Energy transport in an open system. Thermal boundary conditions in a fractional superdiffusion of energy

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Motivation

- Motivation: understanding energy transport in an open one dimensional chain of oscillators subject to (three) conservation laws,

  • harmonic chain with stochastic noise - momenta exchange between atoms; in contact with heat baths - Langevin thermostats at the endpoints;

  • closed system: the case of an infinite chain (M. Jara, T.K., S. Olla 15') discussed earlier: fractional Lévy superdiffusion of energy with index of stability 3/4,

  • open system, question: boundary conditions?

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Existing results

- two step limit in an infinite chain with a Langevin, or Poisson thermostat attached at a single site,
  - kinetic limit: from a microscopic chain to a kinetic equation;
    - (LT) Langevin thermostat: T.K., S. Olla, H. Spohn, L. Ryzhik 20', T.K., S. Olla, 20',
    - (PT) Poisson thermostat: T.K., S. Olla 22', interpolation between flip and Langevin
  - superdiffusive limit: from the kinetic equation to a fractional superdiffusion:
    - (ND) non-degenerate killing rate: T.K., S. Olla, L. Ryzhik 20', relevant to the case discussed in (LT);
    - (D) degenerate killing rate: K. Bogdan, T.K., L. Marino 25';

Existing results cont'd

- the direct hydrodynamic limit
  - heuristic results:
    - (S) at stationarity: S. Lepri, C. Mejia-Monasterio, A. Politi, 09'
    - (NS) non-stationary case with the microscopic Dirichlet boundary: A. Kundu, C. Bernardin, K. Saito, A. Kundu, A. Dhar 19',

Hydrodynamic limit for SSE with long jumps - open system

- C. Bernardin, P Cardoso, P Gonçalves, S. Scotta, 23': the hydrodynamic limit for symmetric exclusion processes with heavy-tailed long jumps and in contact with infinitely extended reservoirs
- various regimes for the limit
- the regional fractional Laplacian (censored stable process) with either:
  - **(N)** (homogeneous) fractional Neumann, or
  - **(D)** (non-homogeneous) Dirichlet boundary conditions

The model: stochastically perturbed harmonic chains

- Microscopic dynamics: unpinned harmonic chain with random exchange of momenta:

$$\dot{q}_x(t) = p_x(t), \qquad x \in \mathbb{Z}_n = \{0, \dots, n\}$$

- in the bulk: $x = 1, \dots, n-1$

$$\begin{aligned} \mathrm{d}p_x(t) = \Delta_{\mathrm{N}} q_x(t)\mathrm{d}t &+ \big[p_{x+1}(t-) - p_x(t-)\big]\mathrm{d}N_{x,x+1}(\gamma t) \\ &+ \big[p_{x-1}(t-) - p_x(t-)\big]\mathrm{d}N_{x-1,x}(\gamma t) \end{aligned}$$

$\{N_{x,x+1}(t), x = 0, \dots, n-1\}$ independent Poisson processes of intensity 1, $\gamma > 0$ - intensity of the exchange

- Notation: $\nabla f_x = f_{x+1} - f_x, \quad \nabla^\star f_x = f_x - f_{x-1},$

$$\Delta_{\mathrm{N}} f_x = f_{x+1} + f_{x-1} - 2f_x = \nabla \nabla^\star f_x,$$

$$f_{n+1} := f_n \quad \text{and} \quad f_{-1} = f_0.$$

Boundaries $x = 0, n$

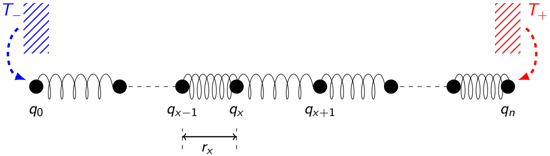
- **the energy exchanged** with two **Langevin heat baths** at temperatures $T_L > 0$ and $T_R > 0$:

$$\mathrm{d}p_0(t) = \big(q_1(t) - q_0(t)\big)\mathrm{d}t + [p_1(t-) - p_0(t-)]\mathrm{d}N_{0,1}(\gamma t) - \tilde{\gamma} p_0(t)\mathrm{d}t + \sqrt{2T_L\tilde{\gamma}}\mathrm{d}w_L$$

$$\mathrm{d}p_n(t) = \big(q_{n-1}(t) - q_n(t)\big)\mathrm{d}t + [p_{n-1}(t-) - p_n(t-)]\mathrm{d}N_{n-1,n}(\gamma t) - \tilde{\gamma} p_n(t)\mathrm{d}t + \sqrt{2T_R\tilde{\gamma}}\mathrm{d}w_R(t)$$

- $w_L(t)$ and $w_R(t)$ - **independent standard Brownian motions**
- $\tilde{\gamma} > 0$ - **the strength of the thermostats**

Stochastically perturbed harmonic chains



Rysunek: Oscillator chain: thermostats at both endpoints

- the dynamics is **unpinned**, **invariant under translations:** ($q_x \to q_x + a$, $a \in \mathbb{R}$).

- convenient to work with **the inter-particle stretches**

$$r_x := q_x - q_{x-1}, \quad x = 1, \ldots, n.$$

Dynamics in terms of the momentum/stretch

- $\dot{r}_x(t) = \nabla^\star p_x(t)$, $x \in \{1, \ldots, n\}$,
- in **the bulk:** $x = 1, \ldots, n-1$

$$\mathrm{d}p_x(t) = \nabla r_x \mathrm{d}t + [\nabla^\star p_{x+1}(t-)\mathrm{d}N_{x,x+1}(\gamma t) \\ -\nabla^\star p_x(t-)\mathrm{d}N_{x-1,x}(\gamma t)]$$

- and **at the boundaries:**

$$\mathrm{d}p_0(t) = r_1 \mathrm{d}t + \nabla^\star p_1(t-)\mathrm{d}N_{0,1}(\gamma t) - \tilde{\gamma} p_0(t)\mathrm{d}t + \sqrt{2\tilde{\gamma} T_L}\mathrm{d}w_L(t),$$

$$\mathrm{d}p_n(t) = -r_n \mathrm{d}t - \nabla^\star p_n(t-)\mathrm{d}N_{n-1,n}(\gamma t) - \tilde{\gamma} p_n(t)\mathrm{d}t + \sqrt{2\tilde{\gamma} T_R}\mathrm{d}w_R(t).$$

Conserved quantities

- Local conservation laws (without boundaries):

(L) stretch r_x

(M) momentum p_x

(E) energy

$$\mathcal{E}_x(\mathbf{r},\mathbf{p}) := \frac{1}{2}(p_x^2 + r_x^2), \qquad x = 0, \ldots, n,$$

superdiffusive time scaling

- **Scaling** $t := n^{3/2}t'$, $x = nu$, $u \in [0,1]$

$$r_x^{(n)}(t) = r_x(n^{3/2}t), \quad p_x^{(n)}(t) = p_x(n^{3/2}t)$$

$$\begin{aligned}
\dot r_x^{(n)}(t) &= n^{3/2}\nabla^\star p_x^{(n)}(t),\\
\mathrm{d}p_x^{(n)}(t) &= n^{3/2}\nabla r_x^{(n)}\mathrm{d}t + \Big[\nabla^\star p_{x+1}^{(n)}(t-)\mathrm{d}N_{x,x+1}(n^{3/2}\gamma t)\\
&\quad -\nabla^\star p_x^{(n)}(t-)\mathrm{d}N_{x-1,x}(n^{3/2}\gamma t)\Big]\\
&\quad + \sum_{z=0,n}\delta_{x,z}\Big[-n^{3/2}\tilde\gamma p_z^{(n)}(t)\mathrm{d}t + \sqrt{2n^{3/2}\tilde\gamma T_z}\mathrm{d}w_z(t)\Big]
\end{aligned}$$

- **Conventions:**

$$T_0 = T_L, \quad T_n = T_R, \quad w_0 = w_L, \quad w_n = w_R,$$
$$r_x^{(n)}(t) \mapsto r_x(t), \quad p_x^{(n)}(t) \mapsto p_x(t)$$

Initial profile of energy

(IP) the existence of **an initial profile** of **the energy functional**

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{x=0}^{n} \varphi\left(\frac{x}{n}\right) \mathbb{E}_n\left[\mathcal{E}_{n,x}(0)\right] = \int_0^1 T_{\mathrm{ini}}(u)\varphi(u)\mathrm{d}u,$$

for all $\varphi \in C[0,1]$.

Informal statement of the main result

Theorem (The limit of thermal energy and equipartition)

- *Under* some assumptions *about the initial data,* $\exists\, T(t,u)$

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{x=0}^{n} \varphi\left(\frac{x}{n}\right)\mathbb{E}_n\left[\mathcal{E}_{n,x}(t)\right] = \int_0^1 T(t,u)\varphi(u)\mathrm{d}u,\ \forall \varphi \in C[0,1],$$

- $\forall \Phi : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ - compactly supported, cont. function

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{x=0}^{n} \int_0^{+\infty} \Phi\left(t, \frac{x}{n}\right)\mathbb{E}_n\left[p_x^2(t)\right]\mathrm{d}t$$

$$= \lim_{n\to+\infty} \frac{1}{n}\sum_{x=0}^{n} \int_0^{+\infty} \Phi\left(t, \frac{x}{n}\right)\mathbb{E}_n\left[\mathcal{E}_x(t)\right]\mathrm{d}t$$

$$= \int_0^{+\infty} \mathrm{d}t \int_0^1 T(t,u)\Phi\left(t,u\right)\mathrm{d}u.$$

Informal description of the limit

- $T(t, u)$ is the solution of

$$\partial_t T(t,u) = -c_{\rm bulk}|\Delta|^{3/4} T(t,u)$$
$$+ c_{\rm bd} \sum_{v=0,1} \int_0^{+\infty} \Big\{ V_\varrho(u,v) \int_0^1 V_\varrho(u',v)[T_v - T(t,u')]{\rm d}u' \Big\} \frac{{\rm d}\varrho}{\varrho^{3/4}},$$

with the initial data $T(0,u) = T_{\rm ini}(u)$ and the boundary conditions: $T(t,0) = T_L(= T_0)$, $T(t,1) = T_R(= T_1)$.

