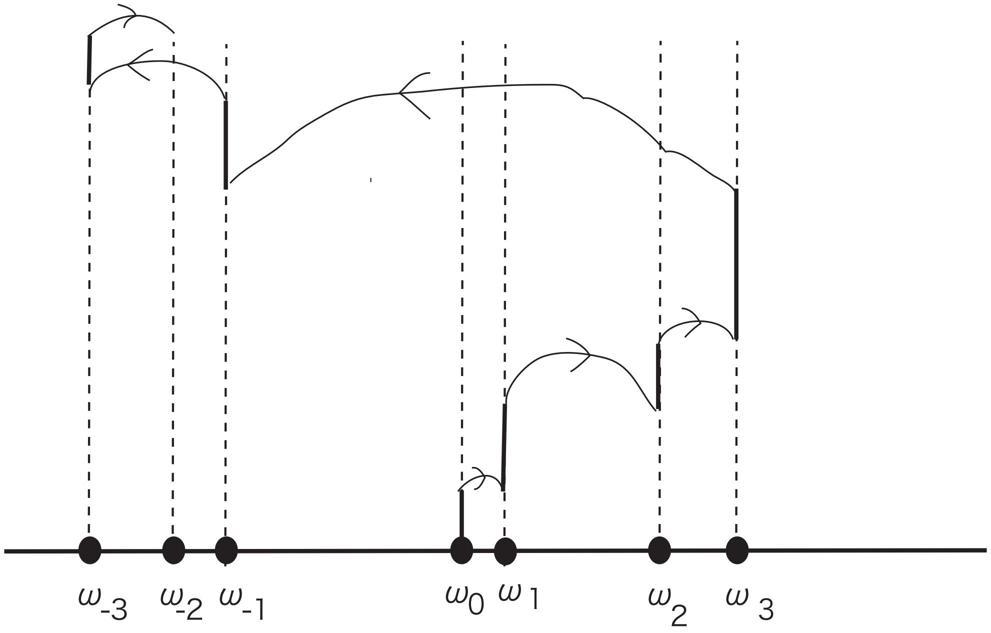
$\mathcal{C}^N$: largest con. comp. E.g. $N = 200, c = 0.8$ $N = 200, c = 1.2$ Pictures by C. Goldschmidt.



Critical window: $p = 1/N + \lambda N^{-4/3}$ for fixed $\lambda \in \mathbb{R}$ $\Rightarrow |\mathcal{C}^N| \asymp N^{2/3}$. (Aldous '97)

- [Addario-Berry, Broutin, Goldschmidt '12]: $N^{-1/3}\mathcal{C}^N \xrightarrow{d} \exists \mathcal{M} = \mathcal{M}_\lambda$
- [Croydon '12] $\{N^{-1/3} Y^{\mathcal{C}^N}_{[Nt]}\}_{t \geq 0} \xrightarrow{d} \{B^{\mathcal{M}}_t\}_{t \geq 0}$: BM on $\mathcal{M}$

  (Q) What happens for the "heavy tailed" conductance?



Example 5: Mott variable-range hopping (special case)

$\{\omega_i : i \in \mathbb{Z}\}$: Atoms of 1-dim PPP (intensity $\rho > 0$), conditioned to $\omega_0 = 0$.

Conductance $C_{\omega_i,\omega_j} = \exp(-|\omega_i - \omega_j|)$

$\{X_t\}_{t\geq 0}$: cont. time MC

Jump rate 1, tran. prob. from ω_i to ω_j is $C_{\omega_i,\omega_j}/(\sum_{\mathrm{J}\neq i} C_{\omega_i,\omega_j})$.

(Q) What is the scaling limit to this MC?

2 Framework and Theorems

Resistance form (J. Kigami): Definition skipped.

F: set, $(\mathcal{E}, \mathcal{F})$: *resistance form* on F

Assume $\mathcal{F} \subset L^2(F, \mu)$ for some Radon meas. μ, and $(\mathcal{E}, \mathcal{F})$ is a reg. Dirichlet form.

(The corresponding process has positive capacity on each point.)

