Anomalous diffusion and anomalous energy transport in classical models of statistical mechanics

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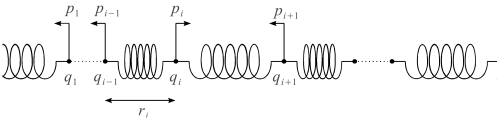
SNS
16 Marzo, 2026

Introduction: Historical comments

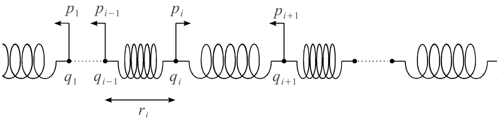


The Fermi-Pasta-Ulam-Tsingou numerical experiment (1955):

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = \sum_j \left(\frac{p_j^2}{2} + V(q_j - q_{j-1}) \right), \qquad V(r) = \frac{c_2}{2} r^2 + \frac{c_3}{3} r^3 + \frac{c_4}{4} r^4$$

Fermi wanted to check if the non-linearities ($c_3 \neq 0, c_4 > 0$) will generate equipartition (i.e. 'Thermalization' or 'ergodicity' in a generic sense).
The experiment was done at very low energy, and it 'failed'. But repeating it at higher energies, numerical evidence confirm equipartition at relatively short time.
Questions remain open, and it motivated wide physical and mathematical research in many directions. Completely integrable dynamics (like Toda lattice) provide conterexamples.

The hamiltonian dynamics



A chain of n particles connected by *springs*. Phase space is given by $\mathbb{R}^{2n}$, $(\mathbf{q},\mathbf{p}) = (q_1,\dots,q_n,p_1,\dots,p_n)$,
q_j denotes the position of the particle j and p_j denotes its velocity.
Particle j is connected to particles $j \pm 1$ by springs with potential energy $V(r)$.

$$V(r) \to +\infty \qquad |r| \to \infty$$

The dynamics is therefore given for any $j = 1,\dots,n$, $\quad n+1 \sim n$ (periodic BC) by

$$\begin{aligned} \dot{q}_j(t) &= p_j(t), \\ \dot{p}_j(t) &= V'(q_{j+1}(t) - q_j(t)) - V'(q_j(t) - q_{j-1}(t)) \end{aligned} \tag{1}$$

Hamiltonian (total energy)

$$\mathcal{H}_n(t) := \sum_{j\in\mathbb{T}_n} \mathfrak{e}_j(t), \qquad \mathfrak{e}_j := V(q_j - q_{j-1}) + \frac{1}{2}p_j^2.$$

The equation (1) defines the Hamiltonian dynamics associated with $\mathcal{H}_n$, and $\mathcal{H}_n(t) \equiv \mathcal{H}_n(0)$ is constant in time. The generator of this dynamics is is

$$\mathcal{A}_n = \sum_{j\in\mathbb{T}_n} \left\{ p_j\partial_{q_j} + \left[V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}) \right] \partial_{p_j} \right\}.$$

Since the dynamics is translation invariant, we introduce the variables

$$r_j = q_j - q_{j-1}, \qquad j = 1, \dots, n, \qquad (q_0 = q_n).$$

Conserved quantities

$$\mathcal{H}_n := \sum_{j\in\mathbb{T}_n} \mathfrak{e}_j, \qquad \mathcal{P}_n := \sum_{j\in\mathbb{T}_n} \mathfrak{p}_j, \qquad \mathcal{R}_n := \sum_{j\in\mathbb{T}_n} \mathfrak{r}_j.$$

Consequently the trajectories must ly in the *microcanonical surface* determined by these quantities at initial time:

$$\Sigma^{(n)}_{E,P,R} = \left\{(\mathbf{r},\mathbf{p}) : \mathcal{H}_n = E, \mathcal{P}_n = P, \mathcal{R}_n = R\right\}.$$

Notice that this surface is of finite volume. When $V(r) = \frac{r^2}{2}$ (harmonic chain), this is just a sphere.

Of course there are other conserved quantities, and the motion will happen in some submanifold of $\Sigma^{(n)}_{E,P,R}$. The *ergodic property* is that in some sense, as $n \to \infty$, the other submanifold *fold densely* inside $\Sigma^{(n)}_{E,P,R}$. This does not always happens as we will see later.

Microcanonical Gibbs Measure

Given E, P, R there exists a stationary measure for the dynamics given by the projection of the Lebesgue measure of $\mathbb{R}^{2n}$ on $\Sigma_{E,P,R}^{(n)}$. We call this measure the *microcanonical Gibbs measure* $d\sigma_{E,P,R}^{(n)}$:

$$\int F(\mathbf{q},\mathbf{p}) G(\mathcal{H}_n,\mathcal{P}_n,\mathcal{R}_n) d\mathbf{q} d\mathbf{p} = \int dE\; dP\; dR\; G(E,P,R) \int_{\Sigma_{E,P,R}^{(n)}} F(\mathbf{q},\mathbf{p}) d\sigma_{E,P,R}^{(n)}$$

The stationarity follows from the stationarity of the Lebesgue measure and the conservation of $(\mathcal{H}_n,\mathcal{P}_n,\mathcal{R}_n)$. Since $\Sigma_{E,P,R}^{(n)}$ has finite Lebesgue measure, the microcanonical measure can be normalized to a probability measure.

Equivalence Microcanonical - Canonical

Taking $E = ne$, $P = n\bar{p}$, $R = nr$, and $n \to \infty$ one can prove that the microcanonical measure converges to the canonical Gibbs measure (in the sense of the convergence of the finite dimensional distributions)

$$d\mu_{\beta,p,\tau} = \prod_j e^{-\beta(e_j + \bar{p}p_j + \tau r_j) - \mathcal{G}(\beta,\bar{p},\tau)} dr_j dp_j$$

where β, τ are function of e, r (we will see later the explicit expression). This is called *equivalence of ensembles*.
$\beta^{-1} = T$ is called the *temperature*, while τ is the *tension*. In fact we have

$$T = \beta^{-1} = \int (p_j - \bar{p})^2 d\mu_{\beta,p,\tau}, \qquad \tau = \int V'(r) d\mu_{\beta,p,\tau}.$$

A precise statement of the ergodicity: *The only "regular" stationary measures for the infinite dynamics are the canonical Gibbs measures.*
Thermalization, 0-principle of Thermodynamics, chaotic dynamics, etc