- $V_\varrho(u', u) = \varrho G_\varrho(u', u)$, where $G_\varrho = (\rho - \Delta)^{-1}$ - the Green's function of the Neumann Laplacian Δ on $[0, 1]$
- $|\Delta|^{3/4}$ - the spectral power of the Neumann Laplacian

Boundary condition and the coefficients

- with $T_0 = T_L$, $T_1 = T_R$: for any $t > 0$

$$\int_0^{+\infty} \Big\{ \int_0^t \mathrm{d}s \Big(\int_0^1 V_\varrho(u', v)\big(T_v - T(s, u')\big)\mathrm{d}u' \Big)^2 \Big\} \frac{\mathrm{d}\varrho}{\varrho^{3/4}} < +\infty$$

- Coefficients:

$$c_{\rm bulk} = \frac{1}{(2^3\gamma)^{1/2}},$$

$$c_{\rm bd} = \frac{\tilde\gamma}{2\gamma^{1/2}\pi(1+\tilde\gamma)^2} = \frac{\sqrt{2}\tilde\gamma}{\pi(1+\tilde\gamma)^2} c_{\rm bulk}.$$

$$\frac{c_{\rm bd}}{c_{\rm bulk}} \to 0, \quad \tilde\gamma \to 0, \quad \text{or} \quad \tilde\gamma \to +\infty$$

maximum $\dfrac{c_{\rm bd}}{c_{\rm bulk}} = \dfrac{1}{2\sqrt{2}\pi}$ at $\tilde\gamma = 1.$

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maximum $\dfrac{c_{\mathrm{bd}}}{c_{\mathrm{bulk}}} = \dfrac{1}{2\sqrt{2}\pi}$ at $\tilde\gamma = 1.$

Rigorous definition of $T(t, u)$

Definition (Weak solution)

Suppose that $c_{\rm bulk}, c_{\rm bd} > 0$, T_0, $T_1 > 0$ and $T_{\rm ini} \in L^2[0,1]$;

i) $T \in C\Big([0,+\infty); L^2_w[0,1]\Big)$, where $L^2_w[0,1]$ is equipped with the weak topology,

ii) for any $t > 0$ and $v = 0, 1$ we have

$$\int_0^t {\rm d}s \int_0^{+\infty} \Big(\int_0^1 V_\varrho(u', v)\big(T_v - T(s, u')\big){\rm d}u' \Big)^2 \frac{{\rm d}\varrho}{\varrho^{3/4}} < +\infty,$$

Weak formulation

iii) for any $\varphi \in C_c^\infty(0,1)$:

$$
\begin{aligned}
&\langle \varphi, T(t)\rangle_{L^2[0,1]} - \langle \varphi, T_{\rm ini}\rangle_{L^2[0,1]} \\
&= -c_{\rm bulk}\int_0^t \langle |\Delta|^{3/4}\varphi, T(s)\rangle_{L^2[0,1]} {\rm d}s \\
&+ c_{\rm bd}\int_0^t {\rm d}s \int_0^{+\infty} \langle V_\varrho(\cdot,0),\varphi\rangle_{L^2[0,1]} \\
&\qquad\qquad \times \langle V_\varrho(\cdot,0), T_L - T(s)\rangle_{L^2[0,1]} \frac{{\rm d}\varrho}{\varrho^{3/4}} \\
&+ c_{\rm bd}\int_0^t {\rm d}s \int_0^{+\infty} \langle V_\varrho(\cdot,1),\varphi\rangle_{L^2[0,1]} \\
&\qquad\qquad \times \langle V_\varrho(\cdot,1), T_R - T(s)\rangle_{L^2[0,1]} \frac{{\rm d}\varrho}{\varrho^{3/4}}.
\end{aligned}
$$

Uniqueness

Theorem (Uniqueness of weak solutions)

Suppose that $T_{\rm ini} \in L^2[0,1]$. Then, the equation has **a unique solution** $T(\cdot,\cdot)$. *In addition,*

$$\begin{aligned}&\int_0^t T(s,\cdot){\rm d}s \in C[0,1] \quad \textit{and}\\ &\int_0^t T(s,0){\rm d}s = T_0 t, \quad \int_0^t T(s,1){\rm d}s = T_1 t, \quad t \ge 0.\end{aligned} \tag{1}$$

Regularity of solutions

Theorem

Suppose that $T_{\rm ini} \in H^{3/4}[0,1]$ *is such that* $T_{\rm ini}(v) = T_v$, $v = 0, 1$. *Then:*

i) *the solution* $T(t)$ *belongs to the space*

$$C\Big([0,+\infty); L^2[0,1]\Big) \cap L^\infty_{\rm loc}\Big([0,+\infty); H^{3/4}[0,1]\Big)$$

and $\int_0^t T(s)\mathrm{d}s$ *belongs to* $C\Big([0,+\infty); H^{3/4}[0,1]\Big)$, *where the target spaces are considered with* **the strong topologies**,

ii) *we have*

$$T_0 = T(t,0) \quad \textit{and} \quad T_1 = T(t,1), \quad \textit{for a.e. } t \geq 0, \qquad (2)$$

iii) *for any* $\varphi \in H^{3/4}_0[0,1]$ **the weak formulation** *holds.*

- **In the bulk:** for $\varphi \in C^\infty[0,1]$ we have

$$|\Delta|^{3/4}\varphi(u) = \int_0^1 q(u',u)[\varphi(u') - \varphi(u)]\mathrm{d}u',$$

with

$$q(u,u') := \frac{3}{2^{5/2}\pi^{1/2}} \sum_{n\in\mathbb{Z}} \left(\frac{1}{|u+u'+2n|^{5/2}} + \frac{1}{|u-u'+2n|^{5/2}} \right),$$

Boundary behavior

- **At the boundary:** $v = 0, 1$

$$\int_0^{+\infty} V_\varrho(u,v)V_\varrho(u',v)\frac{\mathrm{d}\varrho}{\varrho^{3/4}} = g(u,u';v),$$

$$g(u,u';v) = \sum_{n,n'\in\mathbb{Z}} W(u+v+2n, u'+v+2n'), \qquad \text{where}$$

$$W(u,u') = W(u',u) := \frac{5\Gamma^2\left(\frac{1}{4}\right)}{2^5\pi}\int_0^{\pi/2}\Big(\frac{\sin^2(2\theta)}{(u\sin\theta)^2+(u'\cos\theta)^2}\Big)^{5/4}\mathrm{d}\theta.$$

Here $\Gamma(\cdot)$ is **the Euler gamma function.**

$$\int_0^1 g(u,u';v)\mathrm{d}u' = \sqrt{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{|u+v+2n|^{3/2}}, \qquad v = 0, 1.$$

Probabilistic interpretation

- Informal **reformulation** of the equation:

$$\partial_t T(t,u) = \int_0^1 r(u,u')\left[T(t,u') - T(t,u)\right] du' + \sum_{v=0,1} b(u;v)\left[T_v - T(t,u)\right],$$

$$r(u,u') := c_{\rm bulk} q(u',u) - c_{\rm bd} \sum_{v=0,1} g(u,u';v),$$

$$b(u;v) := c_{\rm bd} \int_0^1 g(u,u';v){\rm d}u'.$$

Probabilistic interpretation, cont'd

- **If $r(u,u') \geq 0$,** then $T(t,u)$ - **the density** of a Markov process with **creation** and **annihilation:**

(J) a particle **jumps** from u to u' with rate $r(u,u')$

(A) at time t and position u the particle gets **annihilated** at rate $b(u,0)+b(u,1)$;

(C) at time t and position u a particle is **created** at the rate $b(u,0)T_L + b(u,1)T_R$.

Remark about microscopic Dirichlet boundary condition

- **microscopic Dirichlet boundary condition:** $q_{-1} = 0$, $q_{n+1} = 0$
- **dynamics:** $\dot{q}_x(t) = p_x(t), \qquad x \in \mathbb{Z}_n = \{0, \dots, n\}$
- **momentum:**

$$\mathrm{d}p_x(t) = \Delta_{\mathrm{D}} q_x(t)\mathrm{d}t + \big[p_{x+1}(t-) - p_x(t-)\big]\mathrm{d}N_{x,x+1}(\gamma t)$$
$$+ \big[p_{x-1}(t-) - p_x(t-)\big]\mathrm{d}N_{x-1,x}(\gamma t)$$
$$+ \sum_{z=0,n} \delta_{x,z}\big[- n^{3/2}\tilde{\gamma} p_z^{(n)}(t)\mathrm{d}t + \sqrt{2n^{3/2}\tilde{\gamma} T_z}\mathrm{d}w_z(t)\big]$$

- **microscopic Dirichlet laplacian:**

$$\Delta_{\mathrm{D}} f_x = f_{x+1} + f_{x-1} - 2f_x, \quad f_{n+1} := 0 \quad \text{and} \quad f_{-1} = 0.$$

Energy limit

Theorem

Suppose that $T_{\rm ini} \in H^{3/4}[0,1]$*. Then, for* any test function $\varphi \in C_c^\infty(0,1)$ *we have*

$$\int_0^1 T(t,u)\varphi(u)\mathrm{d}u - \int_0^1 T_{\rm ini}(u)\varphi(u)\mathrm{d}u$$
$$= -c_{\rm bulk}\int_0^t \mathrm{d}s \int_0^1 T(s,u)|\Delta_{\rm N}|^{3/4}\varphi(u)\mathrm{d}u,$$

with $T \in C\Big([0,+\infty); L^2[0,1]\Big) \cap L^\infty_{\rm loc}\Big([0,+\infty); H^{3/4}[0,1]\Big)$, the Dirichlet boundary condition:

$$T(t,0) = T_L, \quad T(t,1) = T_R$$

and $c_{\rm bulk} = \frac{1}{(2^3\gamma)^{1/2}}$.