On resistance forms,

$$\begin{aligned} R(x, y) \ &:= \Big(\inf\{\mathcal{E}(f, f) : f(x) = 1, f(y) = 0, f \in \mathcal{F}\} \Big)^{-1} \\ &= \ \sup \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} : \ f \in \mathcal{F}, \ \mathcal{E}(f, f) > 0 \right\} < \infty. \end{aligned}$$

Note that $R(\cdot, \cdot)$ is a metric and

$$|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f), \quad \forall f \in \mathcal{F}, x, y \in F.$$

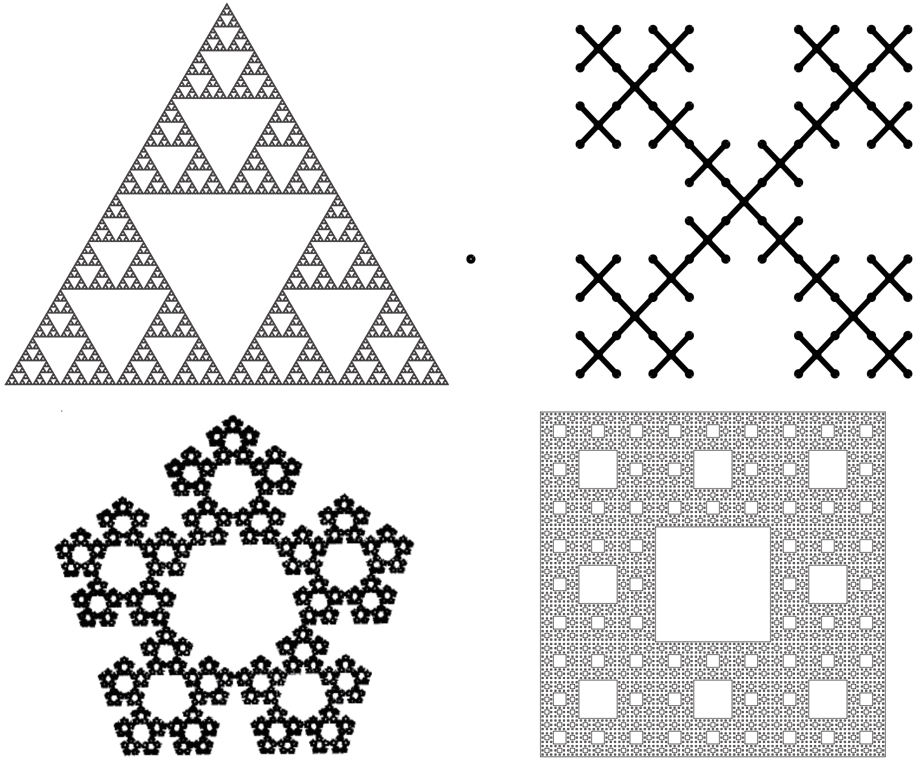


Figure 1: Examples; High dim Sierpinski gaskets are also included.

For the above examples, $(\mathcal{E}, \mathcal{D})$ is a resistance form and a reg. Dirichlet form.

$\Rightarrow \exists \{X_t\}_{t \geq 0}$ corresponding Hunt proc.

Gromov-Hausdorff-vague topology (GH-vague top.)

$\boxed{\mathbb{F}_c := \{(F, R, \mu, \rho) : (F, R) \text{ is compact, } \mu \text{ Radon meas., } \rho \in F\text{: distinguished pt}\}.}$

For $(F, R, \mu, \rho), (F', R', \mu', \rho') \in \mathbb{F}_c$, we set $\Delta_c((F, R, \mu, \rho), (F', R', \mu', \rho'))$ to be

$$\inf_{M, \psi, \psi'} \left\{ d_M^H(\psi(F), \psi'(F)) + d_M^P\left(\mu \circ \psi^{-1}, \mu' \circ \psi'^{-1}\right) + d_M(\rho, \rho') \right\},$$

where inf is taken over all $M = (M, d_M)$ and

all isometric embeddings $\psi : (F, R) \to (M, d_M)$, $\psi' : (F', R') \to (M, d_M)$.

d_M^H: Hausdorff distance, d_M^P: Prohorov distance

$\Rightarrow$ Δ_c defines a metric on (the equiv. classes of) $\mathbb{F}_c$.

