Superdiffusion of energy transport in a harmonic chain with a stochastic perturbation - the case of a closed system

Tomasz Komorowski, IMPAN, Warszawa,
S. Olla, CEREMADE, GSSI

19 marca 2026

Derivation of thermodynamics from statistical mechanics

Basic goal:

Mathematical derivation of principles of thermodynamics, starting from the laws of classical mechanics on the microscopic (atomic) level derive the macroscopic laws of of heat propagation

Derivation of the heat equation

Conservation law

$$\partial_t T(t,x) = -\nabla_x \cdot \mathbf{J}(t,x)$$

$T(t,x)$ -temperature, $\mathbf{J}(t,x)$ - heat current

Fourier's law - phenomenological

$$\mathbf{J}(t,x) = -\kappa(T)\nabla_x T(t,x)$$

$\kappa(T)$ - heat conductivity

- describes the macroscale observation (heat flows from hot to cold), not derived from first principles

Heat equation

Putting things together...

$$\begin{cases} \partial_t T(t,x) = \nabla_x \cdot \{\kappa(T) \nabla_x T(t,x)\}, \\ \\ T(0,x) = T_0(x), \end{cases}$$

Classical model of heat conduction: the chain of interacting oscillators

- Dimension $d = 1$ (both for simplicity and to arrive at **superdiffusion**).

- Atoms modelled by a **chain** of interacting oscillators; evolving according to the **Hamiltonian system**

$$\dot{\mathfrak{q}}_y(t) = \partial_{\mathfrak{p}_y}\mathcal{H}(\mathfrak{p}(t), \mathfrak{q}(t)) \tag{1}$$

$$\dot{\mathfrak{p}}_y(t) = -\partial_{\mathfrak{q}_y}\mathcal{H}(\mathfrak{p}(t), \mathfrak{q}(t)), \quad y \in \mathbb{Z}.$$

- $\mathfrak{p}(t) = (\mathfrak{p}_y(t))$ - **momenta**, $\mathfrak{q}(t) = (\mathfrak{q}_y(t))$ - **positions**

Classical model of heat conduction: the chain of interacting oscillators

- Dimension $d = 1$ (both for simplicity and to arrive at **superdiffusion**).

- Atoms modelled by a **chain** of interacting oscillators; evolving according to the **Hamiltonian system**

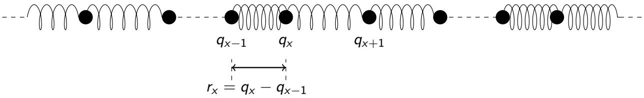
$$\dot{\mathfrak{q}}_y(t) = \partial_{\mathfrak{p}_y} \mathcal{H}(\mathfrak{p}(t), \mathfrak{q}(t)) \tag{1}$$

$$\dot{\mathfrak{p}}_y(t) = -\partial_{\mathfrak{q}_y} \mathcal{H}(\mathfrak{p}(t), \mathfrak{q}(t)), \quad y \in \mathbb{Z}.$$

- $\mathfrak{p}(t) = (\mathfrak{p}_y(t))$ - **momenta**, $\mathfrak{q}(t) = (\mathfrak{q}_y(t))$ - **positions**

Oscillator chain

(Formal) Hamiltonian: $\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \sum_{y\in\mathbb{Z}} \mathfrak{e}_y$,



Rysunek: Infinite oscillator chain

- $\mathfrak{e}_y := \frac{1}{2}\mathfrak{p}_y^2 + V(\mathfrak{q}_y - \mathfrak{q}_{y-1}) + U(\mathfrak{q}_y)$ - microscopic energy density (energy per atom),
- $V(\cdot)$ - interaction potential between molecules
- $U(\cdot)$ - the pinning potential

Unpinned system

- $V \equiv 0$ unpinned chain vs pinned chain, if otherwise
- in the unpinned case the system is translation invariant:

$$\mathfrak{q}_x \mapsto \mathfrak{q}_x + q, \quad q \in \mathbb{R}.$$

- coordinate to follow - stretch $\mathfrak{r}_x := \nabla^\star \mathfrak{q}_x$;

$$\nabla^\star f_x := f_x - f_{x-1}, \quad \nabla f_x = f_{x+1} - f_x, \quad \Delta f_x := f_{x+1} + f_{x-1} - 2f_x.$$

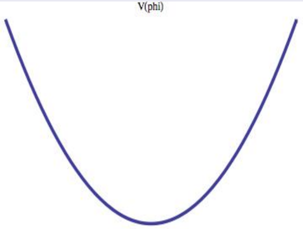
$$\frac{d\mathfrak{r}_y(t)}{dt} = \nabla^\star \mathfrak{p}_y(t), \quad y \in \mathbb{Z},$$
$$\frac{d\mathfrak{p}_y(t)}{dt} = \nabla U'\left(\mathfrak{q}_y(t) - \mathfrak{q}_{y-1}(t)\right)$$

Potentials

Typical potentials

- harmonic oscillator chain:

$$U(r) = \frac{1}{2} r^2 \qquad \text{system is linear!}$$

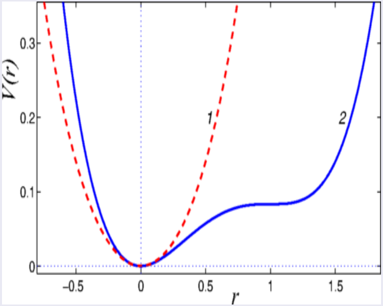


quadratic potential

FPUT - Potentials

$\alpha - \beta - FPUT$ (Fermi-Pasta-Ulam-Tsingou potential)

$$U(r) = \frac{1}{2}r^2 + \frac{\alpha}{3}r^3 + \frac{\beta}{4}r^4$$



red: $\alpha = 0$, $\beta = 0.25$ blue: $\alpha = -1$, $\beta = 0.25$

Constants of motion

In **unpinned case**, ($V \equiv 0$) the following quantities are constant:

$$\sum_x \mathfrak{r}_x(t) \qquad \text{- volume (mass),}$$

$$\sum_x \mathfrak{p}_x(t) \qquad \text{- momentum,}$$

$$\sum_x \mathfrak{e}_x(t) \qquad \text{- energy}$$

Ergodic conjecture

Generic conjecture: FPUT chain

For an anharmonic chain with a **quartic interaction potential** U (**FPUT chain**), as $n \longrightarrow +\infty$, **only these three quantities are conserved**

Macroscopic description

- **Scaled empirical distribution**

$$\varepsilon \sum_x J(\varepsilon x)\mathfrak{F}_x\left(\frac{t}{\varepsilon^\delta}\right), \quad J \in C_0^\infty(\mathbb{R})$$

where ε $(\ll 1)$ - the ratio of **micro/macroscopic** spatial scales

$$\mathfrak{F}_x(t) = (\mathfrak{r}_x(t), \mathfrak{p}_x(t), \mathfrak{e}_x(t))$$

- **Macroscopic scales:** existence of the limit

$$\lim_{\varepsilon \to 0+} \varepsilon \sum_x J(\varepsilon x)\mathfrak{F}_x\left(\frac{t}{\varepsilon^\delta}\right) = \int_{\mathbb{R}} J(u)\bar{F}(t,u)\,du,$$

Equations for $\bar{F}(t,u) := (\bar{r}(t,u), \bar{p}(t,u), \bar{e}(t,u))$

Stochastically perturbed chain of harmonic oscillators

- Chain of harmonic oscillators with a noise preserving the 3 constants of motion (Basile, Bernardin, Olla 09')
- **Hamiltonian**

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}}\mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}}\alpha_{y-y'}\mathfrak{q}_y\mathfrak{q}_{y'}$$

$$\begin{aligned} \frac{d\mathfrak{q}_y(t)}{dt} &= \mathfrak{p}_y(t), \\ d\mathfrak{p}_y(t) &= -(\alpha * \mathfrak{q}(t))_y dt \\ &+ \gamma^{1/2}\sum_{z=-1,0,1}(Y_{y+z}\mathfrak{p}_y(t))\circ dw_{y+z}(t), \quad y\in\mathbb{Z}. \end{aligned} \tag{2}$$

(in the Stratonovich sense) $\gamma > 0$ - noise intensity, $\{w_y(t),\ t \geqslant 0\}$, $y \in \mathbb{Z}$ i.i.d. 1-dim. standard Brownian motions

Stochastically perturbed chain of harmonic oscillators

- Chain of harmonic oscillators with a noise preserving the 3 constants of motion (Basile, Bernardin, Olla 09')
- Hamiltonian

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}}\mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}}\alpha_{y-y'}\mathfrak{q}_y\mathfrak{q}_{y'}$$

$$\begin{aligned} \frac{d\mathfrak{q}_y(t)}{dt} &= \mathfrak{p}_y(t), \\ d\mathfrak{p}_y(t) &= -(\alpha * \mathfrak{q}(t))_y dt \\ &+ \gamma^{1/2}\sum_{z=-1,0,1}(Y_{y+z}\mathfrak{p}_y(t))\circ dw_{y+z}(t), \quad y\in\mathbb{Z}. \end{aligned} \tag{2}$$

(in the Stratonovich sense) $\gamma > 0$ - noise intensity, $\{w_y(t),\, t \geqslant 0\}$, $y\in\mathbb{Z}$ i.i.d. 1-dim. standard Brownian motions