- the fractional heat equation with the 3/4-power of the Neumann laplacian and the Dirichlet boundary condition
- no boundary layer
- heuristics:
  - G. Basile, T. K., S. Olla 15'
  - A. Kundu, C. Bernardin, K. Saito, A. Kundu, A. Dhar 19',

Assumptions about the initial data

- the distribution $\mu_n(0)$ of initial data $(\mathbf{r}(0), \mathbf{p}(0))$ is of zero mean,
- for a given $T > 0$, define

$$\nu_T(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := g_T(\mathbf{r}, \mathbf{p})\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{p}, \quad \text{where}$$

$$g_T(\mathbf{r}, \mathbf{p}) = \frac{e^{-\mathcal{E}_0/T}}{\sqrt{2\pi T}} \prod_{x=1}^{n} \frac{e^{-\mathcal{E}_x/T}}{2\pi T}$$

- the relative entropy with respect to ν_T bounded, up to a constant, by the size of the system:

$$0 \le \mathbf{H}_{n,T}(0) = \int_{\Omega_n} f_n(0) \log f_n(0) \mathrm{d}\nu_T \le C_{H,T}\, n.$$

Assumptions: square summability of covariances

- Define

$$\mathcal{H}_n^{(2)}(t) = \frac{1}{2n} \sum_{x,x'=0}^{n} \left\{ \left(\mathbb{E}_n\left[p_x(t)p_{x'}(t)\right]\right)^2 + \left(\mathbb{E}_n\left[r_x(t)r_{x'}(t)\right]\right)^2 + 2\left(\mathbb{E}_n\left[p_x(t)r_{x'}(t)\right]\right)^2 \right\}.$$

- there exists $C_{2,\mathcal{H}} > 0$ such that

$$\mathcal{H}_n^{(2)}(0) \leq C_{2,\mathcal{H}}.$$

Entropy estimate

Theorem (Entropy bound)

Under the assumption presented above, for any $t_* > 0$ there exists a constant $C_{H,t_*} > 0$ such that

$$\mathbf{H}_{n,T}(t) \leq C_{H,t_*} n, \qquad t \in [0, t_*].$$

- **Entropy bound - easy case:** $T_L = T_R = T$. Measure ν_T is **invariant** under the dynamics $\Rightarrow t \mapsto \mathbf{H}_{n,T}(t)$ is **not increasing**.

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Entropy inequality

- a version of Young's inequality, with the pair of **convex conjugate** functions: e^f and $g \log g$,

- for any $\alpha > 0$, probability measure μ on $(\Omega, \mathcal{F})$ and measurable $f, g : \Omega \to [0, +\infty)$:

$$\int_\Omega fg \mathrm{d}\mu \leq \frac{1}{\alpha} \left\{ \log \left(\int_\Omega e^{\alpha f} \mathrm{d}\mu \right) + \int_\Omega g \log g \mathrm{d}\mu \right\}$$

- **the entropy inequality** $\Rightarrow$

$$\mathbb{E}_{\mu_n} \left[\frac{1}{2} \sum_{x=0}^{n} \left(p_x^2(t) + r_x^2(t) \right) \right] = \int_{\Omega_n} \left(\frac{1}{2} \sum_{x=0}^{n} (p_x^2 + r_x^2) \right) f_n(t) \mathrm{d}\nu_T$$

$$\leq \frac{1}{\alpha} \left\{ \log \left(\int_{\Omega_n} \exp \left\{ \frac{\alpha}{2} \sum_{x=0}^{n} (p_x^2 + r_x^2) \right\} \mathrm{d}\nu_T \right) + \mathbf{H}_n(t) \right\}$$

Energy bound

- the entropy inequality $\Rightarrow$ the energy bound: $\exists\, C, C'$

$$\mathbb{E}_n\left[\mathcal{H}_n(t)\right] \le C\left(n + \mathbf{H}_{n,\beta}(t)\right) \le C' n, \qquad t \ge 0,\ n = 1, 2, \ldots.$$

Corollary

Energy bound For any $t_* > 0$ there exists $C_{\mathcal{H},t_*} > 0$ such that

$$\mathbb{E}_n\left[\mathcal{H}_n(t)\right] \le C_{\mathcal{H},t_*} n, \qquad t \in [0, t_*],\ n = 1, 2, \ldots. \tag{3}$$

Energy currents

- the energy currents:

$$\frac{d}{dt}\mathbb{E}_n\left[\mathcal{E}_x(t)\right] = -n^{3/2}\nabla^\star j_{x,x+1}(t), \quad x = 0,\ldots,n,$$

$$j_{x,x+1}(t) = j^{(a)}_{x,x+1}(t) + j^{(s)}_{x,x+1}, \quad \text{where}$$

$$j^{(a)}_{x,x+1}(t) := -p_x(t)r_{x+1}(t), \quad j^{(s)}_{x,x+1} = -\frac{\gamma}{2}(p^2_{x+1} - p^2_x)$$

- from thermostats:

$$j_{-1,0} := \tilde{\gamma}\left(T_L - p_0^2\right), \qquad j_{n,n+1} := \tilde{\gamma}\left(p_n^2 - T_R\right).$$

Bound on the time interval of energy currents

Theorem

For any $t_* > 0$ *there exists* $C_{\mathcal{J},t_*} > 0$ *such that*

$$\sup_{x=0,\ldots,n+2} \left| \int_0^t \mathbb{E}_n \left[j_{x-1,x}(s) \right] \mathrm{d}s \right| \le \frac{C_{\mathcal{J},t_*}}{\sqrt{n}}, \quad t \in [0, t_*],\ n = 1, 2, \ldots .$$

Identity for covariances

$$\mathcal{H}_n^{(2)}(t) + \underbrace{\text{Bulk}(t)}_{\geq 0} + \underbrace{\text{Bulk–boundary}(t)}_{\geq 0} = \mathcal{H}_n^{(2)}(0)$$

$$+ \frac{2\tilde{\gamma} n^{3/2}}{n+1} \int_0^t \left[T_L\Big(T_L - \mathbb{E}_n p_0^2(s)\Big) + T_R\Big(T_R - \mathbb{E}_n p_n^2(s)\Big)\right] \mathrm{d}s.$$

Bulk-boundary terms

$$\text{Bulk–boundary}(t) = \frac{2\tilde{\gamma} n^{3/2}}{n+1} \sum_{v=0,n} \int_0^t \left\{ T_v - \mathbb{E}_n \left[p_v^2(s) \right] \right\}^2 \mathrm{d}s$$

$$+ \frac{2\tilde{\gamma} n^{3/2}}{n+1} \sum_{v=0,n} \sum_{x=1}^{n} \int_0^t \left\{ \mathbb{E}_n \left[p_v(s) p_x(s) \right] \right\}^2 \mathrm{d}s$$

$$+ \frac{2\tilde{\gamma} n^{3/2}}{n+1} \sum_{v=0,n} \sum_{x'=1}^{n} \int_0^t \left\{ \mathbb{E}_n \left[p_v(s) r_{x'}(s) \right] \right\}^2 \mathrm{d}s$$

Bulk terms

$$\mathrm{Bulk}(t) := \frac{\gamma n^{3/2}}{n+1} \sum_{x=0}^{n-1} \int_0^t \left[\nabla \mathbb{E}_n p_x^2(s)\right]^2 \mathrm{d}s$$

$$+ \frac{2\gamma n^{3/2}}{n+1} \sum_{x=1}^{n} \sum_{\substack{x'=0 \\ x' \notin \{x-1,x\}}}^{n} \int_0^t \left\{\mathbb{E}_n \left[\nabla^\star p_x(s) p_{x'}(s)\right]\right\}^2 \mathrm{d}s$$

$$+ \frac{2\gamma n^{3/2}}{n+1} \sum_{x=1}^{n} \sum_{x'=1}^{n} \int_0^t \left\{\mathbb{E}_n \left[\nabla^\star p_x(s) r_{x'}(s)\right]\right\}^2 \mathrm{d}s$$

The special case $T_L = T_R = T$

- A direct calculation $\Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_n\mathcal{H}_n(t) = n^{3/2}\mathbb{E}_n\left(j_{-1,0}(t) - j_{n,n+1}(t)\right)\mathrm{d}s.$$

- $T_L = T_R = T \Rightarrow$

$$\mathcal{H}_n^{(2)}(t) \le \mathcal{H}_n^{(2)}(0) + \frac{2Tn^{3/2}}{n+1}\int_0^t \mathbb{E}_n\left(j_{-1,0}(s) - j_{n,n+1}(s)\right)\mathrm{d}s.$$

Corollary (In the case $T_L = T_R$)

Suppose that $T_L, T_R > 0$. Then, for any $t_* > 0$ there exists $C > 0$ s.t.

$$\mathcal{H}_n^{(2)}(t) \le C \tag{4}$$

- $\Rightarrow$ L^2 estimate for **the average energy functional**

Limit identification

- $\varphi \in C_c^\infty(0,1)$ - a test function,

$$\mathrm{E}_n(t;\varphi) = \frac{1}{n+1}\sum_{x=0}^{n}\varphi_x \mathbb{E}_n\left[\mathcal{E}_x(t)\right],$$

$$\varphi_x = \varphi(u_x), \quad u_x = \frac{x}{n+1},$$

- Summation by parts

$$\mathrm{E}_n(t,\varphi) - \mathrm{E}_n(0,\varphi) = -\frac{n^{3/2}}{n+1}\sum_{x=0}^{n}\int_0^t \varphi_x \mathbb{E}_n\left[\nabla^\star j_{x,x+1}(s)\right]\mathrm{d}s$$

$$= \frac{n^{1/2}}{n+1}\sum_{x=0}^{n-1}\varphi'_{n,x}\int_0^t \mathbb{E}_n\left[j_{x,x+1}(s)\right]\mathrm{d}s.$$

Here $\varphi'_{n,x} := n(\varphi_{x+1} - \varphi_x)$.