* One can generalize it to non-cpt case.

Uniform volume growth (UVD) $(F_n, R_n, \mu_n, \rho_n)_{n\geq 1} \subset \mathbb{F}$ satisfies (UVD) if

$$c_1 v(r) \leq \mu_n\left(B_n(x,r)\right) \leq c_2 v(r), \qquad \forall x \in F_n,\ r \leq \operatorname{diam}_{R_n} F_n,\ \forall n \geq 1,$$

where $v : (0,\infty) \to (0,\infty)$ is non-dec. with $v(2r) \leq c_3 v(r)$, $\forall r \in \mathbb{R}_+$.

Assumption 2.1 $(F_n, R_n, \mu_n, \rho_n)_{n\geq 1} \subset \mathbb{F}$ *satisfies UVD, and also*

$$\left(F_n, R_n, \mu_n, \rho_n\right) \to \left(F, R, \mu, \rho\right), \tag{2.1}$$

in GH-vague top., where $(F, R, \mu, \rho) \in \mathbb{F}$.

Theorem 2.2 *Suppose Assumption 2.1 holds. Then one can isometrically embed* (F_n, R_n), $n \geq 1$, *and* (F, R) *into* $\exists (M, d_M)$ *so that if* $X_0^n = \rho_n$ *and* $X_0 = \rho$, *then*

$$\left(X_t^n\right)_{t \geq 0} \to \left(X_t\right)_{t \geq 0}, \quad \textit{in distri. in} \quad D(\mathbb{R}_+, M). \tag{2.2}$$

Moreover, $\{L^n\}_n$ *(local times) are equicontinuous, and if* $(x_i^n)_{i=1}^k \subset F_n$, $n \geq 1$, *are s.t.* $d_M(x_i^n, x_i) \to 0$ *for* $\exists (x_i)_{i=1}^k \subset F$, *then*

$$\left(L_t^n\left(x_i^n\right)\right)_{i=1,\dots,k, t \geq 0} \to \left(L_t\left(x_i\right)\right)_{i=1,\dots,k, t \geq 0}, \quad \textit{in distri. in} \quad C(\mathbb{R}_+, \mathbb{R}^k). \tag{2.3}$$

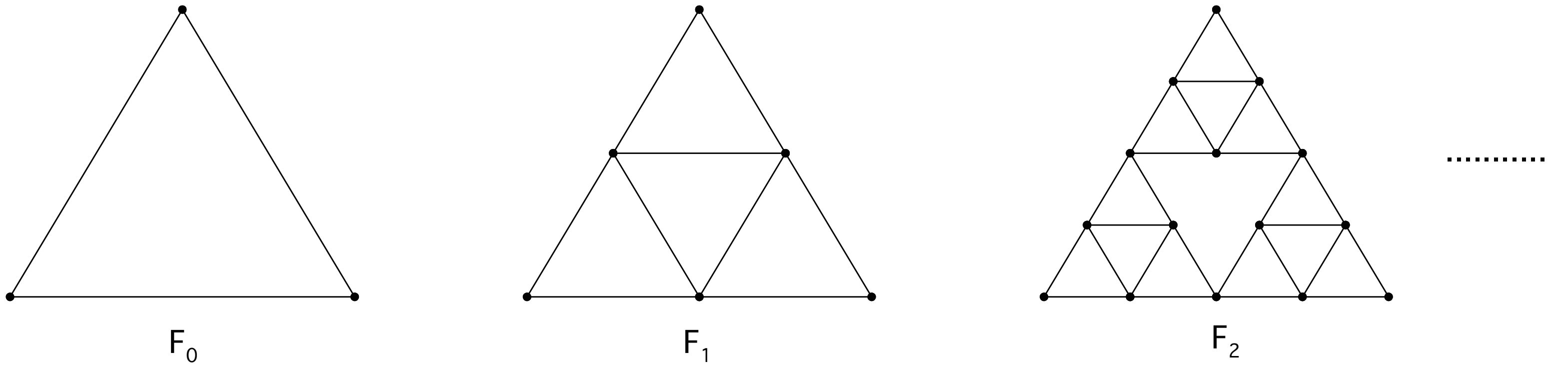
Example: Sierpinski gasket and $\mathbb{R}_+$.