Fourier Modes

$$\widehat{q}(k) := \sum_{j\in\mathbb{T}_n} q_j\, e^{-2\pi ikj}, \qquad \widehat{p}(k) := \sum_{j\in\mathbb{T}_n} p_j\, e^{-2\pi ikj}, \qquad k \in \widehat{\mathbb{T}}_n := \left\{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}\right\}.$$

$$q_j = \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} e^{2\pi ijk}\widehat{q}(k), \qquad p_j = \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} e^{2\pi ijk}\widehat{p}(k) \qquad j \in \mathbb{T}_n.$$

By Parseval's identity

$$\sum_{j\in\mathbb{T}_n} p_j^2 = \frac{1}{n}\sum_{k\in\widehat{\mathbb{T}}_n} |\widehat{p}(k)|^2$$

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The canonical distribution of the velocities (with $\bar{p} = 0$) is the Gaussian

$$\frac{1}{(2\pi\beta^{-1})^{n/2}} e^{-\frac{\beta}{2}\sum_j p_j^2} = \frac{1}{(2\pi\beta^{-1})^{n/2}} e^{-\frac{\beta}{2n}\sum_{k\in\widehat{\mathbb{T}}_n} |\widehat{p}(k)|^2}$$

i.e. all the frequencies are equidistributed.

Equidistribution ~ thermalization

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At low temperatures (large β) / low energies, this is also approximatively true of the positions.
By the equivalence of ensembles, the microcanonical is well approximated by the canonical, so if the system is close to the microcanonical distribution the frequencies are (approximatively) equidistributed.
This is what the Fermi-Pasta-Ulam-Tsingou experiment was about.

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It turns out to be a *failure* (the most succesful failure in physics).

Fourier Modes interaction

Consider

$$V(r) = \frac{r^2}{2} + c_3 \frac{r^3}{3} + c_4 \frac{r^4}{4}, \qquad c_4 \geq 0.$$

This was the original choice made in FPUT, and the model is usually referred as α-FPU, and if $c_3 = 0$ as β-FPU. Notice that we always need $c_4 > 0$ if $c_3 \neq 0$ *(This point it was actually dismissed in FPUT)*. For simplicity let us consider $c_3 = 0$:

$$\begin{aligned}
\dot{q}_j(t) &= p_j(t), \\
\dot{p}_j(t) &= \Delta q_j(t) + C_4 \left[(q_{j+1}(t) - q_j(t))^3 - (q_j(t) - q_{j-1}(t))^3 \right],
\end{aligned}$$

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$$\partial_t \widehat{p}(t,k) = -\omega(k)^2 \widehat{q}(t,k) + c_4 \sum_{k_1,k_2,k_3,k_4} C(k,k_1,k_2,k_3,k_4) \widehat{q}(t,k_1) \widehat{q}(t,k_2) \widehat{q}(t,k_3) \widehat{q}(t,k_4),$$

with some complicated function $C(k, k_1, k_2, k_3, k_4)$.

Fermi-Pasta-Ulam-Tsingou experiment

Approved for reprint
Lucia O. Conolly

LOS ALAMOS SCIENTIFIC LABORATORY
of the
UNIVERSITY OF CALIFORNIA

Report written:
May 1955

Report distributed:

LA-1940

STUDIES OF NONLINEAR PROBLEMS. I

This report was prepared as a scientific account of Government-sponsored work. [illegible] States, nor the Commission [illegible] the Commission [illegible] respect [illegible], with [illegible] formation [illegible] the information, [illegible] information may not [illegible] report assumes no [illegible] Commission resulting from [illegible] of, or from damages [illegible] any information, apparatus, method, or process disclosed in this report.

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PHYSICS

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ABSTRACT

A one-dimensional dynamical system of 64 particles with forces between neighbors containing nonlinear terms has been studied on the Los Alamos computer MANIAC I. The nonlinear terms considered are quadratic, cubic, and broken linear types. The results are analyzed into Fourier components and plotted as a function of time.

The results show very little, if any, tendency toward equipartition of energy among the degrees of freedom.

The last few examples were calculated in 1955. After the untimely death of Professor E. Fermi in November, 1954, the calculations were continued in Los Alamos.

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Fermi-Pasta-Ulam-Tsingou experiment

This report is intended to be the first one of a series dealing with the behavior of certain nonlinear physical systems where the non-linearity is introduced as a perturbation to a primarily linear problem. The behavior of the systems is to be studied for times which are long compared to the characteristic periods of the corresponding linear problems.

The problems in question do not seem to admit of analytic solutions in closed form, and heuristic work was performed numerically on a fast electronic computing machine (MANIAC I at Los Alamos).* The ergodic behavior of such systems was studied with the primary aim of establishing, experimentally, the rate of approach to the equipartition of energy among the various degrees of freedom of the system. Several problems will be considered in order of increasing complexity. This paper is devoted to the first one only.

We imagine a one-dimensional continuum with the ends kept fixed and with forces acting on the elements of this string. In addition to the usual linear term expressing the dependence of the force on the displacement of the element, this force contains higher order terms. For

*We thank Miss Mary Tsingou for efficient coding of the problems and for running the computations on the Los Alamos MANIAC machine.

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the purposes of numerical work this continuum is replaced by a finite number of points (at most 64 in our actual computation) so that the partial differential equation defining the motion of this string is replaced by a finite number of total differential equations. We have, therefore, a dynamical system of 64 particles with forces acting between neighbors with fixed end points. If x_i denotes the displacement of the i-th point from its original position, and α denotes the coefficient of the quadratic term in the force between the neighboring mass points and β that of the cubic term, the equations were either

$$x_i = (x_{i+1} + x_{i-1} - 2x_i) + \alpha\left[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2\right] \qquad (1)$$

$$i = 1, 2, \ldots 64,$$

or

$$x_i = (x_{i+1} + x_{i-1} - 2x_i) + \beta\left[(x_{i+1} - x_i)^3 - (x_i - x_{i-1})^3\right] \qquad (2)$$

$$i = 1, 2, \ldots 64.$$

α and β were chosen so that at the maximum displacement the nonlinear term was small, e. g., of the order of one-tenth of the linear term. The corresponding partial differential equation obtained by letting the number of particles become infinite is the usual wave equation plus non-linear terms of a complicated nature.