Stochastically perturbed chain of harmonic oscillators

- Chain of harmonic oscillators with a noise preserving the 3 constants of motion (Basile, Bernardin, Olla 09')
- Hamiltonian

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}}\mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}}\alpha_{y-y'}\mathfrak{q}_y\mathfrak{q}_{y'}$$

$$\begin{aligned}\frac{d\mathfrak{q}_y(t)}{dt} &= \mathfrak{p}_y(t),\\ d\mathfrak{p}_y(t) &= -(\alpha * \mathfrak{q}(t))_y dt\\ &+ \gamma^{1/2}\sum_{z=-1,0,1}(Y_{y+z}\mathfrak{p}_y(t))\circ dw_{y+z}(t), \quad y\in\mathbb{Z}.\end{aligned} \tag{2}$$

(in the Stratonovich sense) $\gamma > 0$ - noise intensity, $\{w_y(t),\ t \geqslant 0\}$, $y \in \mathbb{Z}$ i.i.d. 1-dim. standard Brownian motions

Stochastically perturbed chain of harmonic oscillators

- Chain of harmonic oscillators with a noise preserving the 3 constants of motion (Basile, Bernardin, Olla 09')
- Hamiltonian

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}}\mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}}\alpha_{y-y'}\mathfrak{q}_y\mathfrak{q}_{y'}$$

$$\begin{aligned} \frac{d\mathfrak{q}_y(t)}{dt} &= \mathfrak{p}_y(t), \\ d\mathfrak{p}_y(t) &= -(\alpha * \mathfrak{q}(t))_y dt \\ &+ \gamma^{1/2}\sum_{z=-1,0,1}(Y_{y+z}\mathfrak{p}_y(t))\circ dw_{y+z}(t), \quad y\in\mathbb{Z}. \end{aligned} \tag{2}$$

(in the Stratonovich sense) $\gamma > 0$ - noise intensity, $\{w_y(t),\ t\geqslant 0\}$, $y\in\mathbb{Z}$ i.i.d. 1-dim. standard Brownian motions

Stochastically perturbed chain of harmonic oscillators

- Chain of harmonic oscillators with a noise preserving the 3 constants of motion (Basile, Bernardin, Olla 09')
- Hamiltonian

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}}\mathfrak{p}_y^2 + \frac{1}{2}\sum_{y,y'\in\mathbb{Z}}\alpha_{y-y'}\mathfrak{q}_y\mathfrak{q}_{y'}$$

$$\begin{aligned} \frac{d\mathfrak{q}_y(t)}{dt} &= \mathfrak{p}_y(t), \\ d\mathfrak{p}_y(t) &= -(\alpha * \mathfrak{q}(t))_y dt \\ &+ \gamma^{1/2}\sum_{z=-1,0,1}(Y_{y+z}\mathfrak{p}_y(t)) \circ dw_{y+z}(t), \quad y\in\mathbb{Z}. \end{aligned} \tag{2}$$

(in the Stratonovich sense) $\gamma > 0$ - noise intensity, $\{w_y(t),\, t \geqslant 0\}$, $y \in \mathbb{Z}$ i.i.d. 1-dim. standard Brownian motions

- Vector fields

$$
\begin{aligned}
Y_x &:= \nabla\mathfrak{p}_x\nabla\partial_{\mathfrak{p}_{x-1}} - \nabla\mathfrak{p}_{x-1}\nabla\partial_{\mathfrak{p}_x}\\
&= (\mathfrak{p}_x - \mathfrak{p}_{x+1})\partial_{\mathfrak{p}_{x-1}} + (\mathfrak{p}_{x+1} - \mathfrak{p}_{x-1})\partial_{\mathfrak{p}_x} + (\mathfrak{p}_{x-1} - \mathfrak{p}_x)\partial_{\mathfrak{p}_{x+1}}
\end{aligned}
$$

tangent to the circle and the plane

$$
\mathfrak{p}_{x-1}^2 + \mathfrak{p}_x^2 + \mathfrak{p}_{x+1}^2 \equiv \text{const}, \qquad \mathfrak{p}_{x-1} + \mathfrak{p}_x + \mathfrak{p}_{x+1} \equiv \text{const},
$$

- **Nearest neighbor interaction:** $\alpha_0 = 2 + \omega_0^2$, $\alpha_1 = \alpha_{-1} = -1$, $\alpha_y = 0$, if $|y| \geqslant 2$

$$
\mathcal{H}(\mathfrak{p}, \mathfrak{q}) := \frac{1}{2}\sum_{y\in\mathbb{Z}} \mathfrak{p}_y^2 + \frac{1}{2}\sum_{y\in\mathbb{Z}} \left(\omega_0^2\mathfrak{q}_y^2 + \mathfrak{r}_y^2\right)
$$

$\omega = 0$ - the chain is **unpinned**

Itô form...

$$
\begin{aligned}
\frac{d\mathfrak{q}_y(t)}{dt} &= \mathfrak{p}_y(t), \qquad (3)\\
d\mathfrak{p}_y(t) &= \left[-(\alpha * \mathfrak{q}(t))_y - \frac{\gamma}{2}(\beta * \mathfrak{p}(t))_y\right] dt,\\
&+\gamma^{1/2} \sum_{z=-1,0,1} (Y_{y+z}\mathfrak{p}_y(t)) dw_{y+z}(t), \quad y \in \mathbb{Z}.
\end{aligned}
$$

$$\beta_y = \Delta\beta_y^{(0)} = \beta_{y-1}^{(0)} + \beta_{y-1}^{(0)} - 2\beta_y^{(0)}$$

$$\beta_y^{(0)} = \begin{cases} -4, & y = 0 \\ -1, & y = \pm 1 \\ 0, & \text{if otherwise.} \end{cases}$$

Itô form...

$$\begin{aligned} \frac{d\mathfrak{q}_y(t)}{dt} &= \mathfrak{p}_y(t), \\ d\mathfrak{p}_y(t) &= \left[-(\alpha * \mathfrak{q}(t))_y - \frac{\gamma}{2}(\beta * \mathfrak{p}(t))_y\right] dt, \\ &+\gamma^{1/2} \sum_{z=-1,0,1} (Y_{y+z}\mathfrak{p}_y(t))dw_{y+z}(t), \quad y \in \mathbb{Z}. \end{aligned} \tag{3}$$

$$\beta_y = \Delta\beta_y^{(0)} = \beta_{y-1}^{(0)} + \beta_{y-1}^{(0)} - 2\beta_y^{(0)}$$

$$\beta_y^{(0)} = \begin{cases} -4, & y = 0 \\ -1, & y = \pm 1 \\ 0, & \text{if otherwise.} \end{cases}$$

Other linear models. Random momentum exchange

$$\begin{aligned} d\mathfrak{p}_y(t) &= -(\alpha * \mathfrak{q}(t))_y dt \\ &+ \left[\nabla \mathfrak{p}_y(t-) dN_{y,y+1}(\gamma t) + \nabla^* \mathfrak{p}_y(t-) dN_{y-1,y}(\gamma t)\right], \quad y \in \mathbb{Z}. \end{aligned} \tag{4}$$

$(N_{x,x+1}(t))$ - i.i.d. Poisson processes with intensity 1, the exchange intensity $\gamma > 0$.

Assumptions about potential

After Basile, Olla, Spohn (2009):

a1) $\alpha_y \in \mathbb{R}$, $y \in \mathbb{Z}$ and $\exists C > 0 : |\alpha_y| \leqslant Ce^{-|y|/C}$, $y \in \mathbb{Z}$,

a2) $\alpha_{-y} = \alpha_y$,

a3) $\hat{\alpha}(k) = \sum_{y\in\mathbb{Z}} \alpha_y e^{2\pi iky}$ satisfies $\hat{\alpha}(k) > 0$ for $k \neq 0$ and **if** $\hat{\alpha}(0) = 0$ **then** $\hat{\alpha}''(0) > 0$.

both $y \mapsto \alpha_y$ and $k \mapsto \hat{\alpha}(k)$ are even, $\hat{\alpha} \in C^\infty(\mathbb{T})$.

Dispersion relation

$$\omega(k) := \hat{\alpha}^{1/2}(k).$$

Assumptions about potential

After Basile, Olla, Spohn (2009):

a1) $\alpha_y \in \mathbb{R}$, $y \in \mathbb{Z}$ and $\exists C > 0 : |\alpha_y| \leqslant Ce^{-|y|/C}$, $y \in \mathbb{Z}$,

a2) $\alpha_{-y} = \alpha_y$,

a3) $\hat{\alpha}(k) = \sum_{y\in\mathbb{Z}} \alpha_y e^{2\pi iky}$ satisfies $\hat{\alpha}(k) > 0$ for $k \neq 0$ and **if** $\hat{\alpha}(0) = 0$ **then** $\hat{\alpha}''(0) > 0$.

both $y \mapsto \alpha_y$ and $k \mapsto \hat{\alpha}(k)$ are even, $\hat{\alpha} \in C^\infty(\mathbb{T})$.

Dispersion relation

$$\omega(k) := \hat{\alpha}^{1/2}(k).$$

Assumptions about potential

After Basile, Olla, Spohn (2009):

a1) $\alpha_y \in \mathbb{R}$, $y \in \mathbb{Z}$ and $\exists C > 0 : |\alpha_y| \leqslant Ce^{-|y|/C}$, $y \in \mathbb{Z}$,

a2) $\alpha_{-y} = \alpha_y$,

a3) $\hat{\alpha}(k) = \sum_{y \in \mathbb{Z}} \alpha_y e^{2\pi i k y}$ satisfies $\hat{\alpha}(k) > 0$ for $k \neq 0$ and
if $\hat{\alpha}(0) = 0$ **then** $\hat{\alpha}''(0) > 0$.

both $y \mapsto \alpha_y$ and $k \mapsto \hat{\alpha}(k)$ are even, $\hat{\alpha} \in C^\infty(\mathbb{T})$.