Current decomposition

$$\mathrm{E}_n(t;\varphi) - \mathrm{E}_n(0;\varphi) = J_n^{(a)}(t;\varphi') + J_n^{(s)}(t;\varphi') + o_n(1), \quad \text{where}$$

$$J_n^{(a)}(t;\varphi') := \frac{1}{\sqrt{n}} \sum_{x=1}^{n} \varphi'_x \int_0^t \mathbb{E}_n \left[j_{x-1,x}^{(a)}(s) \right] \mathrm{d}s$$

$$= -\frac{1}{\sqrt{n}} \sum_{x=1}^{n} \varphi'_x \int_0^t \mathbb{E}_n \left[p_{x-1}(s) r_x(s) \right] \mathrm{d}s,$$

$$J_n^{(s)}(t;\varphi') := -\frac{\gamma}{2\sqrt{n}} \sum_{x=0}^{n-1} \varphi'_x \int_0^t \mathbb{E}_n \left[\nabla p_x^2(s) \right] \mathrm{d}s$$

$$= \frac{\gamma}{2n^{3/2}} \sum_{x=0}^{n-1} \varphi''_x \int_0^t \mathbb{E} \left[p_x^2(s) \right] \mathrm{d}s + o_n(1) = o_n(1), \quad \text{and}$$

$$\varphi''_x := \varphi''(u_x).$$

Matrix of covariances

- $S(t)$ - the covariance matrix

$$S(t) = \begin{bmatrix} S^{(r)}(t) & S^{(r,p)}(t) \\ S^{(p,r)}(t) & S^{(p)}(t) \end{bmatrix},$$

where

$$S^{(r)}(t) = \Big[\mathbb{E}_n[r_x(t)r_y(t)]\Big]_{x,y=1,\dots,n},$$
$$S^{(p)}(t) = \Big[\mathbb{E}_n[p_x(t)p_y(t)]\Big]_{x,y=0,\dots,n}$$
$$S^{(r,p)}(t) = \Big[\mathbb{E}_n[r_x(t)p_y(t)]\Big]_{x=1,\dots,n,y=0,\dots,n} \quad \text{and}$$
$$S^{(p,r)}(t) = \Big[S^{(r,p)}(t)\Big]^T.$$

Dynamics in the matrix form

- $\mathbf{X}(t)$ - vector of **stretches and momenta**
- the solution of

$$d\mathbf{X}(t) = -n^{3/2}A\mathbf{X}(t)dt + \Sigma\Big(\mathbf{p}(t-)\Big)\mathrm{d}M_n(t),$$

- A is a 2×2 **block matrix**

$$A = \left(\begin{array}{cc} 0_n & -\nabla^\star \\ -\nabla & -\gamma\Delta_{\mathrm{N}} + \tilde{\gamma}E \end{array}\right),$$

where $E = [\delta_{x,0}\delta_{y,0} + \delta_{x,n}\delta_{y,n}]_{x,y=0,\ldots,n}$

- $\mathrm{d}M_n(t)$ - **zero mean vector, martingale**

$$\mathrm{d}M(s)^T = \Big(0_{n,1}, n^{3/4}\mathrm{d}w_L(s), \mathrm{d}\tilde{N}_{0,1}^{(n)}(\gamma s), \ldots, \mathrm{d}\tilde{N}_{n-1,n}^{(n)}(\gamma s), n^{3/4}\mathrm{d}w_R(s)\Big).$$

Matrix of covariances, cont'd

- using the dynamics:

$$\underbrace{A\langle\langle S\rangle\rangle_t + \langle\langle S\rangle\rangle_t A^T}_{\text{covariances, incl. boundary}} = \underbrace{\Sigma_2\left(\langle\langle\overline{(\nabla\mathbf{p})^2}\rangle\rangle_t\right)}_{\text{kin. energy flow in the bulk}} + \underbrace{\frac{1}{n^{3/2}}\delta_{0,t}S}_{\text{time fluctuating term}} ,$$

$$\langle\langle f\rangle\rangle_t := \int_0^t \mathbb{E}_n f(s)\mathrm{d}s,$$

$$\delta_{0,t}f := \mathbb{E}_n f(0) - \mathbb{E}_n f(t),$$

$$\overline{(\nabla\mathbf{p})^2}(s) = \left[\mathbb{E}_n\big(\nabla^\star p_1(s)\big)^2, \ldots, \mathbb{E}_n\big(\nabla^\star p_n(s)\big)^2\right].$$

Fourier transforms of covariances

- the Fourier transforms

$$\tilde{S}^{(r)}(t) = \left[\langle\langle \tilde{r}_j \tilde{r}_{j'} \rangle\rangle_t\right]_{j,j'=1,\ldots,n}, \quad \tilde{S}^{(rp)}(t) = \left[\langle\langle \tilde{r}_j \tilde{p}_{j'} \rangle\rangle_t\right]_{j=1,\ldots,n, j'=0,\ldots,n},$$

$$\tilde{S}^{(p)}(t) = \left[\langle\langle \tilde{p}_j \tilde{p}_{j'} \rangle\rangle_t\right]_{j,j'=0,\ldots,n} \quad \text{and} \quad \tilde{S}^{(pr)}(t) = \left[\tilde{S}^{(rp)}(t)\right]^T,$$

Fourier transforms: the bases

$$\tilde{r}_j(t) := \sum_{x=1}^{n} \phi_j(x) r_x(t) \quad \text{and} \quad \tilde{p}_j(t) := \sum_{x=0}^{n} \psi_j(x) p_x(t),$$

$$\psi_j(x) = \left(\frac{2-\delta_{0,j}}{n+1}\right)^{1/2} \cos\left(\frac{\pi j(2x+1)}{2(n+1)}\right),$$

$$\phi_j(x) = \left(\frac{2}{n+1}\right)^{1/2} \sin\left(\frac{jx\pi}{n+1}\right),$$

$$\nabla^\star \psi_j = -\gamma_j \phi_j \quad \text{and} \quad \nabla \phi_j = \gamma_j \psi_j, \quad \gamma_j = 2\sin\left(\frac{j\pi}{2(n+1)}\right)$$

$$\lambda_j = \gamma_j^2 = 4\sin^2\left(\frac{j\pi}{2(n+1)}\right).$$

Resolution of covariances

- for $\iota \in I = \{p, r, pr, r\}$

$$\tilde{S}^{(\iota)}_{j,j'} = \underbrace{\Theta_\iota(\lambda_j, \lambda_{j'}) F_{j,j'}}_{\text{bulk}} + \underbrace{\sum_{\iota' \in I} \Pi^{(\iota)}_{\iota'}(\lambda_j, \lambda_{j'}) B^{(\iota')}_{j,j'}}_{\text{boundary}} + \underbrace{\sum_{\iota' \in I} \Xi^{(\iota)}_{\iota'}(\lambda_j, \lambda_{j'}) R^{(\iota')}_{j,j'}}_{\text{fluctuations}=o_n(1)}$$

$$F_{j,j'} = \gamma \sum_{y=1}^{n} \phi_j(y)\phi_{j'}(y)\langle\langle(\nabla^\star p_y)^2\rangle\rangle_t,$$

$$R^{(\iota)}_{j,j'} = \frac{1}{n^{3/2}}\delta_{0,t}\tilde{S}^{(\iota)}_{j,j'}$$

- $p - r$ boundary-bulk covariances:

$$B^{(pr)}_{j,j'} = \psi_j(0)\tilde{s}^{(p,\tilde{r})}_{0,j'} + \psi_j(n)\tilde{s}^{(p,\tilde{r})}_{n,j'}, \quad B^{(rp)}_{j,j'} = B^{(pr)}_{j',j},$$

$$\tilde{s}^{(p,\tilde{r})}_{z,j} = \langle\langle \tilde{r}_j p_z\rangle\rangle_t,$$

Resolution of covariances

- for $\iota \in I = \{p, r, pr, r\}$

$$\tilde{S}^{(\iota)}_{j,j'} = \underbrace{\Theta_\iota(\lambda_j, \lambda_{j'}) F_{j,j'}}_{\text{bulk}} + \underbrace{\sum_{\iota' \in I} \Pi^{(\iota)}_{\iota'}(\lambda_j, \lambda_{j'}) B^{(\iota')}_{j,j'}}_{\text{boundary}} + \underbrace{\sum_{\iota' \in I} \Xi^{(\iota)}_{\iota'}(\lambda_j, \lambda_{j'}) R^{(\iota')}_{j,j'}}_{\text{fluctuations}=o_n(1)}$$

$$F_{j,j'} = \gamma \sum_{y=1}^{n} \phi_j(y)\phi_{j'}(y)\langle\langle(\nabla^\star p_y)^2\rangle\rangle_t,$$

$$R^{(\iota)}_{j,j'} = \frac{1}{n^{3/2}} \delta_{0,t} \tilde{S}^{(\iota)}_{j,j'}$$