$\bar{R}_n(\cdot,\cdot)$: effective resistance on F_n with unit resistor on each bond

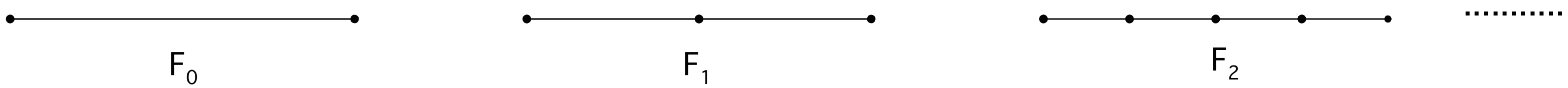
$\bar{\mu}_n$: meas. on F_n s.t. $\mu_n(\{x\}) = 1, \ \forall x \in F_n$.

Gasket: Let $X^{(n)}$ be a SRW on F_n. Then $X^{(n)}([5^n t]) \to B_t$ (BM on the gasket F)

$R_n(\cdot,\cdot) := (3/5)^n \bar{R}_n(\cdot,\cdot), \mu_n := 3^{-n}\bar{\mu}_n(\cdot)$. Then $(F_n, R_n, \mu_n, \rho_n) \to (F, R, \mu, \rho)$



$\mathbb{R}_+$: $X^{(n)}$ SRW on F_n, then $X^{(n)}([4^n t]) \to B_t$ (BM on $\mathbb{R}_+$) $R_n := 2^{-n}\bar{R}_n, \mu_n := 2^{-n}\bar{\mu}_n$.



Uniform volume growth (UVD) $(F_n, R_n, \mu_n, \rho_n)_{n\geq 1} \subset \mathbb{F}$ satisfies (UVD) if

$$c_1 v(r) \leq \mu_n\left(B_n(x,r)\right) \leq c_2 v(r), \qquad \forall x \in F_n,\ r \leq \mathrm{diam}_{R_n} F_n,\ \forall n \geq 1.$$

Remark. • Croydon ('18) extends the result (2.2) by relaxing Assumption 2.1. (Ext. of Athreya-Löhr-Winter.) Namely, instead of the UVD condition he assumes

$$\lim_{r\to\infty} \limsup_{n\to\infty} R_n(\rho_n, B_n(\rho_n, r)^c) = \infty. \tag{2.4}$$

(When all the spaces are compact, (2.4) always holds.)

⇒ Applicable to random media!

• R. Noda ('25 AIHP): convergence of local times under weaker condition.

C.f. Heavily related recent (series of) work: Ben Arous-Cabezas-Fribergh

Assumption 2.3 *Assumption 2.1 holds with (2.1) replaced by*

$$\Big(F_n, R_n, \mu_n, \nu_n, \rho_n\Big) \to \Big(F, R, \mu, \nu, \rho\Big),$$

in the (extended) GH-vague top., where ν_n*: loc. finite Borel meas. on* F_n*, and* ν*: loc. finite Borel meas. on* (F, R) *with* $\nu(F) > 0$.

The time-change additive functional that we consider is the following:

$$A_t := \int_F L_t(x)\nu(dx). \tag{2.5}$$

Let $A_t^{-1} := \inf\{s > 0 : \ A_s > t\}$ (right-cont. inverse of A), and define a process X^ν by

$$X_t^\nu := X_{A_t^{-1}}. \tag{2.6}$$

Theorem 2.4 *(a) Suppose Assumption 2.3 holds, and* ν *has full support . one can isometrically embed* (F_n, R_n), $n \geq 1$, *and* (F, R) *into* $\exists (M, d_M)$ *so that*

$$X^{n,\nu_n} \to X^{\nu}, \quad \textit{in distri. in} \quad D(\mathbb{R}_+, M), \tag{2.7}$$

where $X_0^n = \rho_n$ *and* $X_0 = \rho$.