Dispersion relation

$$\omega(k) := \hat{\alpha}^{1/2}(k).$$

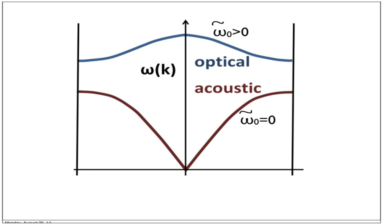
Dispersion relation in the case of nearest neighbor interaction

Example: $\alpha_{-1} = \alpha_1 = -1$, $\alpha_0 := 2 + \omega_0^2$, $\alpha_y = 0$, $|y| \geqslant 2$,

$$\omega^2(k) = \hat{\alpha}(k) = 2 + \omega_0^2 - (e^{2\pi ik} + e^{-2\pi ik}) = \omega_0^2 + 4\sin^2(\pi k)$$

Dispersion relation for nearest neighbor interaction on $\mathbb{Z}$

$\omega(k) = \sqrt{\omega_0^2 + 4\sin^2(\pi k)}$ - the Fourier symbol of $\omega_0^2 - \Delta$



Monday, August 25, 14
sobota, 13 września 2014

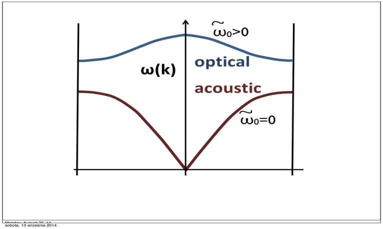
Dispersion relation in the case of nearest neighbor interaction

Example: $\alpha_{-1} = \alpha_1 = -1$, $\alpha_0 := 2 + \omega_0^2$, $\alpha_y = 0$, $|y| \geqslant 2$,

$$\omega^2(k) = \hat{\alpha}(k) = 2 + \omega_0^2 - (e^{2\pi ik} + e^{-2\pi ik}) = \omega_0^2 + 4\sin^2(\pi k)$$

Dispersion relation for nearest neighbor interaction on $\mathbb{Z}$

$$\omega(k) = \sqrt{\omega_0^2 + 4\sin^2(\pi k)}$$ - the Fourier symbol of $\omega_0^2 - \Delta$



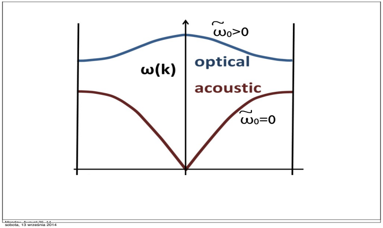
Dispersion relation in the case of nearest neighbor interaction

Example: $\alpha_{-1} = \alpha_1 = -1$, $\alpha_0 := 2 + \omega_0^2$, $\alpha_y = 0$, $|y| \geqslant 2$,

$$\omega^2(k) = \hat{\alpha}(k) = 2 + \omega_0^2 - (e^{2\pi ik} + e^{-2\pi ik}) = \omega_0^2 + 4\sin^2(\pi k)$$

Dispersion relation for nearest neighbor interaction on $\mathbb{Z}$

$$\omega(k) = \sqrt{\omega_0^2 + 4\sin^2(\pi k)}$$ - the Fourier symbol of $\omega_0^2 - \Delta$



Microscopic energy density

$$\mathfrak{e}_x(t) := \frac{\mathfrak{p}_x^2(t)}{2} - \frac{1}{4}\sum_y \alpha_{x-y}(\mathfrak{q}_x(t) - \mathfrak{q}_y(t))^2 + \frac{\hat{\alpha}(0)}{2}\mathfrak{q}_x^2(t).$$

- $\hat{\alpha}(0) = 0$ - unpinned chain ,
- $\hat{\alpha}(0) > 0$ - pinned chain

- the nearest neighbor interaction ($\mathfrak{r}_x = \mathfrak{q}_x - \mathfrak{q}_{x-1}$ - stretch)

$$\mathfrak{e}_x(t) := \frac{\mathfrak{p}_x^2(t)}{2} + \frac{\omega_0^2\mathfrak{q}_x^2(t)}{2} + \frac{\mathfrak{r}_x^2(t)}{2}.$$

Initial data I

- initial data **random**, distributed according to μ_ε on $\Omega := \left(\mathbb{R}^{\mathbb{Z}}\right)^2$

$$\limsup_{\varepsilon\to 0+} \varepsilon \sum_x \langle \mathfrak{e}_x^2(0)\rangle_{\mu_\varepsilon} < +\infty,$$

$\langle\cdot\rangle_{\mu_\varepsilon}$ - the expectation w.r.t. μ_ε,

- the existence of the **initial profile** of energy

$$\lim_{\varepsilon\to 0+} \varepsilon \sum_y J(\varepsilon y)\langle \mathfrak{e}_y(0)\rangle_{\mu_\varepsilon} = \int_{\mathbb{R}} J(u)\bar{e}_0(u)du.$$

Pinned chain, diffusive scaling: $t' \sim t/\varepsilon^2$, $x' := x/\varepsilon$, $\hat{\alpha}(0) > 0$

Theorem 2

Suppose that:

- i) $\hat{\alpha}(0) > 0$ (then $\omega(k)$ is differentiable at 0 and $\omega'(0) = 0$)
- ii) $\gamma > 0$,
- iii) some additional assumptions about the initial data

Then:

$$\lim_{\varepsilon \to 0+} \varepsilon \sum_x \int_0^{+\infty} J(t, \varepsilon x)\mathbb{E}_\varepsilon \mathfrak{e}_x\left(\frac{t}{\varepsilon^2}\right) dt = \int_0^{+\infty} \int_{\mathbb{R}} W(t,u)J(t,u)\,dt\,du,$$

and $W(t,u)$ satisfies:

$$\partial_t W(t,u) = D\partial_u^2 W(t,u), \qquad W(0,u) = \bar{e}_0(u)$$

Pinned chain, diffusive scaling: $t' \sim t/\varepsilon^2$, $x' := x/\varepsilon$, $\hat{\alpha}(0) > 0$

Theorem 2

Suppose that:

- i) $\hat{\alpha}(0) > 0$ (then $\omega(k)$ is differentiable at 0 and $\omega'(0) = 0$)
- ii) $\gamma > 0$,
- iii) some additional assumptions about the initial data

Then:

$$\lim_{\varepsilon \to 0+} \varepsilon \sum_x \int_0^{+\infty} J(t, \varepsilon x) \mathbb{E}_\varepsilon \mathfrak{e}_x \left(\frac{t}{\varepsilon^2} \right) dt = \int_0^{+\infty} \int_{\mathbb{R}} W(t, u) J(t, u) dt du,$$

and $W(t, u)$ satisfies:

$$\partial_t W(t, u) = D \partial_u^2 W(t, u), \qquad W(0, u) = \bar{e}_0(u)$$

Pinned chain, diffusive scaling

Theorem 2, cont.

Here

$$D = \frac{\hat{\sigma}^2}{\gamma} + 8\gamma\pi^2 \qquad \text{dla} \quad s = 0,$$

and

$$\hat{\sigma}^2 := \int_{\mathbb{T}} \frac{[2\omega'(k)]^2}{\hat{\beta}(k)} dk, \tag{5}$$

$$\hat{\beta}(k) = 8\sin^2(\pi k)\left[1 + 2\cos^2(\pi k)\right] \approx k^2, \quad |k| \ll 1. \tag{6}$$

Unpinned chain, superdiffusive scaling: $t' \sim t/\varepsilon^{3/2}$, $x' \sim x/\varepsilon$

If $\hat{\alpha}(0) = 0$ and $\hat{\alpha}''(0) > 0$ (acoustic chain) then

$$\omega(k) = 2\sqrt{\tau_1}|\sin(\pi k)|\varphi(\sin^2(\pi k)), \tag{7}$$

where $\varphi : [0, +\infty) \to (0, +\infty)$ is of class C^2 and $\varphi(0) = 1$. Then $\omega'(k) \approx \operatorname{sign} k$ and

$$\hat{\sigma}^2 := \int_{\mathbb{T}} \frac{[2\omega'(k)]^2}{\hat{\beta}(k)} dk \approx \int_{0+} \frac{dk}{k^2} = +\infty,$$

$\Rightarrow$ indicates superdiffusive scaling

Unpinned chain, superdiffusive scaling

Theorem 3

Suppose that $(\gamma > 0$ - **noise intensity**$)$. Then, under appropriate assumptions about the initial data:

$$\lim_{\varepsilon\to 0+} \varepsilon \sum_y \int_0^{+\infty} J(t,\varepsilon y)\mathbb{E}_\varepsilon \mathfrak{e}_y\left(\frac{t}{\varepsilon^{3/2}}\right)dt = \int_0^{+\infty}\int_{\mathbb{R}} W(t,u)J(t,u)dtdu, \tag{8}$$

where $W(t,u)$ satisfies **the fractional diffusion equation**:

$$\begin{cases} \partial_t W(t,u) = -\hat{c}|\Delta_u|^{3/4}W(t,u), \\ W(0,u) = \bar{e}_0(u), \end{cases}$$

$$\hat{c} := \frac{[\alpha''(0)]^{3/4}}{2^{9/4}(3\gamma)^{1/2}}. \tag{9}$$

Initial data II

- **the initial data is random**, distributed according to μ_ε and

$$K(0) := \sup_{\varepsilon\in(0,1]} \varepsilon \sum_x \langle \mathfrak{e}_x(0)\rangle_{\mu_\varepsilon} < +\infty. \tag{10}$$