Another case studied recently was

$$\ddot{x}_i = \delta_1(x_{i+1} - x_i) - \delta_2(x_i - x_{i-1}) + c \qquad (3)$$

where the parameters δ_1, δ_2, c were not constant but assumed different values depending on whether or not the quantities in parentheses

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Let us say here that the results of our computations show features which were, from the beginning, surprising to us. Instead of a gradual, continuous flow of energy from the first mode to the higher modes, all of the problems show an entirely different behavior. Starting in one problem with a quadratic force and a pure sine wave as the initial position of the string, we indeed observe initially a gradual increase of energy in the higher modes as predicted (e. g., by Rayleigh in an infinitesimal analysis). Mode 2 starts increasing first, followed by mode 3, and so on. Later on, however, this gradual sharing of energy among successive modes ceases. Instead, it is one or the other mode that predominates. For example, mode 2 decides, as it were, to increase rather rapidly at the cost of all other modes and becomes predominant. At one time, it has more energy than all the others put together! Then mode 3 undertakes this role. It is only the first few modes which exchange energy among themselves and they do this in a rather regular fashion. Finally, at a later time mode 1 comes back to within one per cent of its initial value so that the system seems to be almost periodic. All our problems have at least this one feature in common. Instead of gradual increase of all the higher modes, the energy is exchanged, essentially, among only a certain few. It is, therefore, very hard to observe the rate of "thermalization" or mixing in our problem, and this was the initial purpose of the calculation.

If one should look at the problem from the point of view of statistical mechanics, the situation could be described as follows: the

phase space of a point representing our entire system has a great number of dimensions. Only a very small part of its volume is represented by the regions where only one or a few out of all possible Fourier modes have divided among themselves almost all the available energy. If our system with nonlinear forces acting between the neighboring points should serve as a good example of a transformation of the phase space which is ergodic or metrically transitive, then the trajectory of almost every point should be everywhere dense in the whole phase space. With overwhelming probability this should also be true of the point which at time $t = 0$ represents our initial configuration, and this point should spend most of its time in regions corresponding to the equipartition of energy among various degrees of freedom. As will be seen from the results this seems hardly the case. We have plotted (Figs. 1 to 7) the ergodic sojourn times in certain subsets of our phase space. These may show a tendency to approach limits as guaranteed by the ergodic theorem. These limits, however, do not seem to correspond to equipartition even in the time average. Certainly, there seems to be very little, if any, tendency towards equipartition of energy among all degrees of freedom at a given time. In other words, the systems certainly do not show mixing.*

The general features of our computation are these: in each problem, the system was started from rest at time $t = 0$. The

*One should distinguish between metric transitivity or ergodic behavior and the stronger property of mixing.

Fermi-Pasta-Ulam-Tsingou experiment

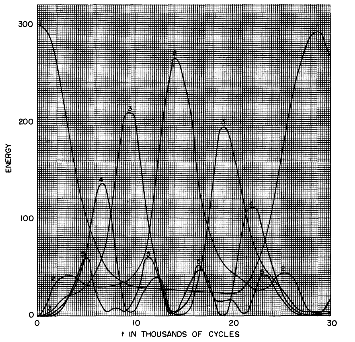

Fig. 1. The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. $N = 32$; $\alpha = 1/4$; $\delta t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.

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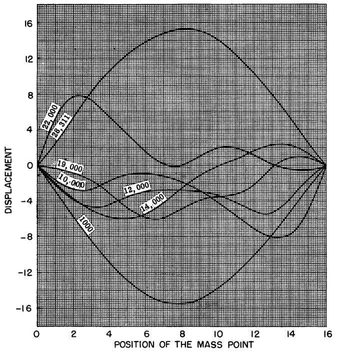

Fig. 8. This drawing shows not the energy but the actual shapes, i.e., the displacement of the string at various times (in cycles) indicated on each curve. The problem is that of Fig. 1.

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Fermi-Pasta-Ulam-Tsingou experiment

Today we can repeat the numerical experiment for larger systems and much longer times, and in particular at higher energies, and we observe thermalization! FPUT experiment was at very low energy, the dynamics was stuck in some kind of metastable quasi–periodic state. But their result showed that the problem of thermalization is much more complex than expected and non-linearity alone cannot guarantee such ergodic behaviour. Two main remarks following the FPUT experiment:

1. Even if the system should eventually thermalize, the time needed could be enourmous, the time scale for thermalization will be irrelevant for thermodynamic behaviour. The systems seems to behave like a completey integrable systems for a long time.
2. It was discovered later that non-linearity does not prevent the dynamics to be completely integrable. The most famous non-linear complete integrable dynamics is give by the **Toda Lattice**, that correspond to an exponential interaction

$$V(r) = ae^{br} - cr + d.$$

Ballistic behaviour (under ergodicity assumption)

We assume now that the dynamics is ergodic (in the sense defined above).
Then the only relevant conserved quantities (energy, momentum, volume) evolve macroscopically at different time scales.
The first time scale is the ballistic (or hyperbolic or Euler) time scale: space and time are rescaled equally.

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$G \in \mathcal{C}^1(\mathbb{T})$,

$$\mathcal{R}_{n,t}(G) = \frac{1}{n}\sum_{j=1}^{n} G\left(\frac{j}{n}\right) r_j(nt) \quad \mathcal{P}_{n,t}(G) = \frac{1}{n}\sum_{j=1}^{n} G\left(\frac{j}{n}\right) p_j(nt) \quad \mathcal{E}_{n,t}(G) = \frac{1}{n}\sum_{j=1}^{n} G\left(\frac{j}{n}\right) e_j(nt),$$

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$$\mathcal{R}_{n,t}(G) \longrightarrow \int_{\mathbb{T}} G(x) r(t,x) dx$$
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Ballistic behaviour preliminaries: thermodynamic relations

$$d\mu^{(n)}_{\beta,\bar p,\tau} = \prod_j e^{-\beta(e_j - \bar p p_j - \tau r_j) - \mathcal{G}(\beta,\bar p,\tau)} dr_j dp_j, \qquad e_j = \frac{p_j^2}{2} + V(r_j).$$

$$\mathcal{G}(\beta,\bar p,\tau) = \log \iint e^{-\beta(\frac{p^2}{2} + V(r) - \bar p p - \tau r)} dr\, dp = \beta \frac{\bar p^2}{2} + \frac{1}{2}\log(\pi\beta^{-1}) + \mathcal{G}(\beta,0,\tau),$$