(b) Suppose Assumption 2.3 holds, and that X *is continuous . Then* (2.7) *holds in distribution in* $L^1_{\rm loc}(\mathbb{R}_+, M)$.

3 Applications

1. Liouville BM (LBM) on F Skip in this talk.

2. Symmetric Bouchaud trap model (BTM) on F

A $\boxed{\text{trapping landscape}}$ $\xi = (\xi_x)_{x \in F_n}$: *i.i.d.* str. pos. on $\mathbf{P}$.

$X^\xi = (X_t^\xi)_{t \geq 0}$: jump rate at x being $1/\xi_x$.

Suppose a trapping landscape $\xi^n = (\xi_x^n)_{x \in F_n}$ satisfies

$$\exists \alpha \in (0,1) \quad \text{s.t.} \quad \mathbf{P}\left(\xi_x^n > u\right) = u^{-\alpha}, \qquad \forall u \geq 1. \tag{3.1}$$

Z^ν: α-FIN diffusion on (F, R, μ). Here we let ν be s.t.

$$\nu(dx) := \sum_i v_i \delta_{x_i}(dx),$$

where (v_i, x_i) are the points of a PP on $(0, \infty) \times F$ with intensity $\alpha v^{-1-\alpha} dv \mu(dx)$.

Proposition 3.1 *Suppose Assumption 2.1 holds with* $\bar{\mu}_n$ *being a counting meas. One can then isometrically embed* $(F_n, a_n\bar{R}_n, b_n\bar{\mu}_n, \rho_n)$, $n \geq 1$, *and* (F, R, μ, ρ) *into* $\exists(M, d_M)$ *so that*

$$\mathbb{P}^{\mathrm{BTM}_n}_{\rho_n}\left(\left(X^{n,\xi^n}_{t/a_n b_n^{1/\alpha}}\right)_{t\geq 0} \in \cdot\right) \to \mathbb{P}^{\mathrm{FIN}}_{\rho}, \quad \textit{in distri. in} \quad D(\mathbb{R}_+, M).$$

Why $\nu(dx) := \sum_i v_i \delta_{x_i}(dx)$ appears in the limit?

Recall (v_i, x_i): points of a PP on $(0,\infty) \times F$ with intensity $\alpha v^{-1-\alpha} dv \mu(dx)$.

Answer: $b_n^{1/\alpha}\bar{\nu}_n(B) \to \nu(B)$ in distri. , where $\bar{\nu}_n(\{x\}) = \xi^n_x$. Indeed,

$$\mathbf{E}\left(e^{-\lambda b_n^{1/\alpha}\bar{\nu}_n(B)}\right) = \mathbf{E}\left(e^{-\lambda b_n^{1/\alpha}\sum_{x\in B}\xi^n_x}\right) = (1 - \lambda^\alpha b_n \Gamma(1-\alpha) + o(b_n))^{\mu_n(B)} \to e^{-\lambda^\alpha \Gamma(1-\alpha)\mu(B)},$$

$$\mathbf{E}\left(e^{-\lambda\nu(B)}\right) = \mathbf{E}\left(e^{-\sum_{i:x_i\in B}\lambda v_i}\right) = e^{-\int_{(0,\infty)\times B}\left(1-e^{-\lambda v}\right)\alpha v^{-1-\alpha} dv \mu(dx)} = e^{-\lambda^\alpha\Gamma(1-\alpha)\mu(B)}. \text{ (Campbell's thm)}$$

3. Random conductance model (RCM) on F

Random conductance $(\omega_e)_{e\in E_G}$: not-necessarily bounded from above.

Much more complicated because now the tran. prob. also changes!

For simplicity, here we consider nested fractals and the approximated graphs.

(F_n, E_n); approximated graphs of the nested fractals, $\bar{\mu}_n$: counting measure.