- due to the energy conservation: $K(t) \equiv K(0)$ for all $t \geqslant 0$,
- the pinned case; the energy spectrum of a configuration $(\mathfrak{q}, \mathfrak{p})$

$$\mathfrak{w}_\varepsilon(k) := \left\langle |\hat{\mathfrak{p}}(k)|^2 \right\rangle_{\mu_\varepsilon} + \hat{\alpha}(k) \left\langle |\hat{\mathfrak{q}}(k)|^2 \right\rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$$

$\hat{\mathfrak{p}}(k) = \sum_x \mathfrak{p}_x e^{2\pi ikx}$ and $\hat{\mathfrak{q}}(k) = \sum_x \mathfrak{q}_x e^{2\pi ikx}$

- in the unpinned case;

$\mathfrak{w}_\varepsilon(k) = \langle |\hat{\mathfrak{p}}(k)|^2 \rangle_{\mu_\varepsilon} + \frac{\hat{\alpha}(k)}{4\sin^2(\pi k)} \langle |\hat{\mathfrak{r}}(k)|^2 \rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$

Initial data II

- the initial data is random, distributed according to μ_ε and

$$K(0) := \sup_{\varepsilon\in(0,1]} \varepsilon \sum_x \langle \mathfrak{e}_x(0)\rangle_{\mu_\varepsilon} < +\infty. \tag{10}$$

- due to the energy conservation: $K(t) \equiv K(0)$ for all $t \geqslant 0$,
- the pinned case; the energy spectrum of a configuration $(\mathfrak{q}, \mathfrak{p})$

$$\mathfrak{w}_\varepsilon(k) := \left\langle |\hat{\mathfrak{p}}(k)|^2 \right\rangle_{\mu_\varepsilon} + \hat{\alpha}(k) \left\langle |\hat{\mathfrak{q}}(k)|^2 \right\rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$$

$\hat{\mathfrak{p}}(k) = \sum_x \mathfrak{p}_x e^{2\pi i k x}$ and $\hat{\mathfrak{q}}(k) = \sum_x \mathfrak{q}_x e^{2\pi i k x}$

- in the unpinned case;

$\mathfrak{w}_\varepsilon(k) = \langle |\hat{\mathfrak{p}}(k)|^2 \rangle_{\mu_\varepsilon} + \frac{\hat{\alpha}(k)}{4\sin^2(\pi k)} \langle |\hat{\mathfrak{r}}(k)|^2 \rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$

Initial data II

- the initial data is random, distributed according to μ_ε and

$$K(0) := \sup_{\varepsilon\in(0,1]} \varepsilon \sum_x \langle \mathfrak{e}_x(0)\rangle_{\mu_\varepsilon} < +\infty. \tag{10}$$

- due to the energy conservation: $K(t) \equiv K(0)$ for all $t \geqslant 0$,
- the pinned case; the energy spectrum of a configuration $(\mathfrak{q}, \mathfrak{p})$

$$\mathfrak{w}_\varepsilon(k) := \left\langle |\hat{\mathfrak{p}}(k)|^2 \right\rangle_{\mu_\varepsilon} + \hat{\alpha}(k) \left\langle |\hat{\mathfrak{q}}(k)|^2 \right\rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$$

$\hat{\mathfrak{p}}(k) = \sum_x \mathfrak{p}_x e^{2\pi ikx}$ and $\hat{\mathfrak{q}}(k) = \sum_x \mathfrak{q}_x e^{2\pi ikx}$

- in the unpinned case;

$$\mathfrak{w}_\varepsilon(k) = \left\langle |\hat{\mathfrak{p}}(k)|^2 \right\rangle_{\mu_\varepsilon} + \frac{\hat{\alpha}(k)}{4\sin^2(\pi k)} \left\langle |\hat{\mathfrak{r}}(k)|^2 \right\rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$$

Initial data II

- the initial data is random, distributed according to μ_ε and

$$K(0) := \sup_{\varepsilon\in(0,1]} \varepsilon \sum_x \langle \mathfrak{e}_x(0)\rangle_{\mu_\varepsilon} < +\infty. \tag{10}$$

- due to the energy conservation: $K(t) \equiv K(0)$ for all $t \geqslant 0$,
- the pinned case; the energy spectrum of a configuration $(\mathfrak{q}, \mathfrak{p})$

$$\mathfrak{w}_\varepsilon(k) := \left\langle |\hat{\mathfrak{p}}(k)|^2 \right\rangle_{\mu_\varepsilon} + \hat{\alpha}(k) \left\langle |\hat{\mathfrak{q}}(k)|^2 \right\rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$$

$\hat{\mathfrak{p}}(k) = \sum_x \mathfrak{p}_x e^{2\pi ikx}$ and $\hat{\mathfrak{q}}(k) = \sum_x \mathfrak{q}_x e^{2\pi ikx}$

- in the unpinned case;

$\mathfrak{w}_\varepsilon(k) = \langle |\hat{\mathfrak{p}}(k)|^2 \rangle_{\mu_\varepsilon} + \dfrac{\hat{\alpha}(k)}{4\sin^2(\pi k)} \langle |\hat{\mathfrak{r}}(k)|^2 \rangle_{\mu_\varepsilon}, \quad k \in \mathbb{T}.$

Initial data II, square integrability

- $$K(0) = \sup_{\varepsilon\in(0,1]} \varepsilon \int_{\mathbb{T}} \mathfrak{w}_\varepsilon(k)dk < +\infty. \qquad (11)$$

Square integrability condition

$$\sup_{\varepsilon\in(0,1]} \varepsilon^2 \int_{\mathbb{T}} \mathfrak{w}_\varepsilon^2(k)dk < +\infty.$$

- the existence of the macroscopic initial energy profile

$$\lim_{\varepsilon\to 0+} \varepsilon \sum_x J(\varepsilon x)\mathbb{E}_\varepsilon \mathfrak{e}_x(0) = \int_{\mathbb{R}} \bar{e}_0(u)J(u)dy, \quad J\in C_0^\infty(\mathbb{R}). \qquad (12)$$

Initial data II, square integrability

- $$K(0)=\sup_{\varepsilon\in(0,1]}\varepsilon\int_{\mathbb{T}}\mathfrak{w}_{\varepsilon}(k)dk<+\infty. \tag{11}$$

Square integrability condition

$$\sup_{\varepsilon\in(0,1]}\varepsilon^2\int_{\mathbb{T}}\mathfrak{w}^2_{\varepsilon}(k)dk<+\infty.$$

- the existence of the macroscopic initial energy profile

$$\lim_{\varepsilon\to0+}\varepsilon\sum_{x}J(\varepsilon x)\mathbb{E}_{\varepsilon}\mathfrak{e}_x(0)=\int_{\mathbb{R}}\bar{e}_0(u)J(u)dy,\quad J\in C_0^{\infty}(\mathbb{R}). \tag{12}$$

Initial data II, square integrability

- $$K(0) = \sup_{\varepsilon\in(0,1]} \varepsilon \int_{\mathbb{T}} \mathfrak{w}_\varepsilon(k)dk < +\infty. \tag{11}$$

Square integrability condition

$$\sup_{\varepsilon\in(0,1]} \varepsilon^2 \int_{\mathbb{T}} \mathfrak{w}_\varepsilon^2(k)dk < +\infty.$$

- the existence of the **macroscopic initial energy profile**

$$\lim_{\varepsilon\to 0+} \varepsilon \sum_x J(\varepsilon x)\mathbb{E}_\varepsilon \mathfrak{e}_x(0) = \int_{\mathbb{R}} \bar{e}_0(u)J(u)dy, \quad J \in C_0^\infty(\mathbb{R}). \tag{12}$$

Remark: deterministic initial data

- unpinned case, μ_ε - delta type measures concentrated on $\left(r_0(\varepsilon x), p_0(\varepsilon x)\right)_{x\in\mathbb{Z}}$, $r_0, p_0 \in C_c^\infty(\mathbb{R})$.

$$\lim_{\varepsilon\to 0+} \varepsilon \sum_x \mathfrak{e}_x = \frac{1}{2} \lim_{\varepsilon\to 0+} \varepsilon \sum_x \left(p_0^2(\varepsilon x) + r_0^2(\varepsilon x)\right)$$
$$= \frac{1}{2} \int_{\mathbb{R}} \left(p_0^2(u)^2 + r_0^2(u)\right) du = \mathcal{E}_{\mathrm{tot}}.$$

- Parseval identity $\Rightarrow$

$$\frac{\varepsilon}{2} \int_{\mathbb{T}} \mathfrak{w}_\varepsilon(k) = \frac{\varepsilon}{2} \left(\left\langle |\hat{\mathfrak{p}}(k)|^2 \right\rangle_{\mu_\varepsilon} + \underbrace{\frac{\hat{\alpha}(k)}{4\sin^2(\pi k)}}_{=1} \left\langle |\hat{\mathfrak{r}}(k)|^2 \right\rangle_{\mu_\varepsilon} \right)$$
$$= \frac{1}{2} \varepsilon \sum_x \left(p_0^2(\varepsilon x) + r_0^2(\varepsilon x)\right) \to \frac{1}{2} \int_{\mathbb{R}} \left(p_0^2(u) + r_0^2(u)\right) du = \mathcal{E}_{\mathrm{tot}}.$$

Remark: square integrability of the energy spectrum. The case of deterministic initial data.