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$$T = \beta^{-1} = \int_{\mathbb{R}^{2n}} (p_j - \bar p)^2 \, d\mu^{(n)}_{\beta,\bar p,\tau}, \qquad \tau = \int_{\mathbb{R}^{2n}} V'(r_j)\, d\mu^{(n)}_{\beta,\bar p,\tau},$$

$$r(\tau,T) := \int_{\mathbb{R}^{2n}} r_j \, d\mu^{(n)}_{\beta,\bar p,\tau} = \frac{1}{\beta}\frac{\partial \mathcal{G}}{\partial \tau}(\tau,\beta),$$

$$u(\tau,T) := \int_{\mathbb{R}^{2n}} \mathfrak{e}_j \, d\mu^{(n)}_{\beta,0,\tau} = -\frac{\partial \mathcal{G}}{\partial \beta}(\tau,\beta) + \tau r(\tau,T).$$

Thermodynamic entropy

For any bounded, integrable function f on $\mathbb{R} \times \mathbb{R}_+$, we define the *microcanonical volume* $W_n(r,u)$ of all the configurations of total length nr and total energy nu as

$$\int_{\mathbb{R}^{2n}} f(r^{(n)}, \mathfrak{e}^{(n)}) \prod_{i=1}^{n} dr_j dp_j = \int_{\mathbb{R}\times\mathbb{R}_+} f(r,u) W_n(r,u) dr du,$$

where $(r^{(n)}, \mathfrak{e}^{(n)}) := \frac{1}{n} \sum_{j=1}^{n} (r_j, \mathfrak{e}_j)$.
Since $\Sigma_n(r,u) \times \Sigma_m(r,u) \subseteq \Sigma_{n+m}(r,u)$,

$$\log W_n(r,u) + \log W_m(r,u) \leq \log W_{n+m}(r,u).$$

$$S(r,u) := \lim_{n\to\infty} \frac{1}{n} \log W_n(r,u) = \sup_{n\geq 1} \frac{1}{n} \log W_n(r,u) \in \mathbb{R} \cup \{\pm\infty\}. \qquad \text{Thermodynamic entropy}$$

Thermodynamic entropy

$$S(r,u) := \lim_{n\to\infty} \frac{1}{n} \log W_n(r,u) = \inf_{\tau\in\mathbb{R},\beta>0} \left\{\beta u - \beta\tau r + \mathcal{G}(\tau,\beta)\right\}.$$

$$\tau(r,u) = -\frac{1}{\beta(r,u)} \frac{\partial S}{\partial r}(r,u), \qquad \beta(r,u) = \frac{\partial S}{\partial u}(r,u),$$

$$dS(r,u) = -\beta\tau dr + \beta du = \frac{đQ}{T},$$

where $đQ = -\tau dr + du$ is the (non-exact) differential (1st principle of thermodynamic).

Euler equations

$$\mathcal{R}_{n,t}(G) = \frac{1}{n}\sum_{j=1}^{n} G\left(\frac{j}{n}\right) r_j(nt) \longrightarrow \int_{\mathbb{T}} G(x) r(t,x) dx$$

$$\mathcal{P}_{n,t}(G) = \frac{1}{n}\sum_{j=1}^{n} G\left(\frac{j}{n}\right) p_j(nt) \longrightarrow \int_{\mathbb{T}} G(x) p(t,x) dx$$

$$\mathcal{E}_{n,t}(G) = \frac{1}{n}\sum_{j=1}^{n} G\left(\frac{j}{n}\right) e_j(nt) \longrightarrow \int_{\mathbb{T}} G(x) e(t,x) dx$$

$$\partial_t r = \partial_x p, \quad \partial_t p = \partial_x \tau(r,u), \quad \partial_t e = \partial_x \left(p\tau(r,u)\right), \qquad u = e - \frac{p^2}{2}. \tag{2}$$

Heuristics (under ergodicity assumption)

$$\dot{r}_j = p_j - p_{j-1}, \qquad \dot{p}_j = V'(r_{j+1}) - V'(r_j), \qquad \dot{e}_j = p_j V'(r_{j+1}) - p_{j-1} V'(r_j).$$

Heuristics (under ergodicity assumption)

$$\dot r_j = p_j - p_{j-1}, \qquad \dot p_j = V'(r_{j+1}) - V'(r_j), \qquad \dot e_j = p_j V'(r_{j+1}) - p_{j-1} V'(r_j).$$

$$\begin{aligned}\frac{d}{dt}\mathcal{P}_{n,t}(G) &= \sum_j G\left(\frac{j}{n}\right)\left(V'(r_{j+1}(nt)) - V'(r_j(nt))\right)\\ &= \sum_j \left[G\left(\frac{j-1}{n}\right) - G\left(\frac{j}{n}\right)\right] V'(r_j(nt)) = -\frac{1}{n}\sum_j G'\left(\frac{j}{n}\right) V'(r_j(ns)) + O\left(\frac{1}{n}\right)\\ &= -\frac{1}{n}\sum_j G'\left(\frac{j}{n}\right)\frac{1}{2k}\sum_{|j'-j|\le k} V'(r_{j'}(nt)) + O\left(\frac{k}{n}\right).\end{aligned}$$

Since the equilibrium expectation of $V'(r_j)$ is τ, if at time nt the local distribution is an equilibrium one, we expect the local averaging is, for large k and large n, with $k << n$,

$$\frac{1}{2k}\sum_{|j'-j|\le k} V'(r_{j'}(nt)) \sim \tau\left(r(t,j/n), u(t,j/n)\right),$$

Energy conservation has a similar argument.

Ingredients for the proof of Euler equations

Under the *ergodicity assumption* on the dynamics, i.e. that the only stationary translation invariant probability measure for the corresponding infinite system are given by the canonical Gibbs measures, it is possible to prove such convergence if Euler equations have smooth strong solutions (Olla-Varadhan-Yau 1993)(Even-Olla 2015). Beside the ergodicity, there are some technical important conditions:

1. The initial distribution must have finite relative entropy (with respect to the equilibrium) bounded by cn for some constant c. This implies some local regularity of the initial distribution with respect to the Lebesgue measure that will be maintained at later times.
2. The interaction V must have second derivative bounded: $\sup_r |V''(r)| \leq C$.
3. Smoothness of the solution of (2). Eventually shock will appear, and the solution should be in a weak sense. Then uniqueness of the weak solution maybe lost and we have to select the right solution by entropy dissipation.