$(\omega_e^n)_{e\in E_n}$: RC s.t. within each n-cell, $(\omega_e^n)_{e\subseteq\Psi_{i_1,\dots,i_n}(F_0)}$, are *i.i.d.* as $(\omega_e^0)_{e\in E_0}$. Assume

$$\exists\alpha\in(0,1)\quad\text{s.t.}\quad \mathbf{P}\left(\sum_{e\in E_0}\omega_e^0>u\right)\sim cu^{-\alpha}\quad\text{as}\quad u\to\infty.$$

Proposition 3.2 *Suppose* $(\omega_e^0)_{e\in E_0}$ *are uniformly bounded from below. Then*

$$\mathbb{P}_{\rho_n}^{\mathrm{CSRW}_n}\left(\left(X^{n,\omega,\nu}_{c_0 t/a_n b_n^{1/\alpha}}\right)_{t\geq 0}\in\cdot\right)\to\mathbb{P}_\rho^{\mathrm{FIN}},\quad \textit{in distri. in}\quad D(\mathbb{R}_+,\mathbb{R}^d).$$

Key Idea: To prove convergence of resistance.

To use renormalization map. (To analyze dynamics of the iteration of the map.)

(Developed late in 90's: Sabot, Metz, Peirone etc. (Very much involved!))

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Example 3: Heavy tailed RCM on the gasket converges to the FIN-diffusion.

Example 4: Heavy tailed RCM on critical E-R random graph

$\mathbf{P}(\mu_e^N \geq 1 \text{ or } 0) = 1, \quad \mathbf{P}(\mu_e^N = 1) = 1/N + \lambda N^{-4/3}, \mathbf{P}\left(\mu_e > u\right) = u^{-\alpha}, \forall u \geq 1.$

Then $\{N^{-1/3} Y^{\mathcal{C}^N}_{[N^{1/3+2/(3\alpha)}t]}\}_{t\geq 0} \xrightarrow{d} \{B^{\mathcal{M}}(\phi_\rho^{-1}(t))\}_{t\geq 0}$: FIN-diffusion on $\mathcal{M}$

(Need Croydon's result to characterize the limit (lack of UVD).)

Example 5:

Theorem 3.3 (Caputo-Faggionato '09) *When* $\rho > 1$*,*

$$\{n^{-1} X_{n^2 t}\}_{t\geq 0} \xrightarrow{d} \{\sigma B_t\}_{t\geq 0},$$

Here $\{B_t\}_{t\geq 0}$ *is Brownian motion and* $\sigma > 0$.

Theorem 3.4 (Croydon-Fukushima-Junk '20) *When* $\rho < 1$*,*

$$\{n^{-1} X_{n^{1+1/\rho} t}\}_{t\geq 0} \xrightarrow{d} \{Z_t\}_{t\geq 0},$$

where $\{Z_t\}_{t\geq 0}$ *is given by*

$$Z_t = S^{-1}(B_{H_t}).$$

Here $\{B_t\}_{t\geq 0}$ *is BM,* $\{S(u)\}_{u\in\mathbb{R}}$ *is a two-sided* ρ*-stable proc. (i.e.* $\{S(u)\}_{u\geq 0}$ *and* $\{-S(-u)\}_{u\geq 0}$ *are indep.* ρ*-stable proc. starting at* 0*).*

$H_t := \inf\{s \geq 0 : \int_{\mathbb{R}} L_s^B(x)\mu(dx) > t\}$*, where* $\{L_s^B(x)\}$ *is the local time of BM and* $\mu((a,b)) = c(S^{-1}(b) - S^{-1}(a))$*.*

In other word, $\{Z_t\}_{t\geq 0}$ is a generalized 1-dim diffusion with scale meas. S and speed measure dr (Note: S is dis-cont.).

- By discussing G-H convergence on (F, R, μ, ρ, Φ), where $\Phi : M \to \mathbb{R}$ is cont., one can obtain generalization of Thm 2.2(1) (Croydon '18).

Thm 3.4 can be proved by applying the theorem, and by proving the convergence of effective resistance.

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Thank you!