- Recall

$$\varepsilon^2 \int_{\mathbb{T}} \mathfrak{w}_\varepsilon^2(k)dk = \varepsilon^2 \int_{\mathbb{T}} \left(\left\langle |\hat{\mathfrak{p}}(k)|^2 \right\rangle_{\mu_\varepsilon} + \left\langle |\hat{\mathfrak{r}}(k)|^2 \right\rangle_{\mu_\varepsilon} \right)^2 dk$$

- Fourier transform of $(p_0(\varepsilon x))$ $(p_0 \in C_c^\infty(\mathbb{R}))$

$$\hat{\mathfrak{p}}_\varepsilon(k) = \sum_{x\in\mathbb{Z}} \exp\{-2\pi ikx\} \, p_0(\varepsilon x) = \frac{1}{\varepsilon} \sum_{m\in\mathbb{Z}} \hat{p}_0\Big(\frac{k+m}{\varepsilon}\Big)$$

$$= \frac{1}{\varepsilon}\hat{p}_0\Big(\frac{k}{\varepsilon}\Big) + o\Big(\frac{1}{\varepsilon^N}\Big).$$

for arbitrary $N \geqslant 1$. Then,

$$\varepsilon^2 \int_{\mathbb{T}} \mathfrak{w}_\varepsilon^2(k)dk \sim \varepsilon^2 \int_{\mathbb{T}} |\hat{\mathfrak{p}}_\varepsilon(k)|^4 dk \sim \frac{1}{\varepsilon^2}.$$

- For deterministic data the square integrability of the energy spectrum necessarily fails!

Remark: square integrability of the energy spectrum. The case of random initial data.

- suppose that under μ_ε - random variables $(r_x, p_x)_{x\in\mathbb{Z}}$ independent (also from each other) of mean 0 and with the second moments

$$\langle \mathfrak{p}_x^2\rangle_{\mu_\varepsilon} = p_0^2(\varepsilon x), \quad \langle \mathfrak{r}_x^2\rangle_{\mu_\varepsilon} = r_0^2(\varepsilon x), \quad x\in\mathbb{Z},$$

where $r_0, p_0 \in C_c^\infty(\mathbb{R})$.

$$\lim_{\varepsilon\to 0+} \varepsilon \sum_x \langle \mathfrak{e}_x\rangle_{\mu_\varepsilon} = \frac{1}{2}\int_{\mathbb{R}} \sum_x \left(p_0^2(u) + r_0^2(u)\right) du.$$

Square integrability of the energy spectrum for random initial data.

$$\begin{aligned}
&\varepsilon^2\int_{\mathbb{T}}\langle|\hat{\mathfrak{p}}_\varepsilon(k)|^2\rangle_{\mu_\varepsilon}^2\,dk=\varepsilon^2\int_{\mathbb{T}}\langle|\sum_x e^{-2\pi ikx}\mathfrak{p}_x|^2\rangle_{\mu_\varepsilon}^2\,dk\\
&=\varepsilon^2\int_{\mathbb{T}}\langle|\sum_x e^{-2\pi ikx}\mathfrak{p}_x|^2\rangle_{\mu_\varepsilon}^2\,dk=\varepsilon^2\int_{\mathbb{T}}\Big(\sum_x\langle\mathfrak{p}_x^2\rangle_{\mu_\varepsilon}\Big)^2dk\\
&=\int_{\mathbb{T}}\Big(\varepsilon\sum_x p_0^2(\varepsilon x)\Big)^2dk\to\Big(\int_{\mathbb{R}}p_0^2(u)du\Big)^2<+\infty
\end{aligned}$$

- the square integrability of the energy spectrum condition holds for random initial data with sufficiently fast decorrelation

About the proof

- the Wigner functions (Basile, Olla, Spohn (09'))
- wave function:

$$\psi_x^{(\varepsilon)}(t) := \left(\tilde{\omega} * \mathfrak{q}\left(\frac{t}{\varepsilon^\delta}\right)\right)_x + i\mathfrak{p}_x\left(\frac{t}{\varepsilon^\delta}\right), \tag{13}$$

- $(\tilde{\omega}_y)$, $y \in \mathbb{Z}$ the inverse Fourier transform of the dispersion relation $\omega(k) := \sqrt{\hat{\alpha}(k)}$
- the Fourier transform of the wave function:

$$\hat{\psi}^{(\varepsilon)}(t,k) = \omega(k)\hat{\mathfrak{q}}\left(\frac{t}{\varepsilon^\delta}, k\right) + i\hat{\mathfrak{p}}\left(\frac{t}{\varepsilon^\delta}, k\right). \tag{14}$$

Wigner transform

The (averaged) Wigner function

The (averaged) Wigner transform of the law of $\psi^{(\varepsilon)}(t)$ on ℓ^2

$$\langle W_\varepsilon(t), J\rangle$$
$$:= \frac{\varepsilon}{2}\int_{\mathbb{R}\times\mathbb{T}} \mathbb{E}_\varepsilon\left[(\hat{\psi}^{(\varepsilon)})^*\left(t, k-\frac{\varepsilon p}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t, k+\frac{\varepsilon p}{2}\right)\right]\hat{J}^*(p,k)dpdk.$$

$$\hat{J}(p,k) := \int_{\mathbb{R}} e^{2\pi i p u} J(u,k)du$$

Wigner function cont'd

- Resolution of the energy density in the spatial and mode co-ordinates

$$\frac{\varepsilon}{2}\sum_x J_1(\varepsilon x)\mathbb{E}_\varepsilon|\psi_x^{(\varepsilon)}(t)|^2 = \langle W_\varepsilon(t), J_1\rangle \tag{15}$$

and

$$\int_{\mathbb{T}} J_2(k)\mathbb{E}_\varepsilon|\hat{\psi}^{(\varepsilon)}(t,k)|^2 dk = \langle W_\varepsilon(t), J_2\rangle \tag{16}$$

for any $J_1 \in C_0^\infty(\mathbb{R})$, $J_2 \in C^\infty(\mathbb{T})$.

Equivalence of the averaged energy density functionals

Proposition

- **Assume:**

$$K_0 := \sup_{\varepsilon\in(0,1]} \frac{\varepsilon}{2}\left\langle \|(\mathbf{q},\mathbf{p})\|^2_{\ell^2}\right\rangle_{\mu_\varepsilon} < +\infty, \tag{17}$$

$$\lim_{L\to+\infty}\sup_{\varepsilon\in(0,1]} \frac{\varepsilon}{2}\langle \|(\hat{\mathbf{q}},\hat{\mathbf{p}})_L\|^2_{L^2(\mathbb{T})}\rangle_{\mu_\varepsilon} = 0, \tag{18}$$

where $\hat{f}_L(k) := \hat{f}(k)1_{[|\hat{f}(k)|\geqslant L]}$.

- **Then,**

$$\lim_{\varepsilon\to0+}\varepsilon\sum_x J(\varepsilon x)\mathbb{E}_\varepsilon\left[\mathfrak{e}_x\left(\frac{t}{\varepsilon^\delta}\right)-\frac{1}{2}\left|\psi_x^{(\varepsilon)}(t)\right|^2\right]=0 \tag{19}$$

for any $J\in C_0^\infty(\mathbb{R})$ and $t\geqslant 0$.

Limit identification

- Conservation of energy ⇒

$$\frac{\varepsilon}{2}\sup_{p\in\mathbb{R}}\int_{\mathbb{T}}|W_\varepsilon(t,p,k)|dk \leqslant \frac{\varepsilon}{2}\|\psi_\varepsilon(t)\|^2_{L^2(\mathbb{T})} = \mathfrak{w}_\varepsilon(t,k) = \mathfrak{w}_\varepsilon(0,k).$$

- ⇒ $(W_\varepsilon(t))$ *-weakly sequentially compact in $L^\infty([0,T],\mathcal{A}')$ for any $T>0$, where $\mathcal{A}$ is the completion of $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ in the norm

$$\|J\|_{\mathcal{A}} := \int_{\mathbb{R}}\sup_k|\hat{J}(p,k)|dp, \tag{20}$$

- for any $\varepsilon_n\to 0$, as $n\to+\infty$, we can choose a subsequence such that *-weakly converges to some $W(\cdot)\in L^\infty([0,T],\mathcal{A}')$,

- ⇒ limit identification problem for $(W_\varepsilon(t))$, as $\varepsilon\to 0+$.