- $p - r$ **boundary-bulk covariances:**

$$B^{(pr)}_{j,j'} = \psi_j(0)\tilde{s}^{(p,\tilde{r})}_{0,j'} + \psi_j(n)\tilde{s}^{(p,\tilde{r})}_{n,j'}, \quad B^{(rp)}_{j,j'} = B^{(pr)}_{j',j},$$

$$\tilde{s}^{(p,\tilde{r})}_{z,j} = \langle\langle \tilde{r}_j p_z \rangle\rangle_t,$$

Resolution of covariances - cont'd

- $p-p$ boundary-bulk covariances:

$$B_{j,j'}^{(p)} = B_{j,j'}^{(p,0)} + B_{j,j'}^{(p,n)}, \quad \text{where}$$

$$B_{j,j'}^{(p,z)} = \psi_j(z)\langle\langle\tilde{b}_{z,j'}^{(p)}\rangle\rangle_t + \langle\langle\tilde{b}_{z,j}^{(p)}\rangle\rangle_t\psi_{j'}(z) \quad \text{and}$$

$$\tilde{b}_{z,j}^{(p)}(s) := \sum_{x=0}^{n} b_{z,x}^{(p)}(s)\psi_j(x),$$

$$b_{0,0}^{(p)}(s) = T_L - \mathbb{E}_n p_0^2(s), \quad b_{n,n}^{(p)}(s) = T_R - \mathbb{E}_n p_n^2(s),$$

$$b_{z,x}^{(p)}(s) = -\mathbb{E}_n\big[p_z(s)p_x(s)\big], \quad x \neq z.$$

- coefficients $\Theta_{\iota'}^{(\iota)}(c,c')$, $\Pi_{\iota'}^{(\iota)}(c,c')$ i $\Xi_{\iota'}^{(\iota)}(c,c')$ can be computed explicitly

Coefficients Θ, Π, Ξ

$$\Theta_{pr}(c,c') = \frac{(c-c')\sqrt{c'}}{\theta(c,c')} = \Xi_p^{(pr)}(c,c'),$$

$$\Pi_p^{(pr)}(c,c') = \tilde{\gamma}\Theta_{pr}(c,c'),$$

$$\Pi_{pr}^{(pr)}(c,c') = -\tilde{\gamma}\Xi_{pr}^{(pr)}(c,c'), \quad \Xi_{pr}^{(pr)}(c,c') = \frac{\gamma c'(c+c')}{\theta(c,c')},$$

$$\Pi_{rp}^{(pr)}(c,c') = -\tilde{\gamma}\Xi_{rp}^{(pr)}(c,c'), \quad \Xi_{rp}^{(pr)}(c,c') = -\frac{\gamma(c+c')\sqrt{cc'}}{\theta(c,c')}$$

$$\Pi_r^{(pr)}(c,c') = 0, \quad \Xi_r^{(pr)}(c,c') = \frac{1}{2\sqrt{c}}\left[1+\frac{c^2-(c')^2}{\theta(c,c')}\right],$$

$$\theta(c,c') = (c-c')^2 + 2\gamma^2 cc'(c+c').$$

Limit identification cont'd

- back to the limit identification, $I = \{p, pr, rp, r\}$,

$$\mathrm{E}_n(t;\varphi) - \mathrm{E}_n(0;\varphi) = -\frac{1}{n}\sum_{j,j'} \tilde{S}^{(pr)}_{j',j} \mathscr{W}_{j,j'} + o_n(1)$$

$$= \underbrace{-\frac{1}{n}\theta_{pr}(\varphi';n)}_{\text{bulk}} \underbrace{-\frac{1}{n}\sum_{\iota\in I}\pi^{(pr)}_{\iota}(\varphi';n)}_{\text{boundary-bulk}} \underbrace{-\frac{1}{n}\sum_{\iota\in I}\xi^{(pr)}_{\iota}(\varphi';n)}_{\text{fluctuations}} + o_n(1).$$

Coefficients

$$\theta_{pr}(\varphi'; n) = \sum_{j,j'=1}^{n} \mathcal{W}_{j,j'} \sqrt{\lambda_j \lambda_{j'}} \Theta_{pr}(\lambda_j, \lambda_{j'}) F_{j,j'},$$

$$\pi_\iota^{(pr)}(\varphi'; n) = \sum_{j=0}^{n} \sum_{j'=1}^{n} \mathcal{W}_{j,j'} \Pi_\iota^{(pr)}(\lambda_j, \lambda_{j'}) B_{j,j'}^{(\iota)}, \quad \iota \in I := \{p, pr, rp, r\}$$

$$\xi_\iota^{(pr)}(\varphi'; n) = \sum_{j=0}^{n} \sum_{j'=1}^{n} \mathcal{W}_{j,j'} \Xi_\iota^{(pr)}(\lambda_j, \lambda_{j'}) R_{j,j'}^{(\iota)},$$

$$\mathcal{W}_{j,j'} := \sqrt{n} \sum_{x=1}^{n} \phi_{j'}(x) \psi_j(x-1) \varphi'(u_x),$$

$$F_{j,j'} = \gamma \sum_{y=1}^{n} \phi_j(y) \phi_{j'}(y) \langle\langle (\nabla^\star p_y)^2 \rangle\rangle_t, \quad R_{j,j'}^{(\iota)} = \frac{1}{n^{3/2}} \delta_{0,t} \tilde{S}_{j,j'}^{(\iota)}$$

Asymptotics of the bulk term

Proposition

For any $\varphi \in C_c^\infty(0,1)$ we have

$$\frac{1}{n}\theta_{pr}(\varphi'; n) = \frac{1}{(2^3\gamma)^{1/2} n}\sum_{y=1}^{n}\langle\langle \mathcal{E}_y\rangle\rangle_t \sum_{\ell=1}^{+\infty}(\pi\ell)^{3/2}\hat{\varphi}_c(\ell)c_\ell(u_y) + o_n(1)$$

$$= \frac{1}{(2^3\gamma)^{1/2}}\int_0^t \mathrm{E}_n\left(s, |\Delta_{\mathrm{N}}|^{3/4}\varphi\right) \mathrm{d}s + o_n(1),$$

Here: $c_0(u) := 1, \qquad c_\ell(u) := \sqrt{2}\cos(\pi\ell u),$

$$\hat{\varphi}_c(\ell) := \int_0^1 \varphi(u)c_\ell(u)du,$$

$$|\Delta_N|^{3/4}\varphi(u) = \sum_{n=0}^{+\infty}(n\pi)^{3/2}\hat{\varphi}_c(n)c_n(u)$$

Boundary layer estimates

- thanks to the L^2 control of covariances

$$(n+1)^{1/2}\sum_{j=0}^{n}\left(\tilde{b}^{(p)}_{z,j}(t)\right)^2=(n+1)^{1/2}\sum_{x=0}^{n}\left(b^{(p)}_{z,x}(t)\right)^2\le C,\quad n=1,2,\ldots$$

- **Boundary layer:** $v=0,1,\ \mathfrak{b}^{(p,v)}_n:[0,+\infty)^2\to\mathbb{R}$
- where

$$\mathfrak{b}^{(p,v)}_n(t,\varrho)=0,\quad \varrho\ge(n+1)^{2/3}\pi,$$
$$\mathfrak{b}^{(p,v)}_n(t,\varrho)=(n+1)^{1/2}\tilde{b}^{(p)}_{nv,j}(t),\ 0\le j\le(n+1)^{2/3},\ \varrho\in\left[\varrho_j,\varrho_{j+1}\right),$$
$$\varrho_j:=\frac{j\pi}{(n+1)^{1/2}}$$

- $p - p$ **covariances:**

$$\tilde{b}^{(p)}_{z,j}(s) := \sum_{x=0}^{n} b^{(p)}_{z,x}(s)\psi_j(x),$$

$$b^{(p)}_{0,0}(s) = T_L - \mathbb{E}_n p_0^2(s), \quad b^{(p)}_{n,n}(s) = T_R - \mathbb{E}_n p_n^2(s),$$

$$b^{(p)}_{z,x}(s) = -\mathbb{E}_n\left[p_z(s)p_x(s)\right], \quad x \neq z.$$

Boundary layer estimates

- for any $t > 0$ **there exists** $C > 0$ such that

$$\int_0^t \mathrm{d}s \int_0^{+\infty} \left[\mathfrak{b}_n^{(p,v)}(s,\varrho)\right]^2 \mathrm{d}\varrho \le C, \qquad v = 0, 1,\ n = 1, 2, \ldots,$$

Asymptotics of the boundary terms

Proposition

As $n \to +\infty$:

$$\pi_r^{(pr)}(\varphi'; n) = 0,$$

$$\frac{1}{n}\left(\pi_{pr}^{(pr)}(\varphi'; n) + \pi_{pr}^{(pr)}(\varphi'; n)\right) = o_n(1),$$

$$\frac{1}{n}\pi_p^{(pr)}(\varphi'; n)(n) = \frac{2^{1/2}\tilde{\gamma}}{\pi} \sum_{v=0,1} \sum_{\ell=1}^{+\infty} c_\ell(nv)(\pi\ell)^2 \hat{\varphi}_c(\ell) \times \int_0^t \mathrm{d}s \int_0^{+\infty} \frac{\mathfrak{b}_n^{(p,v)}(s,\varrho)\mathrm{d}\varrho}{(\pi\ell)^2 + \gamma^2\varrho^4} + o_n(1).$$