Entropy evolution

denoting $u(t,x) = e(t,x) - p(t,x)^2/2$ (local internal entropy),

$$\frac{d}{dt} S(r(t,x), u(t,x)) = (\partial_r S)\partial_t r + (\partial_u S)\left(\partial_t e - p\partial_t p\right)$$
$$= -\beta\tau\partial_x p + \beta\partial_x(p\tau) - \beta p\partial_x \tau = 0.$$

This means that smooth solution do not increase the thermodynamic entropy. It implies that, in order to get to a global equilibrium, shocks have to appear and produce thermodynamic entropy, so that to attain the maximum entropy state. This suggest that a positive entropy production is the criteria to select weak solution of (2). At the moment there are no results about the uniqueness of the *entropy weak solution* in the analysis of (2), this is one of the main open problems in hyperbolic partial differential equations.

Mechanical equilibrium

The *mechanical equilibrium* is reached when $\tau(u(t,x), r(t,x)) = \bar{\tau} = \text{const}$ and $p(t,x) = \bar{p} = const$.
Recall $\beta = T^{-1} = \partial_u S(u,r)$, $u = e - \frac{p^2}{2}$.
The inverse temperature macroscopic profile is given by

$$\partial_t \beta(t,x) = \partial_u^2 S(u,r)\partial_t u + \partial_r \partial_u S(u,r)\partial_t r = \partial_u^2 S(u,r)\tau(u,r)\partial_x p + \partial_r \partial_u S(u,r)\partial_x p = 0,$$

i.e. the temperature profile does not move in the *mechanical equilibrium* in the Euler time scale.

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i.e. the temperature profile does not move in the *mechanical equilibrium* in the Euler time scale.

The question is in which time scale it evolves (diffusive $\Longrightarrow$ heat equation, superdiffusive $\Longrightarrow$ fractional heat equation).

Equilibrium Fluctuations

We start in equilibrium with $P = 0$ and $\tau = 0$ (for simplicity) and temperature $T = \beta^{-1}$. Define the fluctuating fields for the conserve quantities

$$\widetilde{\mathcal{R}}_{n,t}(G) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} G\left(\frac{j}{n}\right)(r_j(nt) - \bar{r}) \longrightarrow \int_{\mathbb{T}} G(x)\tilde{\mathfrak{r}}(t,x)dx$$

$$\widetilde{\mathcal{P}}_{n,t}(G) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} G\left(\frac{j}{n}\right) p_j(nt) \longrightarrow \int_{\mathbb{T}} G(x)\tilde{\mathfrak{p}}(t,x)dx$$

$$\widetilde{\mathcal{E}}_{n,t}(G) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} G\left(\frac{j}{n}\right)(e_j(nt) - \bar{e}) \longrightarrow \int_{\mathbb{T}} G(x)\tilde{\mathfrak{e}}(t,x)dx$$

The initial fluctuations of the three conserved quantities are gaussian (CLT) and evolve deterministically following the linearized Euler equations:

$$\partial_t \tilde{\mathfrak{r}} = \partial_x \tilde{\mathfrak{p}}, \qquad \partial_t \tilde{\mathfrak{p}} = c^2 \partial_x \tilde{\mathfrak{r}}, \qquad \partial_t \tilde{\mathfrak{e}} = 0,$$

$c = \sqrt{\partial_r \tau(\bar{r}, \bar{u})}$ is the *sound velocity*.

Normal Modes

$$\chi^{\pm} := c\tilde{\mathfrak{r}} \pm \tilde{\mathfrak{p}} \qquad \text{two phonon or sound modes}$$
$$\chi^{0} := \tilde{\mathfrak{e}} \qquad \text{energy or heat mode}$$

they evolve deterministically as

$$\chi^{\sigma}(t,x) = \chi^{\sigma}(x+\sigma ct), \qquad \sigma = -1, 0, 1.$$

Phonon modes evolves ballistically with opposite velocities $\pm c$, heat mode does not evolve in the Euler scale.

Beyond Euler time scale?

Numerical evidence since the 90's showed that the energy mode evolves in a superdiffusive time scale (cf. Lepri, Livi, Politi PRL 1997).

Nonlinear Fluctuating Hydrodynamics Theory

Van Beijeren PRL 2012, Spohn JSP 2014

This is a mesoscopic approach to catch the superdiffusive macroscopic broadening of the modes.

Develop Euler equations up to second order and add a dissipative randomness (a gradient of space-time white noises). This is a system of *stochastic Burgers equations*.

$$\partial_t \chi^\sigma = \sigma c \partial_x \chi^\sigma + \partial_x \sum_{\sigma'} G_{\sigma',\sigma} \chi^{\sigma'} \chi^\sigma + \sum_{\sigma'} \left(D_{\sigma',\sigma} \partial_x^2 \chi^{\sigma'} + B_{\sigma',\sigma} \partial_x W_{\sigma'} \right), \qquad \sigma = -1, 0, 1,$$

where $\{W_\sigma(t,x)\}$ are independent space-time white noises.

Beyond Euler time scale

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- phonon modes $\Longrightarrow$ KPZ universal scaling function,
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- phonon modes $\Longrightarrow$ KPZ universal scaling function,
- energy mode $\Longrightarrow$ 5/3-Levy distribution

If $c_3 = 0$ and $\tau = 0$ (even potential)

- phonon modes $\Longrightarrow$ diffusive
- energy mode $\Longrightarrow$ 3/2-Levy distribution

Mathematical Results

Rigorous mathematical results have been obtained by adding noise directly on the microscopic hamiltonian dynamics. Noise is such that conserves the three modes, destroying all other conserved quantities, for example exchanging momentum between nearest neighbor particles at random times.

- This gives the required ergodicity to the infinite system. (Fritz, Funaki, Lebowitz, PTRF, 1994).
- With such stochastic perturbation, compressible Euler Equations can be proven in the smooth regime (Olla, Varadhan, Yau, CMP 1993; Even, Olla, ARMA, 2014).
- Linearized fluctuations on the Euler scale can be proven with a smoother noise (Olla, Xu, Nonlinearity, 2020, includes also mechanical boundary conditions on the tension).