Limit identification

- Conservation of energy ⇒

$$\frac{\varepsilon}{2}\sup_{p\in\mathbb{R}}\int_{\mathbb{T}}|W_\varepsilon(t,p,k)|dk \leqslant \frac{\varepsilon}{2}\|\psi_\varepsilon(t)\|^2_{L^2(\mathbb{T})} = \mathfrak{w}_\varepsilon(t,k) = \mathfrak{w}_\varepsilon(0,k).$$

- ⇒ $(W_\varepsilon(t))$ *-weakly sequentially compact in $L^\infty([0,T],\mathcal{A}')$ for any $T>0$, where $\mathcal{A}$ is the completion of $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ in the norm

$$\|J\|_{\mathcal{A}} := \int_{\mathbb{R}}\sup_k|\hat{J}(p,k)|dp, \qquad (20)$$

- for any $\varepsilon_n\to 0$, as $n\to+\infty$, we can choose a subsequence such that *-weakly converges to some $W(\cdot)\in L^\infty([0,T],\mathcal{A}')$,

- ⇒ **limit identification problem** for $(W_\varepsilon(t))$, as $\varepsilon\to 0+$.

Dynamics

- the dynamics of the **(Averaged) Fourier-Wigner functions**

$$\partial_t \widehat{W}_\varepsilon = -\frac{i\delta_\varepsilon\omega}{\varepsilon^{\delta-1}}\widehat{W}_\varepsilon + \frac{\gamma}{\varepsilon^{\delta}}\left(\mathcal{L} + \frac{(\varepsilon p)^2}{2}(\delta^2\mathcal{L})\right)\Big[\widehat{W}_\varepsilon - \underbrace{\frac{1}{2}\left(\widehat{Y}_{\varepsilon,+} + \widehat{Y}_{\varepsilon,-}\right)}_{=\widehat{U}_{\varepsilon,+}}$$

$$+\frac{\gamma}{\varepsilon^{\delta}}R'(\varepsilon p)\underbrace{\frac{1}{2}(\widehat{Y}_{\varepsilon,-} - \widehat{Y}_{\varepsilon,+})}_{=-i\widehat{U}_{\varepsilon,-}} + \underbrace{\varepsilon^{3-\delta}\mathfrak{R}_\varepsilon}_{\text{remainder}},$$

- $\mathfrak{R}_\varepsilon$ of order $O(1)$ as $\varepsilon \ll 1$.

Notation

- **Dispersion:**

$$\delta_\varepsilon\omega(p,k) := \frac{1}{\varepsilon}\left[\omega\left(k+\frac{\varepsilon p}{2}\right)-\omega\left(k-\frac{\varepsilon p}{2}\right)\right] \approx \omega'(k)p,$$

$$(p,k)\in\mathbb{R}\times\mathbb{T}.$$

- **Scattering:**

$$\mathcal{L}f(k) := 2\int_{\mathbb{T}} R(k,k')f(k')dk' - 2R(k)f(k),$$

Scattering operator

- **Scattering kernel**

$$R(k,k') := 16\sin^2(\pi k)\sin^2(\pi k')[\sin^2(\pi k)\cos^2(\pi k') + \cos^2(\pi k)\sin^2(\pi k')] = \frac{3}{4}\sum_{\iota\in\{-,+\}} \mathfrak{e}_\iota(k)\otimes\mathfrak{e}_{-\iota}(k'),$$

$$\mathfrak{e}_+(k) := \frac{8}{3}\sin^4(\pi k), \quad \mathfrak{e}_-(k) := \sin^2(2\pi k)$$

- **Total scattering**

$$R(k) := \int_{\mathbb{T}} R(k,k')dk' \approx \sin^2(\pi k), \quad k\in\mathbb{T}.$$

Complete set of Fourier-Wigner functions

$$\widehat{W}_{\varepsilon,+}(t,p,k) := \frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\left(\hat{\psi}^{(\varepsilon)}\right)^*\left(t,k-\frac{\varepsilon p}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,k+\frac{\varepsilon p}{2}\right)\right],$$

$$\widehat{Y}_{\varepsilon,+}(t,p,k) := \frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\hat{\psi}^{(\varepsilon)}\left(t,-k+\frac{\varepsilon p}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,k+\frac{\varepsilon p}{2}\right)\right],$$

$$\widehat{Y}_{\varepsilon,-}(t,p,k) := \widehat{Y}^*_\varepsilon(t,-p,k),$$

$$\widehat{W}_{\varepsilon,-}(t,p,k) := \widehat{W}_\varepsilon(t,p,-k).$$

$$\widehat{U}_{\varepsilon,+}(t,p,k) := \frac{1}{2}\left[\widehat{Y}_{\varepsilon,+}(t,p,k)+\widehat{Y}_{\varepsilon,-}(t,p,k)\right]$$

$$\widehat{U}_{\varepsilon,-}(t,p,k) := \frac{1}{2i}\left[\widehat{Y}_{\varepsilon,+}(t,p,k)-\widehat{Y}_{\varepsilon,-}(t,p,k)\right]$$

Complete set of Fourier-Wigner functions

$$\widehat{W}_{\varepsilon,+}(t,p,k) := \frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\left(\hat{\psi}^{(\varepsilon)}\right)^*\left(t,k-\frac{\varepsilon p}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,k+\frac{\varepsilon p}{2}\right)\right],$$

$$\widehat{Y}_{\varepsilon,+}(t,p,k) := \frac{\varepsilon}{2}\mathbb{E}_\varepsilon\left[\hat{\psi}^{(\varepsilon)}\left(t,-k+\frac{\varepsilon p}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,k+\frac{\varepsilon p}{2}\right)\right],$$

$$\widehat{Y}_{\varepsilon,-}(t,p,k) := \widehat{Y}^*_\varepsilon(t,-p,k),$$

$$\widehat{W}_{\varepsilon,-}(t,p,k) := \widehat{W}_\varepsilon(t,p,-k).$$

$$\widehat{U}_{\varepsilon,+}(t,p,k) := \frac{1}{2}\Big[\widehat{Y}_{\varepsilon,+}(t,p,k)+\widehat{Y}_{\varepsilon,-}(t,p,k)\Big]$$

$$\widehat{U}_{\varepsilon,-}(t,p,k) := \frac{1}{2i}\Big[\widehat{Y}_{\varepsilon,+}(t,p,k)-\widehat{Y}_{\varepsilon,-}(t,p,k)\Big]$$

Complete set of equations

$$
\begin{aligned}
\partial_t \widehat{W}_{\varepsilon,+} &= -\frac{i\delta_\varepsilon \omega}{\varepsilon^{\delta-1}} \widehat{W}_{\varepsilon,+} - \frac{i\gamma R' p}{\varepsilon^{\delta-1}} \widehat{U}_{\varepsilon,-} \\
&+ \left(\frac{\gamma}{\varepsilon^{\delta}} \mathcal{L} + \frac{\gamma p^2}{2\varepsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) \left(\widehat{W}_{\varepsilon,+} - \widehat{U}_{\varepsilon,+} \right) + \varepsilon^{3-\delta} \bar{\mathfrak{R}}_{\varepsilon}^{(1)}, \\
\partial_t \widehat{W}_{\varepsilon,-} &= \frac{i\delta_\varepsilon \omega}{\varepsilon^{\delta-1}} \widehat{W}_{\varepsilon,-} + \frac{i\gamma R' p}{\varepsilon^{\delta-1}} \widehat{U}_{\varepsilon,-} \\
&+ \left(\frac{\gamma}{\varepsilon^{\delta}} \mathcal{L} + \frac{\gamma p^2}{2\varepsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) \left(\widehat{W}_{\varepsilon,-} - \widehat{U}_{\varepsilon,+} \right) + \varepsilon^{3-\delta} \bar{\mathfrak{R}}_{\varepsilon}^{(4)}
\end{aligned}
$$

$$\partial_t \widehat{U}_{\varepsilon,+} = \frac{2\bar{\omega}}{\varepsilon^{\delta}}\widehat{U}_{\varepsilon,-} + \left(\frac{\gamma}{\varepsilon^{\delta}}\mathcal{L} + \frac{\gamma p^2}{2\varepsilon^{\delta-2}}(\delta^2\mathcal{L})\right)\left[\widehat{U}_{\varepsilon,+} - \frac{1}{2}\left(\widehat{W}_{\varepsilon,+} + \widehat{W}_{\varepsilon,-}\right)\right]$$

$$+\,\varepsilon^{3-\delta}\bar{\mathfrak{R}}^{(2)}_{\varepsilon},$$

$$\partial_t \widehat{U}_{\varepsilon,-} = -\frac{2\bar{\omega}}{\varepsilon^{\delta}}\widehat{U}_{\varepsilon,+} - \frac{2\gamma}{\varepsilon^{\delta}}R_{\varepsilon}\widehat{U}_{\varepsilon,-} - \frac{i\gamma R' p}{2\varepsilon^{\delta-1}}(\widehat{W}_{\varepsilon,-} - \widehat{W}_{\varepsilon,+}) + \varepsilon^{3-\delta}\bar{\mathfrak{R}}^{(3)}_{\varepsilon},$$

$$R_{\varepsilon} := R(k) + \frac{(\varepsilon p)^2}{8}R''(k), \quad \bar{\omega} := \bar{\omega}(k,\varepsilon p), \quad \delta_{\varepsilon}\omega := \delta_{\varepsilon}\omega(k,p).$$

$$\partial_t \widehat{U}_{\varepsilon,+} = \frac{2\bar{\omega}}{\varepsilon^{\delta}} \widehat{U}_{\varepsilon,-} + \left(\frac{\gamma}{\varepsilon^{\delta}} \mathcal{L} + \frac{\gamma p^2}{2\varepsilon^{\delta-2}} (\delta^2 \mathcal{L}) \right) \left[\widehat{U}_{\varepsilon,+} - \frac{1}{2} \left(\widehat{W}_{\varepsilon,+} + \widehat{W}_{\varepsilon,-} \right) \right]$$