Identification of the limit of $\mathfrak{b}_n^{(p,v)}(s,\varrho)$ - closure of the equation for the energy

Proposition

For any test function $f \in L^2[0,+\infty)$, $t>0$ and $v=0,1$ we have

$$
\begin{aligned}
&(1+2\tilde{\gamma})\int_0^t \mathrm{d}s\int_0^{+\infty} \mathfrak{b}_n^{(p,v)}(s,\varrho)f(\varrho)\mathrm{d}\varrho \\
&= \sqrt{2}\int_0^t \mathrm{d}s\int_0^{+\infty}\left(T_v-\sum_{\ell=0}^{+\infty}\frac{\gamma^2\varrho^4 c_\ell(v)\widehat{\mathfrak{E}}_n(s,\ell)}{(\ell\pi)^2+\gamma^2\varrho^4}\right)f(\varrho)\mathrm{d}\varrho \\
&+\frac{\tilde{\gamma}}{\pi}\int_0^t \mathrm{d}s\int_0^{+\infty}\mathfrak{b}_n^{(pr,v)}(s,\varrho)\mathfrak{T}f(\varrho)\mathrm{d}\varrho+o_n(1).
\end{aligned}
$$

Energy Fourier coefficients & operator $\mathfrak{T}$

- **Fourier cosine coefficients** of the energy functional

$$\widehat{\mathfrak{E}}_n(s,\ell) := \frac{1}{n+1}\sum_{y=0}^{n} \mathbb{E}_n \mathcal{E}_y(s) c_\ell(u_y),$$

$$c_0(u) := 1, \qquad c_\ell(u) := \sqrt{2}\cos(\pi \ell u), \quad u_y = \frac{y}{n+1},$$

- **operator** $\mathfrak{T} : C_c^1[0,+\infty) \to L^p[0,+\infty)$:

$$\mathfrak{T}f(\varrho) = 2\int_0^{+\infty} \frac{[f(\varrho') - f(\varrho)]\varrho}{(\varrho-\varrho')(\varrho+\varrho')} \mathrm{d}\varrho', \quad f \in C_c^1[0,+\infty).$$

It **extends continuously** to $L^p[0,+\infty)$ for any $p \in (1,+\infty)$

Boundary layer estimates: $p - r$ covariances

- $p - r$ **covariances:**

$$\tilde{b}^{(p,\tilde{r})}_{z,j}(s) = \mathbb{E}_n\big[\tilde{r}_j(s)p_z(s)\big] = \sum_{x=1}^{n} \phi_j(x)\mathbb{E}_n\big[p_z(s)r_x(s)\big],$$

- **Boundary layer:** $v = 0, 1,\ \mathfrak{b}^{(pr,v)}_n : [0, +\infty)^2 \to \mathbb{R}$

- where

$$\mathfrak{b}^{(pr,v)}_n(t, \varrho) = 0, \quad \varrho \ge (n+1)^{2/3}\pi,$$
$$\mathfrak{b}^{(pr,v)}_n(t, \varrho) = (n+1)^{1/2}\tilde{b}^{(pr)}_{nv,j}(t),\ 0 \le j \le (n+1)^{2/3},\ \varrho \in \big[\varrho_j, \varrho_{j+1}\big),$$
$$\varrho_j := \frac{j\pi}{(n+1)^{1/2}}$$

L^2 estimate

- thanks to the L^2 control of covariances

$$(n+1)^{1/2}\sum_{j=0}^{n}\left(\tilde{b}_{z,j}^{(pr)}(t)\right)^2=(n+1)^{1/2}\sum_{x=0}^{n}\left(b_{z,x}^{(pr)}(t)\right)^2\le C$$

for $n=1,2,\ldots,\ z=0,n$.

- $\Rightarrow$ for any $t>0$ there exists $C>0$ such that

$$\int_0^t \mathrm{d}s\int_0^{+\infty}\left[\mathfrak{b}_n^{(pr,v)}(s,\varrho)\right]^2\mathrm{d}\varrho\le C,\qquad v=0,1,\ n=1,2,\ldots,$$

L^2 estimate

- thanks to the L^2 control of covariances

$$(n+1)^{1/2}\sum_{j=0}^{n}\left(\tilde{b}_{z,j}^{(pr)}(t)\right)^2=(n+1)^{1/2}\sum_{x=0}^{n}\left(b_{z,x}^{(pr)}(t)\right)^2\le C$$

for $n=1,2,\ldots,\ z=0,n$.

- $\Rightarrow$ for any $t>0$ there exists $C>0$ such that

$$\int_0^t \mathrm{d}s\int_0^{+\infty}\left[\mathfrak{b}_n^{(pr,v)}(s,\varrho)\right]^2\mathrm{d}\varrho\le C,\qquad v=0,1,\ n=1,2,\ldots,$$

Proposition

For any test function $f \in L^2[0,+\infty)$, $t > 0$ and $v = 0, 1$ we have

$$\int_0^t \mathrm{d}s \int_0^{+\infty} \mathscr{b}_n^{(pr,v)}(s,\varrho) f(\varrho) \mathrm{d}\varrho$$
$$= -\frac{\tilde{\gamma}}{\pi} \int_0^t \mathrm{d}s \int_0^{+\infty} \mathscr{b}_n^{(p,v)}(s,\varrho) \mathfrak{T}^{\star} f(\varrho) \mathrm{d}\varrho + o_n(1).$$

Properties of $\mathfrak{T}$

- $\mathfrak{T}^\star$ - the formal adjoint of $\mathfrak{T}$

$$\mathfrak{T}^\star f(\varrho) := 2\int_0^{+\infty} \frac{[\varrho' f(\varrho') - \varrho f(\varrho)]\mathrm{d}\varrho'}{(\varrho' - \varrho)(\varrho' + \varrho)}, \quad f \in C_c^1[0,+\infty).$$

- It extends continuously to the space $L^2[0,+\infty)$ and

$$\mathfrak{T}^\star \mathfrak{T} = \pi^2 \mathrm{Id},$$

where Id is the identity operator on $L^2[0,+\infty)$.

Theorem

For any test function $f \in L^2[0,+\infty)$, $t>0$ *and* $v=0,1$ *we have*

$$\int_0^t \mathrm{d}s \int_0^{+\infty} \mathscr{b}_n^{(p,v)}(s,\varrho) f(\varrho)\mathrm{d}\varrho = \frac{\sqrt{2}}{(1+\tilde{\gamma})^2} \times \int_0^t \mathrm{d}s \int_0^{+\infty} \left(T_v - \sum_{\ell=0}^{+\infty} \frac{\gamma^2 \varrho^4 c_\ell(v) \widehat{\mathfrak{E}}_n(s,\ell)}{(\ell\pi)^2 + \gamma^2\varrho^4} \right) f(\varrho)\mathrm{d}\varrho + o_n(1)$$

where $\widehat{\mathfrak{E}}_n(t,\ell)$ the Fourier cosine coeffcients *of* the energy functional.

- Change of variables $\varrho' := \sqrt{\gamma}\varrho$. This ends the proof.

Energy bound - the case of arbitrary $T_L, T_R > 0$

- relative entropy w.r.t. the tilted Gibbs measure

$$\mathbf{H}_{n,\beta}(t) := \int_{\Omega_n} \tilde{f}_n(t) \log \tilde{f}_n(t) \mathrm{d}\nu_\beta,$$

$$\tilde{f}_n(t) := \frac{\mathrm{d}\mu_n(t)}{\mathrm{d}\nu_\beta}$$

- Here for $\beta : [0,1] \to (0, +\infty)$

$$\nu_\beta(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := \frac{e^{-\beta_0 p_0^2/2}}{\sqrt{2\pi\beta_0^{-1}}} \mathrm{d}p_0 \prod_{x=1}^{n} \exp\left\{ -\beta_x \mathcal{E}_x - g(\beta_x) \right\} \mathrm{d}r_x \mathrm{d}p_x,$$

$$g(\beta) := \log \int_{\mathbb{R}^2} e^{-\frac{\beta}{2}(r^2+p^2)} \mathrm{d}p \mathrm{d}r = \log\left(2\pi\beta^{-1}\right), \quad \beta > 0,$$

$$\beta_x := \beta\left(\frac{x}{n+1}\right),$$

Estimates of entropy

• easy estimate: $\beta : [0,1] \to [T_L^{-1}, T_R^{-1}]$ a C^1-smooth function s.t.

$$\beta' \geq 0, \quad \operatorname{supp} \beta' \subset (0,1), \quad \beta(0) = T_L^{-1} \quad \text{and } \beta(1) = T_R^{-1}.$$

$$\mathbf{H}_{n,\beta}(t) \leq \mathbf{H}_{n,\beta}(0) + n^{1/2}|J_n(t;\beta')| + \frac{C}{n^{1/2}} \int_0^t \mathbb{E}_n \mathscr{H}_n(s) \mathrm{d}s,$$

$$J_n(t;\beta') = \sum_{x=1}^{n} \beta'_x \int_0^t \mathbb{E}_n j^{(a)}_{x-1,x}(s) \mathrm{d}s$$ - integral Hamiltonian current