Mathematical Results beyond the Euler Scale

Beyond Euler scaling it is very hard to obtain mathematical results, even in presence of conservative noise. Results can be obtained for the Harmonic Chain ($c_3 = 0 = c_4$):

- Thermal diffusivity diverges: superdiffusion of the energy (Basile, Bernardin, Olla, CMP 2009).
- energy mode $\Longrightarrow$ 3/2-Levy distribution, and also corresponding fractional non–stationary superdiffusion given by

$$\partial_t \mathfrak{e} = -D|\Delta_x|^{3/4}\mathfrak{e} \qquad D = 2^3\gamma^{-1/2}$$

  $\gamma > 0$ rate of the random exchanges. (Jara, Komorowski, Olla, CMP 2015)
- Recentered phonon modes are diffusive (Komorowski, Olla, Nonlinearity, 2016).

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Unlike the Euler equation, here the superdiffusive and diffusive equations depend of the rate γ of the random momentum exchange.

Weak non-linearity

(work with Kohei Hayashi), arXiv:2510.12922

Infinite dynamics ($j \in \mathbb{Z}$), with random nearest neighbor exchanges of velocities with rate γ, in equilibrium with $\tau = 0, P = 0$ and $\beta > 0$.

V is a smooth non-linear and $V(0) = V'(0) = 0$. Then we scale it as $V_\varepsilon(r) = \varepsilon^{-2} V(\varepsilon r)$:

$$V_\varepsilon(r) = \frac{c_2}{2!} r^2 + \frac{c_3}{3!} \varepsilon r^3 + \frac{c_4}{4!} \varepsilon^2 r^4 + O(\varepsilon^3), \qquad c_k = V^{(k)}(0).$$

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Choose $\varepsilon_n = \frac{1}{\sqrt{n}}$.

$$\xi_j^\pm = \sqrt{c_2} r_{j+1} \pm p_j, \qquad \xi_j^0 = e_j = \frac{p_j^2}{2} + V_n(r_j), \qquad \bar{\xi}_j^\sigma = \xi_j^\sigma - \mathbb{E}(\xi_j^\sigma).$$

Recentered Fluctuating fields, for $\sigma = -1, 0, 1$

$$\mathcal{X}_t^{\sigma,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j \in \mathbb{Z}} \overline{\xi_j^\sigma}(t) \varphi\left(\frac{[j + \sigma t]}{n}\right)$$

Evolution of fluctuating fields

$$\mathcal{X}_t^{\sigma,n}(\varphi) = \frac{1}{\sqrt{n}} \sum_{j\in\mathbb{Z}} \overline{\xi_j^\sigma}(t)\varphi\left(\frac{[j+\sigma\sqrt{c_2}t]}{n}\right)$$

At time 0 they converge in law to the white noises

$$\mathcal{X}_0^{\sigma,n}(\varphi) \underset{n\to\infty}{\longrightarrow} \mathcal{X}_0^{\sigma}(\varphi), \qquad \mathbb{E}\left[\mathcal{X}_0^{\sigma}(\varphi)\mathcal{X}_0^{\sigma'}(\varphi)\right] = 0, \text{ if } \sigma \neq \sigma'$$

$$\mathbb{E}\left[(\mathcal{X}_0^{\pm}(\varphi))^2\right] = \frac{2}{\beta}\|\varphi\|_{L^2}^2, \qquad \mathbb{E}\left[\left(\mathcal{X}_0^{0}(\varphi)\right)^2\right] = \frac{3}{\beta^2}\|\varphi\|_{L^2}^2.$$

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At the hyperbolic time scale nothing moves:

$$\mathcal{X}_{nt}^{\sigma,n}(\varphi) \underset{n\to\infty}{\longrightarrow} \mathcal{X}_0^\sigma(\varphi), \qquad \sigma = -1, 0, 1.$$

Non-linear fluctuations of the phonon modes

At diffusive time scale $(n^2 t)$ we prove the convergence to the energy solutions of two uncoupled stochastic Burgers equations with drift:

$$\mathcal{X}^{\pm,n}_{n^2 t}(\varphi) \underset{n\to\infty}{\longrightarrow} \int u^{\pm}(t,x)\varphi(x)dx$$

$$\partial_t u^{\pm} = \frac{\gamma}{4}\partial_x^2 u^{\pm} \pm \frac{c_3}{8c_2^2}\partial_x (u^{\pm})^2 \pm D_V \partial_x u^{\pm} + \sqrt{\gamma\beta^{-1}}\partial_x \dot{W}^{\pm}$$

$$D_V = \frac{2c_2c_4 - c_3^2}{24c_2^3}.$$

Here $\dot{W}^+(t,x), \dot{W}^-(t,x)$ are independent standard white noises, $\gamma > 0$ is the rate of the random exchanges of velocities.

Notice that the nonlinear term depends on the presence of $c_3 \neq 0$ (asymmetric interaction).

Kinetic approach

Recall

$$\widehat{q}(k) := \sum_{j \in \mathbb{T}_n} q_j \, e^{-2\pi ikj}, \qquad \widehat{p}(k) := \sum_{j \in \mathbb{T}_n} p_j \, e^{-2\pi ikj}, \qquad k \in \widehat{\mathbb{T}}_n := \left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}\right\}.$$

And consider first the deterministic harmonic chain, eventually with a pinning potential, i.e.

$$\dot{q}_j(t) = p_j(t),$$
$$\dot{p}_j(t) = (q_{j+1} - q_j)(t) - (q_j - q_{j-1})(t) - \omega_0^2 q_j(t) =: \Delta q_j(t) - \omega_0^2 q_j(t)$$

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$$\begin{aligned} \partial_t \widehat{q}(t,k) &= \widehat{p}(t,k), \\ \partial_t \widehat{p}(t,k) &= -\omega(k)^2\, \widehat{q}(t,k). \end{aligned} \qquad k \in \widehat{\mathbb{T}}_n.$$

$$\omega(k) = \sqrt{\omega_0^2 + 4\sin^2(\pi k)}, \qquad \text{dispersion relation}$$

Kinetic approach: harmonic case

$$\widehat{\phi}(t,k) = \omega(k)\widehat{q}(t,k) + i\widehat{p}(t,k).$$

Then[1]

$$\partial_t \widehat{\phi}(t,k) = -i\omega(k)\Big(i\widehat{p}(t,k) + \omega(k)\widehat{q}(t,k)\Big) = -i\omega(k)\widehat{\phi}(t,k).$$

The explicit solution is given by

$$\widehat{\phi}(t,k) = e^{-it\omega(k)}\widehat{\phi}(0,k), \qquad t \geq 0, \;\; k \in \widehat{\mathbb{T}}_n.$$

[1]Similar to the Schrödinger equation $i\partial_t \phi = \Delta\phi$ that gives $\partial_t \widehat{\phi}(t,k) = -i\omega(k)^2 \widehat{\phi}(t,k)$.