$$+ \varepsilon^{3-\delta} \bar{\mathfrak{R}}^{(2)}_{\varepsilon},$$

$$\partial_t \widehat{U}_{\varepsilon,-} = -\frac{2\bar{\omega}}{\varepsilon^{\delta}} \widehat{U}_{\varepsilon,+} - \frac{2\gamma}{\varepsilon^{\delta}} R_{\varepsilon} \widehat{U}_{\varepsilon,-} - \frac{i\gamma R' p}{2\varepsilon^{\delta-1}} (\widehat{W}_{\varepsilon,-} - \widehat{W}_{\varepsilon,+}) + \varepsilon^{3-\delta} \bar{\mathfrak{R}}^{(3)}_{\varepsilon},$$

$$R_{\varepsilon} := R(k) + \frac{(\varepsilon p)^2}{8} R''(k), \quad \bar{\omega} := \bar{\omega}(k, \varepsilon p), \quad \delta_{\varepsilon}\omega := \delta_{\varepsilon}\omega(k, p).$$

$$\bar{\omega}(k,\varepsilon p) := \frac{1}{2}\left[\omega\left(k+\frac{\varepsilon p}{2}\right)+\omega\left(k-\frac{\varepsilon p}{2}\right)\right],$$

$$(\delta^2\mathcal{L})f(k) := -\pi^2\int_{\mathbb{T}} R_1(k,k')f(k')dk' - \frac{1}{2}R''(k)f(k),$$

$$(\delta^2\mathcal{R})f(k) := \int_{\mathbb{T}}\partial_p^2 R(k,k',0)f(k')dk', \quad R_1(k,k') := -\frac{2}{\pi^2}\partial_p^2 R(k,k',0),$$

$$R(k,k',p) := \frac{1}{2}\sum_{\sigma=\pm 1} r\left(k-\frac{p}{2}, k-\sigma k'\right) r\left(k+\frac{p}{2}, k-\sigma k'\right),$$

$$r(k,k') := 2\mathfrak{s}^2(k)\mathfrak{s}(2(k-k')) + 2\mathfrak{s}(2k)\mathfrak{s}^2(k-k'), \quad k,k'\in\mathbb{T},$$

$$\|\mathfrak{R}^{(i)}_{\varepsilon}(t,p,\cdot)\|_{L^2(\mathbb{T})} \preceq \sum_{\iota\in\{-,+\}}(\|\widehat{W}_{\varepsilon,\iota}(t,p,\cdot)\|_{L^2(\mathbb{T})} + \|\widehat{Y}_{\varepsilon,\iota}(t,p,\cdot)\|_{L^2(\mathbb{T})})$$

Useful identity

- for any $\phi, \psi \in L^2(\mathbb{T})$

$$\begin{aligned}\mathcal{D}(\phi,\psi) &:= \int_{\mathbb{T}} (-\mathcal{L})\phi(k)\psi^*(k)dk \\ &= \int_{\mathbb{T}^2} R(k,k')[\phi(k)-\phi(k')][\psi(k)-\psi(k')]^* dkdk',\end{aligned}$$

$$\mathfrak{E}_\varepsilon(t,p) := \sum_{\iota\in\{-,+\}} \left(\frac{1}{2}\|\widehat{W}_{\varepsilon,\iota}(t,p)\|^2_{L^2(\mathbb{T})} + \|\widehat{U}_{\varepsilon,\iota}(t,p)\|^2_{L^2(\mathbb{T})}\right).$$

- Taking the scalar products of both sides of evolution equations against the respective $\widehat{W}_{\varepsilon,\iota}$, $\widehat{U}_{\varepsilon,\iota}$, $\iota\in\{-,+\}$:

$$\begin{aligned}&\frac{1}{2}\mathfrak{E}_\varepsilon(t,p)+\frac{\gamma}{\varepsilon^\delta}\int_0^t\mathcal{D}\left((\widehat{W}_\varepsilon-\widehat{U}_{\varepsilon,+})(s,p)\right)ds\\&+\frac{2\gamma}{\varepsilon^\delta}\int_0^t ds\int_{\mathbb{T}}R(k)|\widehat{U}_{\varepsilon,-}(s,p,k)|^2dk\\&+2\varepsilon^{1-\delta}\gamma p\int_0^t ds\int_{\mathbb{T}}R'(k)\mathrm{Im}\left(\widehat{U}^*_{\varepsilon,-}\widehat{W}_{\varepsilon,+}\right)(s,p,k)dk\\&\qquad=\frac{1}{2}\mathfrak{E}_\varepsilon(0,p)+\varepsilon^{2-\delta}\int_0^t\mathfrak{R}_\varepsilon(s,p)ds,\end{aligned}\tag{21}$$

where for any $M>0$ we have

$$\mathfrak{R}_\varepsilon(t,p)\preceq\mathfrak{E}_\varepsilon(t,p),\quad t\geqslant 0,\ |p|\leqslant M,\ \varepsilon\in(0,1].\tag{22}$$

Fourier-Laplace-Wigner functions

- limit identification tool - the Fourier-Laplace-Wigner functions

$$\bar{w}_{\varepsilon,\pm}(\lambda, p, k) := \int_0^{+\infty} e^{-\lambda t}\widehat{W}_{\varepsilon,\pm}(t, p, k)dt \qquad (23)$$

- the Dirichlet form:

$$\int_{\mathbb{T}}(-\mathcal{L})f(k)f(k)dk = \int_{\mathbb{T}^2} R(k, k')[f(k') - f(k)]^2 dkdk',$$

Conclusion from the identity + some additional fact

Proposition

For any $M > 0$ and a compact interval $I \subset (\lambda_0, +\infty)$

$$\sup_{\lambda \in I, |p| \leqslant M} \left[\int_{\mathbb{T}} R_\varepsilon(k) |\bar{u}_{\varepsilon,-}(\lambda, p, k)|^2 dk + \mathcal{D}\left(\bar{w}_\varepsilon(\lambda, p)\right) \right.$$
$$\left. + \int_{\mathbb{T}} R_\varepsilon(k) |\bar{u}_{\varepsilon,+}(\lambda, p, k)|^2 \left(\frac{\bar{\omega}(k, \varepsilon p)}{\lambda \varepsilon^\delta + \gamma R_\varepsilon(k)} \right)^2 dk \right] \preceq \varepsilon^{\delta - s}, \quad \varepsilon \in (0, 1].$$

Homogenization Theorem

- projection onto the kernels generating the scattering kernel

$$w^{(\iota)}_{\varepsilon,\pm}(\lambda,p) := \int_{\mathbb{T}} \bar{w}_{\varepsilon,\pm}(\lambda,p,k)\mathfrak{e}_\iota(k)dk, \quad \iota \in \{-,+\}.$$

Theorem 4

For any $M > 0$ and a compact interval $I \subset (0,+\infty)$ we have

$$\lim_{\varepsilon\to 0+} \sup_{\lambda\in I,|p|\leqslant M} \int_{\mathbb{T}} \left| \bar{w}_{\varepsilon,+}(\lambda,p,k) - w^{(\pm)}_{\varepsilon,+}(\lambda,p) \right| dk = 0, \tag{24}$$

and

$$\lim_{\varepsilon\to 0+} \sup_{\lambda\in I,|p|\leqslant M} \int_{\mathbb{T}} |\bar{u}_{\varepsilon,\iota}(\lambda,p,k)|\, dk = 0, \quad \iota \in \{-,+\}. \tag{25}$$

- The Laplace transform of the dynamics

$$
\begin{aligned}
\lambda \bar{w}_{\varepsilon,+} - \widehat{W}^{(0)}_{\varepsilon,+} &= -\frac{i\delta_\varepsilon \omega}{\varepsilon^{\delta-1}} \bar{w}_{\varepsilon,+} - \frac{i\gamma R' \varepsilon p}{\varepsilon^\delta} \bar{u}_{\varepsilon,-} + \frac{\gamma}{\varepsilon^\delta} L_\varepsilon \left(\bar{w}_{\varepsilon,+} - \bar{u}_{\varepsilon,+} \right) \\
&\quad + \varepsilon^{3-\delta} \bar{r}^{(1)}_\varepsilon, \\
\lambda \bar{u}_{\varepsilon,+} - \widehat{U}^{(0)}_{\varepsilon,+} &= \frac{2\bar{\omega}}{\varepsilon^\delta} \bar{u}_{\varepsilon,-} + \frac{\gamma}{\varepsilon^\delta} L_\varepsilon \left[\bar{u}_{\varepsilon,+} - \frac{1}{2} \left(\bar{w}_{\varepsilon,+} + \bar{w}_{\varepsilon,-} \right) \right] + \varepsilon^{3-\delta} \bar{r}^{(2)}_\varepsilon, \\
\lambda \bar{u}_{\varepsilon,-} - \widehat{U}^{(0)}_{\varepsilon,-} &= -\frac{2\bar{\omega}}{\varepsilon^\delta} \bar{u}_{\varepsilon,+} - \frac{2\gamma}{\varepsilon^\delta} R_\varepsilon \bar{u}_{\varepsilon,-} - \frac{i\gamma R' \varepsilon p}{2\varepsilon^\delta} \left(\bar{w}_{\varepsilon,-} - \bar{w}_{\varepsilon,+} \right) \\
&\quad + \varepsilon^{3-\delta} \bar{r}^{(3)}_\varepsilon \\
\lambda \bar{w}_{\varepsilon,-} - \widehat{W}^{(0)}_{\varepsilon,-} &= \frac{i\delta_\varepsilon \omega}{\varepsilon^{\delta-1}} \bar{w}_{\varepsilon,-} + \frac{i\gamma R' \varepsilon p}{\varepsilon^\delta} \bar{u}_{\varepsilon,-} + \frac{\gamma}{\varepsilon^\delta} L_\varepsilon \left(\bar{w}_{\varepsilon,-} - \bar{u}_{\varepsilon,+} \right) \\
&\quad + \varepsilon^{3-\delta} \bar{r}^{(4)}_\varepsilon,
\end{aligned}
$$

where $L_\varepsilon := \mathcal{L} + (1/2)(\varepsilon p)^2 (\delta^2 \mathcal{L})$