Crucial estimate

- crucial estimate, quite technical

$$n^{1/2}|J_n(t;\beta')| \le C\Big[n + n^{3/4}|J_n(t,\beta')|^{1/2} + \mathbb{E}_n\mathcal{H}_n(0) + \mathbb{E}_n\mathcal{H}_n(t)$$
$$+ \int_0^t \mathbb{E}_n\mathcal{H}_n(s)\mathrm{d}s + \Big(n\mathbb{E}_n\mathcal{H}_n(0)\Big)^{1/2} + \Big(n\mathbb{E}_n\mathcal{H}_n(t)\Big)^{1/2}$$
$$+ \Big(n\int_0^t \mathbb{E}_n\mathcal{H}_n(s)\mathrm{d}s\Big)^{1/2}\Big].$$

- Young's inequality

$$ab \le \frac{a^2}{2\gamma} + \frac{\gamma b^2}{2}, \quad a, b, \gamma > 0,$$

$\Rightarrow$

$$n^{1/2}|J_n(t;\beta')| \le C\Big(n + \int_0^t \mathbb{E}_n\mathcal{H}_n(s)\mathrm{d}s + \mathbb{E}_n\mathcal{H}_n(t) + \mathbb{E}_n\mathcal{H}_n(0)\Big).$$

Energy and current bounds

- Recall

$$\mathbf{H}_{n,\beta}(t) \le \mathbf{H}_{n,\beta}(0) + n^{1/2}|J_n(t;\beta')| + \frac{C}{n^{1/2}}\int_0^t \mathbb{E}_n\mathcal{H}_n(s)\mathrm{d}s$$

$\Rightarrow$

$$\mathbf{H}_{n,\beta}(t) \le \mathbf{H}_{n,\beta}(0) + C\Big(n + \int_0^t \mathbb{E}_n\mathcal{H}_n(s)\mathrm{d}s + \mathbb{E}_n\mathcal{H}_n(t) + \mathbb{E}_n\mathcal{H}_n(0)\Big).$$

$\Rightarrow$ **energy bound**

$$\forall\, t_* > 0 \exists\, C > 0 \sup_{t\in[0,t_*]} \mathbb{E}_n\mathcal{H}_n(t) \le C$$

Time integral of the energy current

- $\Rightarrow |J_n(t;\beta')| \le C\sqrt{n}$
- Estimate of **the supremum** of the current

$$\sup_{x=0,\dots,n+1} \left| \int_0^t \mathbb{E}_n j_{x-1,x}(s)\mathrm{d}s \right|$$
$$\le \Big(\sum_{x=1}^n \beta_x\Big)^{-1} \Bigg\{ |J_n(t;\beta)| + \|\beta\|_\infty (T_L + T_R)t$$
$$+ \frac{1}{n^{3/2}}\Big[(n+1) + \sum_{x=1}^n \beta_x\Big]\big(\mathbb{E}_n\mathcal{H}_n(t) + \mathbb{E}_n\mathcal{H}_n(0)\big)$$
$$+ \frac{\|\beta\|_\infty}{n^{3/2}}\mathbb{E}\mathcal{H}_n(0) + \frac{\gamma\|\beta'\|_\infty}{2n}\left|\int_0^t \mathcal{H}_n(s)\mathrm{d}s\right| \Bigg\}.$$

$\Rightarrow \sup_{x=0,\dots,n+1} |\int_0^t \mathbb{E}_n j_{x-1,x}(s)\mathrm{d}s| \le \frac{C}{\sqrt{n}}$

Solving the equation for temperature

- Equations for the Fourier coefficients.

Suppose that $T(t,\cdot) \in H^{3/4}[0,1]$ for $t \geq 0$. Then the Fourier coefficients of $T(t,\cdot)$ satisfy :

$$\sum_{\ell=0}^{+\infty} c_\ell(v)\hat{T}_c(t,\ell)\mathrm{d}s = T_v, \quad v = 0,1,$$

$$\sum_{\ell=0}^{+\infty} \hat{T}_c(t,\ell)\hat{\varphi}_c(\ell) - \sum_{\ell=0}^{+\infty} \hat{T}_{\mathrm{ini},c}(\ell)\hat{\varphi}_c(\ell)$$

$$= -c_{\mathrm{bulk}}\sum_{\ell=0}^{+\infty}(\pi\ell)^{3/2}\hat{\varphi}_c(\ell)\int_0^t \hat{T}_c(s,\ell)\mathrm{d}s$$

$$-\,2^{1/2}\pi c_{\mathrm{bd}}\sum_{v=0,1}\sum_{\ell,\ell'=1}^{+\infty}\widehat{\mathscr{K}}_v(\ell,\ell')\hat{\varphi}_c(\ell)\int_0^t \hat{T}_c(s,\ell')\mathrm{d}s$$

for all $\varphi \in H_0^{3/4}[0,1]$.

- For $\ell, \ell' = 1, 2, \ldots$, $c_\ell(v) = \sqrt{2}\cos(\pi\ell v)$

$$\widehat{\mathcal{K}}_v(\ell, \ell') := \frac{c_\ell(v)c_{\ell'}(v)(\pi\ell)^{1/2}(\pi\ell')^{1/2}(\pi\ell + \pi\ell' + (\pi\ell\pi\ell')^{1/2})}{\big((\pi\ell)^{1/2} + (\pi\ell')^{1/2}\big)(\pi\ell + \pi\ell')}$$

- For trig. polynomials $\varphi, \psi \in P[0,1]$

$$\mathcal{E}_K(\varphi, \psi) := \frac{\pi}{2^{3/2}} \sum_{v=0,1} \sum_{\ell,\ell'=1}^{+\infty} \widehat{\mathcal{K}}_v(\ell, \ell')\hat{\varphi}_c(\ell)\hat{\psi}_c(\ell')$$

Proposition

There exists $C_K > 0$ such that

$$0 \le \mathcal{E}_K(\varphi) \le C_K \|\varphi\|^2_{3/4,0} := \Big(\sum_{\ell=1}^{+\infty} (\pi\ell)^{3/2}\hat{\varphi}_c^2(\ell)\Big)^{1/2}, \quad \varphi \in P[0,1],$$

where $\mathcal{E}_K(\varphi) := \mathcal{E}_K(\varphi, \varphi)$.

- **boundary conditions** $\Rightarrow$

$$\sum_{\ell=0}^{+\infty} \hat{T}_c(t,2\ell)c_{2\ell}(0) = \bar{T} = \frac{1}{2}(T_L + T_R) \quad \text{and}$$

$$\sum_{\ell=1}^{+\infty} \hat{T}_c(t,2\ell-1)c_{2\ell-1}(0) = \frac{1}{2}\Delta T, \quad \text{where} \quad \Delta T = \ T_L - T_R.$$

Even and odd modes

Denote

$$L^2_e[0,1] := \left[\varphi(u) := \sum_{\ell=0}^{+\infty} \hat{\varphi}_c(2\ell) c_{2\ell}(u)\right],$$

$$L^2_o[0,1] := \left[\varphi(u) := \sum_{\ell=1}^{+\infty} \hat{\varphi}_c(2\ell-1) c_{2\ell-1}(u)\right]$$

counterparts $H^{3/4}_\iota[0,1]$, $\iota = e, o$ - subspaces of $H^{3/4}[0,1]$.

- We have

$$\mathcal{E}_K(\varphi,\psi) = \frac{\pi}{2^{3/2}} \sum_{\ell,\ell'=1}^{+\infty} [1+(-1)^{\ell+\ell'}]\widehat{\mathcal{K}}(\ell,\ell')\hat{\varphi}_c(\ell)\hat{\psi}_c(\ell'), \quad \varphi,\psi \in P[0,1],$$

$$\widehat{\mathcal{K}}(\ell,\ell') := \frac{(\pi\ell)^{1/2}(\pi\ell')^{1/2}(\pi\ell+\pi\ell'+(\pi\ell\pi\ell')^{1/2})}{\big((\pi\ell)^{1/2}+(\pi\ell')^{1/2}\big)(\pi\ell+\pi\ell')}, \quad \ell,\ell' = 1,2,\dots \tag{5}$$

- $\Rightarrow$ Equation for evolution of temperature decouples into even and odd modes.

Even modes

- for **even indexed** Fourier coefficients.

$$\sum_{\ell=0}^{+\infty} \hat{T}_c(t,2\ell)\hat{\varphi}_c(2\ell) - \sum_{\ell=0}^{+\infty} \hat{T}_{\mathrm{ini},c}(2\ell)\hat{\varphi}_c(2\ell)$$

$$= -c_{\mathrm{bulk}} \sum_{\ell=1}^{+\infty} (2\pi\ell)^{3/2}\hat{\varphi}_c(2\ell)\int_0^t \hat{T}_c(s,2\ell)\mathrm{d}s$$

$$-\, 2^{3/2}\pi c_{\mathrm{bd}} \sum_{\ell,\ell'=1}^{+\infty} \widehat{\mathcal{K}}_e(\ell,\ell')\hat{\varphi}_c(2\ell)\int_0^t \hat{T}_c(s,2\ell')\mathrm{d}s$$

$$\text{for all} \quad \varphi \in H_e^{3/4}[0,1] \quad \text{s.t.} \sum_{\ell=0}^{+\infty} \hat{\varphi}_c(2\ell)c_{2\ell}(v) = 0, \quad v = 0,1$$

subject to the condition $\sum_{\ell=0}^{+\infty} \hat{T}_c(t,2\ell)c_{2\ell}(0) = \bar{T} = \frac{1}{2}(T_L + T_R)$.
Here $\widehat{\mathcal{K}}_e(\ell,\ell') := \widehat{\mathcal{K}}(2\ell,2\ell')$.