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$$\widehat{\phi}(t,k) = e^{-it\omega(k)}\widehat{\phi}(0,k), \qquad t \geq 0,\ k \in \widehat{\mathbb{T}}_n.$$

$$\mathcal{H}_n = \frac{1}{2}\sum_{j\in\mathbb{T}_n}\Big((q_j - q_{j-1})^2 + \omega_0^2 q_j^2 + p_j^2\Big) = \frac{1}{2n}\sum_{k\in\widehat{\mathbb{T}}_n} |\widehat{\phi}(k)|^2,$$

$$|\widehat{\phi}(k)|^2 = \omega(k)^2|\widehat{q}(k)|^2 + |\widehat{p}(k)|^2 \qquad \text{energy of the frequency k.}$$

[1] Similar to the Schrödinger equation $i\partial_t\phi = \Delta\phi$ that gives $\partial_t\widehat{\phi}(t,k) = -i\omega(k)^2\widehat{\phi}(t,k)$.

Wigner distribution

$$\widehat{W}_n(t,\eta,k) = \widehat{\phi}\left(\frac{nt}{2}, k - \frac{\eta}{n}\right)^* \widehat{\phi}\left(\frac{nt}{2}, k + \frac{\eta}{n}\right), \qquad \eta \in \mathbb{Z}, k \in \widehat{\mathbb{T}}_n.$$

For the deterministic linear evolution

$$\widehat{W}_n(t,\eta,k) = e^{-itn[\omega(k+\frac{\eta}{n})-\omega(k-\frac{\eta}{n})]/2}\widehat{W}_n(0,\eta,k)$$
$$= e^{-it\omega'(k)\eta}\widehat{W}_n(0,\eta,k) + O\left(\frac{1}{n}\right).$$

It can be proven that the limit exists, consequently

$$\widehat{W}(t,\eta,k) := \lim_{n\to\infty} \widehat{W}_n(t,\eta,k) = e^{-it\omega'(k)\eta}\widehat{W}(0,\eta,k), \qquad k \in \mathbb{T},$$

Evolution of the Wigner Distribution: Phonons

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$$W(t,u,k) = \sum_{\eta=1}^{n} e^{i2\pi\eta u}\widehat{W}(t,\eta,k), \qquad u \in \mathbb{T}, k \in \mathbb{T}, \qquad \text{Wigner distribution}$$

$$\partial_t W(t,u,k) + \frac{\omega'(k)}{2\pi}\partial_u W(t,u,k) = 0, \qquad u \in \mathbb{T}, k \in \mathbb{T},$$

i.e. $W(t,u,k) = W(0, u - \bar{\omega}'(k)t, k)$, with $\bar{\omega}'(k) = \frac{\omega'(k)}{2\pi}$.

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k-Phonon moves, without interacting with the other phonons, with velocity

$$\bar{\omega}'(k) = \frac{\omega'(k)}{2\pi} = \frac{8\pi \sin(\pi k)\cos(\pi k)}{4\pi\sqrt{\omega_0^2 + 4\sin^2(\pi k)}} = \frac{\sin(2\pi k)}{\sqrt{\omega_0^2 + 4\sin^2(\pi k)}}.$$

Random exchanges of velocities

$$\mathcal{L}_n = \mathcal{A}_n + \gamma \mathcal{S}_n,$$

$$\mathcal{A}_n = \sum_{j \in \mathbb{T}_n} \left\{ p_j \partial_{q_j} + \left[V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}) \right] \partial_{p_j} \right\}.$$

$$\mathcal{S}_n F(\mathbf{r}, \mathbf{p}) = \sum_{j=1}^{n} \left[F(\mathbf{r}, \mathbf{p}^{j,j+1}) - F(\mathbf{r}, \mathbf{p}) \right],$$

where $\mathbf{p}^{j,j+1}$ is the configuration $\mathbf{p}$ after the exchange $p_j \leftrightarrow p_{j+1}$.

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where $\mathbf{p}^{j,j+1}$ is the configuration $\mathbf{p}$ after the exchange $p_j \leftrightarrow p_{j+1}$.
For every couple $(j, j+1)$ there is a Poisson process $N_{j,j+1}(t)$ that exchanges the velocities $p_j \leftrightarrow p_{j+1}$.

$$\begin{aligned} \dot{q}_j(t) &= p_j(t), \\ \dot{p}_j(t) &= V'(q_{j+1}(t) - q_j(t)) - V'(q_j(t) - q_{j-1}(t)) \\ &\quad + (p_{j+1}(t^-) - p_j(t^-)) dN_{j,j+1}(\gamma t) + (p_{j-1}(t^-) - p_j(t^-)) dN_{j-1,j}(\gamma t). \end{aligned}$$

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\end{aligned}
$$

This stochastic exchanges impose that every stationary measure for $\mathcal{L}_n$ has to be excheangeable in the p_j. This is enough, when $n \to \infty$, for ensuring that only canonical Gibbs measure are the only *regular* stationary translation invariant measure (Fritz-Funaki-Lebowitz 1994). This happens for any interaction V, even those where the ergodicity was not true for the deterministic dynamics (like the harmonic or the Toda lattice).

We should also remark that the corresponding Euler equation will not depend on γ, i.e. are the same as in the deterministic case.

Phonon scattering by random exchange of velocities

We rescale γ with n as $\gamma = \frac{\tilde\gamma}{n}$.

$$
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so that at time nt each couple has a finite number of exchanges.