- After some calculations

$$a_w^{(\varepsilon)} w_\varepsilon^{(+)} = \frac{4\gamma}{3} \int_{\mathbb{T}} \frac{R\widehat{W}_\varepsilon^{(0)}}{D^{(\varepsilon)}} dk + \frac{4\gamma}{3\varepsilon^\delta} \int_{\mathbb{T}} \frac{Rq_\varepsilon}{D^{(\varepsilon)}} dk + o(1),$$

where

$$\begin{aligned} D^{(\varepsilon)} &:= \varepsilon^\delta \lambda + 2\gamma R_\varepsilon + i\varepsilon\delta_\varepsilon \omega, \\ a_w^{(\varepsilon)}(\lambda, p) &:= \frac{4\gamma}{3\varepsilon^\delta} \int_{\mathbb{T}} \left(1 - \frac{2\gamma R}{D^{(\varepsilon)}}\right) R\ dk \\ w_\varepsilon^{(+)}(\lambda, p) &:= \int_{\mathbb{T}} \bar{w}_{\varepsilon,+}(\lambda, p, k)\mathfrak{e}_+(k) dk. \end{aligned}$$

- for any $J \in C_c(\mathbb{R} \times \mathbb{T})$:

$$\lim_{\varepsilon \to 0+} \left(\frac{4\gamma}{3} \int_{\mathbb{R}\times\mathbb{T}} \frac{R\widehat{W}_{\varepsilon}^{(0)}}{D^{(\varepsilon)}} \hat{J} dpdk - \frac{2}{3} \int_{\mathbb{R}\times\mathbb{T}} \widehat{W}_0 \hat{J} dpdk \right) = 0,$$

- If $\hat{\alpha}(0) > 0$ (**pinned chain**), then

$$\lim_{\varepsilon \to 0+} \sup_{\lambda \in I, |p| \leqslant M} \left| a_w^{(\varepsilon)}(\lambda, p) - \frac{2\lambda}{3} - \frac{\hat{\sigma}^2 p^2}{3\gamma} \right| = 0,$$

$$\lim_{\varepsilon \to 0+} \sup_{\lambda \in I, |p| \leqslant M} \left| \frac{4\gamma}{3\varepsilon^{\delta}} \int_{\mathbb{T}} \frac{Rq_{\varepsilon}}{D^{(\varepsilon)}} dk - \frac{8\gamma(\pi p)^2}{3} \bar{w}_{\varepsilon}^{(+)} \right| = 0.$$

Putting things together - in the diffusive case

- If $\hat{\alpha}(0) > 0$, then for any $\lambda > 0$ and $p \in \mathbb{R}$ the limiting point $w(\lambda, p)$ of $\bar{w}_{\varepsilon,+}$, as $\varepsilon \to 0+$ satisfies

$$\left(\lambda + \frac{Dp^2}{2}\right) w(\lambda, p) = \int W_0(p, k) dk, \tag{26}$$

$$D = \frac{\hat{\sigma}^2}{\gamma} + 8\gamma\pi^2 \qquad \text{dla} \quad s = 0,$$

and

$$\hat{\sigma}^2 := \int_{\mathbb{T}} \frac{[\omega'(k)]^2}{R(k)} dk, \tag{27}$$

$$R(k) \approx k^2, \quad |k| \ll 1. \tag{28}$$

The superdiffusive case

- If $\hat{\alpha}(0) = 0$ and $\hat{\alpha}''(0) > 0$, then

$$\lim_{\varepsilon \to 0+} \sup_{\lambda \in I, |p| \leqslant M} \left| a_w^{(\varepsilon)}(\lambda, p) - \frac{2\lambda}{3} - \frac{2\hat{c}|p|^{3/2}}{3} \right| = 0, \qquad (29)$$

where

$$\hat{c} := \frac{[\alpha''(0)]^{3/4}}{2^{9/4}(3\gamma)^{1/2}}.$$

As a result the limiting point satisfies

$$\left(\lambda + \hat{c}|p|^{3/2}\right) w(\lambda, p) = \int W_0(p, k) dk, \qquad (30)$$

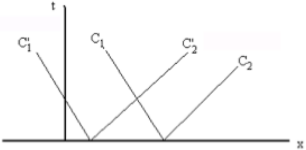
Riemann invariants - phonon modes ($\hat{\alpha}(0) = 0$)

Riemann invariants of the system

$$\begin{cases} \partial_t \bar{r}(t,u) = \partial_u \bar{p}(t,u), \\ \partial_t \bar{p}(t,u) = \tau_1 \partial_u \bar{r}(t,u) \end{cases}$$

$$\bar{f}^{(\pm)}(t,u) = \bar{p}(t,u) \pm \sqrt{\tau_1}\,\bar{r}(t,u).$$

Constant along characteristcs $u \mp \sqrt{\tau_1}\,t = \mathrm{const}$,



characteristics

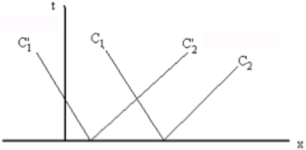
Riemann invariants - phonon modes ($\hat{\alpha}(0) = 0$)

Riemann invariants of the system

$$\begin{cases} \partial_t \bar{r}(t,u) = \partial_u \bar{p}(t,u), \\ \partial_t \bar{p}(t,u) = \tau_1 \partial_u \bar{r}(t,u) \end{cases}$$

$$\bar{f}^{(\pm)}(t,u) = \bar{p}(t,u) \pm \sqrt{\tau_1}\,\bar{r}(t,u).$$

Constant along characteristcs $u \mp \sqrt{\tau_1} t = \mathrm{const}$,



characteristics

$$\bar{f}^{(\pm)}(t,u) = \bar{f}_0^{(\pm)}(u \pm \sqrt{\tau_1}\, t). \tag{31}$$

$$\bar{f}_0^{(\pm)}(u) := \bar{p}_0(u) \pm \sqrt{\tau_1}\,\bar{r}_0(u).$$

phonon modes:

$$\begin{aligned} \mathfrak{f}_y^{(+)}(t) &:= \mathfrak{p}_y(t) + \sqrt{\tau_1}\left[1 + \frac{1}{2}\left(\frac{3\gamma}{\sqrt{\tau_1}} - 1\right)\nabla^*\right]\mathfrak{r}_y(t), \\ \mathfrak{f}_y^{(-)}(t) &:= \mathfrak{p}_y(t) - \sqrt{\tau_1}\left[1 - \frac{1}{2}\left(\frac{3\gamma}{\sqrt{\tau_1}} + 1\right)\nabla^*\right]\mathfrak{r}_y(t), \end{aligned} \tag{32}$$

Diffusive propagation of phonon modes

Theorem 5

Assume:

$$\lim_{\varepsilon\to 0+}\varepsilon\sum_y J(\varepsilon y)\langle \mathfrak{r}_y\rangle_{\mu_\varepsilon}=\int_{\mathbb{R}} J(u)\bar{r}_0(u)du,$$

$$\lim_{\varepsilon\to 0+}\varepsilon\sum_y J(\varepsilon y)\langle \mathfrak{p}_y\rangle_{\mu_\varepsilon}=\int_{\mathbb{R}} J(u)\bar{p}_0(u)du.$$

Then:

$$\lim_{\varepsilon\to 0+}\varepsilon\sum_y J(\varepsilon y)\mathbb{E}_\varepsilon \mathfrak{f}_y^{(\pm)}\left(\frac{t}{\varepsilon}\right)=\int_{\mathbb{R}} J(u)\bar{f}^{(\pm)}(t,u)du,$$

Diffusive propagation of phonon modes, cont

Theorem 5 cont.

$$\lim_{\varepsilon\to 0+}\varepsilon\sum_{y}J\left(\varepsilon y\mp\sqrt{\tau_1}\frac{t}{\varepsilon}\right)\mathbb{E}_\varepsilon\mathfrak{f}_y^{(\pm)}\left(\frac{t}{\varepsilon^2}\right)=\int_{\mathbb{R}}\bar{f}_{\pm}^{(d)}(t,u)J(u)du,$$

where:

$$\bar{f}_{\pm}^{(d)}(t,u):=P_t*\bar{f}_0^{(\pm)}(u),$$

$$P_t(u):=\frac{1}{\sqrt{4\pi\bar{D}t}}\exp\left\{-\frac{u^2}{4\bar{D}t}\right\},\tag{33}$$

and $\bar{D}:=3\gamma$,

$$\bar{f}_0^{(\pm)}(u):=\bar{p}_0(u)\pm\sqrt{\tau_1}\bar{r}_0(u).\tag{34}$$