Odd harmonics

$$
\sum_{\ell=1}^{+\infty} \hat{T}_c(t,2\ell-1)\hat{\varphi}_c(2\ell-1) - \sum_{\ell=0}^{+\infty} \hat{T}_{\mathrm{ini},c}(2\ell-1)\hat{\varphi}_c(2\ell-1)
$$

$$
= -c_{\mathrm{bulk}} \sum_{\ell=1}^{+\infty} \big(\pi(2\ell-1)\big)^{3/2} \hat{\varphi}_c(2\ell-1) \int_0^t \hat{T}_c(s,2\ell-1)
$$

$$
- 2^{3/2}\pi c_{\mathrm{bd}} \sum_{\ell,\ell'=1}^{+\infty} \widehat{\mathcal{K}}_o(\ell,\ell')\hat{\varphi}_c(2\ell-1) \int_0^t \hat{T}_c(s,2\ell'-1)\mathrm{d}s,
$$

for all $\varphi \in H_o^{3/4}[0,1]$ s.t. $\sum_{\ell=1}^{+\infty} \hat{\varphi}_c(2\ell-1)c_{2\ell-1}(v) = 0, \quad v = 0,1,$

subject to the condition $\sum_{\ell=1}^{+\infty} \hat{T}_c(t,2\ell-1)c_{2\ell-1}(0) = \frac{1}{2}\Delta T$, where $\Delta T = T_L - T_R$. Here $\widehat{\mathcal{K}}_o(\ell,\ell') := \widehat{\mathcal{K}}(2\ell-1,2\ell'-1)$.

- $\left(Q_t^{(\iota)}\right)$ a strongly continuous semigroup associated with each equation

$$Q_t^{(\iota)}\varphi(u) = \sum_{m=0}^{+\infty} e^{-\lambda_m^{(\iota)} t} \langle \vartheta_m^{(\iota)}, \varphi \rangle_{L^2[0,1]} \vartheta_m^{(\iota)}(u) \qquad (6)$$

for $\varphi \in L_\iota^2[0,1]$, $\iota = o, e$, $L^{(\iota)}$ the generator.

- $\vartheta_m^{(\iota)}$, $m = 1, 2, \ldots$, the orthonormal bases of eigenvectors of $L^{(\iota)}$, eigenvalues $0 < \lambda_1^{(\iota)} \le \lambda_2^{(\iota)} \le \ldots$,

$$\sum_{m=1}^{+\infty} \frac{1}{\lambda_m^{(\iota)}} < +\infty.$$

- Convention $\vartheta_0^{(e)}(u) \equiv 1$ and $\vartheta_0^{(o)}(u) \equiv 0$ and $\lambda_0^{(\iota)} = 0$.

- Looking for solutions of the form

$$T_\iota(t,u) = Q_t^{(\iota)} T_{\mathrm{ini},e}(u) + \int_0^t \mathrm{c}_\iota(s) Q_{t-s}^{(\iota)} \delta_\iota(u)\mathrm{d}s. \tag{7}$$

where

$$\langle \delta_\iota, \varphi \rangle = \bar{\varphi}^{(\iota)}, \quad \text{where} \quad \bar{\varphi}^{(e)} = \frac{1}{2}\big(\varphi(0) + \varphi(1)\big),$$
$$\bar{\varphi}^{(o)} = \frac{1}{2}\big(\varphi(0) - \varphi(1)\big), \quad \varphi \in H^{3/4}[0,1].$$

- Looking for solutions of the form

$$T_\iota(t,u) = Q_t^{(\iota)} T_{\mathrm{ini},e}(u) + \int_0^t \mathrm{c}_\iota(s) Q_{t-s}^{(\iota)} \delta_\iota(u) \mathrm{d}s. \tag{8}$$

where

$$\langle \delta_\iota, \varphi \rangle = \bar{\varphi}^{(\iota)}, \quad \text{where} \quad \bar{\varphi}^{(e)} = \frac{1}{2}\big(\varphi(0) + \varphi(1)\big),$$
$$\bar{\varphi}^{(o)} = \frac{1}{2}\big(\varphi(0) - \varphi(1)\big), \quad \varphi \in H^{3/4}[0,1].$$

- functions $\mathrm{c}_\iota : [0, +\infty) \to \mathbb{R}$, $\iota = e, o$ determined by the equations

$$\bar{T} = \langle T_e(t), \delta_e\rangle = \langle Q_t^{(e)} T_{\mathrm{ini},e}, \delta_e\rangle + \int_0^t \mathrm{c}_e(s)\langle \delta_e, Q_{t-s}^{(e)}\delta_e\rangle \mathrm{d}s$$

$$\frac{1}{2}\Delta T = \langle \Delta\delta_o, T_{c,o}(t)\rangle = \langle Q_t^{(o)} T_{\mathrm{ini},o}, \delta_o\rangle + \int_0^t \mathrm{c}_o(s)\langle \delta_o, Q_{t-s}^{(o)}\delta_o\rangle \mathrm{d}s.$$

- Performing **the Laplace transform**

$$\frac{\bar{T}}{\lambda} = \langle (\lambda + L^{(e)})^{-1} T_{\mathrm{ini},e}, \delta_e\rangle + \tilde{\mathrm{c}}_e(\lambda)\langle \delta_e, (\lambda + L^{(e)})^{-1}\delta_e\rangle,$$

$$\frac{\Delta T}{2\lambda} = \langle \delta_o, (\lambda + L^{(o)})^{-1} T_{\mathrm{ini},o}\rangle + \tilde{\mathrm{c}}_o(\lambda)\langle \delta_o, (\lambda + L^{(o)})^{-1}\delta_o\rangle,$$

$$\tilde{c}_e(\lambda) = \sum_{m=1}^{+\infty} \frac{\lambda_m^{(e)} \tilde{T}_e(m) \bar{\vartheta}_m^{(e)}}{\lambda(\lambda + \lambda_m^{(e)})} \left\{ \sum_{m=0}^{+\infty} \frac{\left(\bar{\vartheta}_m^{(e)}\right)^2}{\lambda + \lambda_m^{(e)}} \right\}^{-1},$$

$$\tilde{c}_o(\lambda) = \sum_{m=1}^{+\infty} \frac{\lambda_m^{(o)} \tilde{T}_o(m) \Delta\vartheta_m^{(o)}}{\lambda(\lambda + \lambda_m^{(o)})} \left\{ \sum_{m=1}^{+\infty} \frac{\left(\Delta\vartheta_m^{(o)}\right)^2}{\lambda + \lambda_m^{(o)}} \right\}^{-1}.$$

Here

$$\bar{\vartheta}_m^{(e)} = \frac{1}{2}\left(\vartheta_m^{(e)}(0) + \vartheta_m^{(e)}(1)\right) \quad \Delta\vartheta_m^{(e)} = \vartheta_m^{(e)}(0) - \vartheta_m^{(e)}(1).$$

Lemma

Suppose that $T_{\rm ini} \in H^{3/2}[0,1]$. *Then, there exist functions* c_ι *such that* $e^{-\eta t}\mathrm{c}_\iota(t) \in L^2[0,+\infty)$ *for any* $\eta > 0$ *and*

$$\tilde{\mathrm{c}}_\iota(\lambda) = \int_0^{+\infty} e^{-\lambda t}\mathrm{c}_\iota(t)\mathrm{d}t, \quad \mathrm{Re}\,\lambda > 0. \tag{9}$$

In addition,

$$\mathrm{F}_\iota(t) := \int_0^t \mathrm{c}_\iota(s) Q^{(\iota)}_{t-s}\delta_\iota \mathrm{d}s \in C\big([0,+\infty); L^2_\iota[0,1]\big), \quad t \geq 0 \tag{10}$$

and $\int_0^t \mathrm{F}_\iota(s)\mathrm{d}s \in C\big([0,+\infty); H^{3/4}_\iota[0,1]\big)$, *the target spaces with the* **strong topologies**.

Lemma

If we assume that $T_{\rm ini} \in H^{3/4}[0,1]$, *then* $\mathrm{F}_\iota \in L^2_{\rm loc}\big([0,+\infty); L^2_\iota[0,1]\big)$ *and its integral belongs to* $L^2_{\rm loc}\big([0,+\infty); H^{3/4}_\iota[0,1]\big)$.

Stationary solution

Let

$$\vartheta_s(u) := \sum_{m=1}^{+\infty} \frac{\Delta\vartheta_m^{(o)}\vartheta_m^{(o)}(u)}{2\lambda_m^{(o)}},$$

$$T_s(u) := \bar{T} + \frac{\Delta T}{\Delta\vartheta_s}\vartheta_s(u).$$

We have

$$\Delta\vartheta_s = \sum_{m=1}^{+\infty} \frac{\left(\Delta\vartheta_m^{(o)}\right)^2}{2\lambda_m^{(o)}} > 0.$$

We have $T_s(v) = T_v,\ v = 0, 1.$ For any $\varphi \in H_0^{3/4}[0,1]$

$$\langle|\Delta|^{3/4}\varphi, T_s\rangle_{L^2[0,1]} + \mathcal{E}_K(\varphi, T_s) = \frac{\Delta T}{\Delta\vartheta_s} \sum_{m=1}^{+\infty} \langle\delta_o, \vartheta_m^{(o)}\rangle\langle\vartheta_m^{(o)}, \varphi\rangle_{L^2[0,1]}$$

$$= \frac{\Delta T \Delta\varphi}{2\Delta\vartheta_s} = 0.$$

Thank you for your attention!!!