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so that at time nt each couple has a finite number of exchanges.

$$\widehat{\phi}(t,k) = \omega(k)\widehat{q}(t,k) + i\widehat{p}(t,k).$$

is now a random process.

$$\widehat{W}_n(t,\eta,k) = \mathbb{E}\left(\widehat{\phi}\left(\frac{nt}{2}, k - \frac{\eta}{n}\right)^* \widehat{\phi}\left(\frac{nt}{2}, k + \frac{\eta}{n}\right)\right), \qquad \eta \in \mathbb{Z}, k \in \widehat{\mathbb{T}}_n.$$

Kinetic limit

$$\widehat{W}_n(t,\eta,k) = \mathbb{E}\left(\widehat{\phi}\left(\frac{nt}{2}, k - \frac{\eta}{n}\right)^* \widehat{\phi}\left(\frac{nt}{2}, k + \frac{\eta}{n}\right)\right), \qquad \eta \in \mathbb{Z}, k \in \widehat{\mathbb{T}}_n.$$

In (Basile-Olla-Spohn, ARMA 2009) is proven that for $n \to \infty$ the $\widehat{W}_n(t,\eta,k)$ converges to the solution of

$$\partial_t W(t,x,k) + \bar{\omega}'(k)\partial_x W(t,x,k) = \tilde{\gamma} \int_{\mathbb{T}} R(k,k') \left(W(t,x,k') - W(t,x,k)\right) dk', \quad (3)$$

$x \in \mathbb{T}$, $k \in \mathbb{T}$ and

$$R(k,k') = 4\sin^2(\pi k)\sin^2(\pi k'). \quad (4)$$

This is the forward equation for the Markov process of a particle (phonon) of mode k that move with velocity $\bar{\omega}'(k)$, and change mode to k' with rate $R(k,.k')$.
Notice that the uniform measure on k is the stationary measure for the scattering part of the equation (i.e. equidistribution on the frequencies).

Kinetic limit (idea of the proof)

The proof of (48) requires some long but elementary computations, and one mian argument: denoting

$$CF(k) = \tilde{\gamma} \int_{\mathbb{T}} R(k,k') \left(F(k') - F(k)\right) \, dk' \qquad (5)$$

and the *anti-Wigner* distribution

$$\widehat{Y}_n(t,\eta,k) = \mathbb{E}\left[\widehat{\phi}\left(\frac{nt}{2}, k + \frac{\eta}{n}\right)\widehat{\phi}\left(\frac{nt}{2}, -k + \frac{\eta}{n}\right)\right], \qquad \eta \in \mathbb{Z}, k \in \widehat{\mathbb{T}}_n, \qquad (6)$$

the long calculations gives a non closed evolution equation for $\widehat{W}_n$:

$$\partial_t \widehat{W}_n(t,\eta,k) + \omega'(k)\partial_x \widehat{W}_n(t,\eta,k) = C\widehat{W}_n(t,\eta,k) + \frac{1}{2}C\widehat{Y}_n(t,\eta,k) + \frac{1}{2}C\widehat{Y}_n(t,\eta,k)^* + O\left(\frac{1}{n}\right), \qquad (7)$$

$$u \in \mathbb{T}, k \in \mathbb{T}.$$

Kinetic limit (idea of the proof)

Now notice that

$$\widehat{Y}_n(t,0,k) = \mathbb{E}\left[\widehat{\phi}\left(\frac{nt}{2},k\right)\widehat{\phi}\left(\frac{nt}{2},-k\right)\right]$$
$$= \mathbb{E}\left[\omega(k)^2\widehat{q}\left(\frac{nt}{2},k\right)\widehat{q}\left(\frac{nt}{2},-k\right) - \widehat{p}\left(\frac{nt}{2},k\right)\widehat{p}\left(\frac{nt}{2},-k\right)\right]$$
$$= \mathbb{E}\left[\omega(k)^2\left|\widehat{q}\left(\frac{nt}{2},k\right)\right|^2 - \left|\widehat{p}\left(\frac{nt}{2},k\right)\right|^2\right],$$

i.e. is the difference between the potential energy and the kinetic energy of the mode k. In equilibrium these quantities have the same expectations (equipartition) and it turns out that taking the time average

$$\int_0^t \widehat{Y}_n(s,\eta,k)\, ds \underset{n\to\infty}{\longrightarrow} 0.$$

Pinned (optical) harmonic chain: diffusive behavior of the phonons

$$\partial_t W(t,x,k) + \bar{\omega}'(k)\partial_x W(t,x,k) = \tilde{\gamma} \int_{\mathbb{T}} R(k,k') \left(W(t,x,k') - W(t,x,k) \right) dk',$$

$x \in \mathbb{T}$, $k \in \mathbb{T}$ and

$$R(k,k') = 4\sin^2(\pi k)\sin^2(\pi k').$$

$$\bar{\omega}'(k) = \frac{\omega'(k)}{2\pi} = \frac{8\pi \sin(\pi k)\cos(\pi k)}{4\pi\sqrt{\omega_0^2 + 4\sin^2(\pi k)}} = \frac{\sin(2\pi k)}{\sqrt{\omega_0^2 + 4\sin^2(\pi k)}}.$$

If $\omega_0^2 > 0$ (pinned or optical chain), $\bar{\omega}'(k) \to 0$ for $|k| \to 0$.

Pinned (optical) harmonic chain: diffusive behavior of the phonons

Forward equation of a Markov process $(X(t), K(t))$ on $\mathbb{T} \times \mathbb{T}$, where

$$X(t) = X(0) + \int_0^t \bar{\omega}'(K(s))\, ds,$$

while $K(t)$ is an autonomous jump process on $\mathbb{T}$ whose generator is the collision operator C defined by

$$CF(k) = R(k)\tilde{\gamma} \int_{\mathbb{T}} R(k') \left(F(k') - F(k) \right) dk', \qquad R(k) = 2\sin^2(\pi k).$$

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The process $K(t)$ has a unique stationary reversible measure given by the Lebesgue (i.e. uniform) measure on $\mathbb{T}$. Furthermore notice that

$$\int \bar{\omega}'(k) dk = 0,$$

i.e. the average velocity of $X(t)$ is null.

Pinned (optical) harmonic chain: diffusive behavior of the phonons

We can prove that

$$\lim_{\varepsilon \to 0} \left\{ \varepsilon X(\varepsilon^{-2} t), t \geq 0 \right\} = \left\{ B(t), t \geq 0 \right\}, \tag{8}$$

where $B(t)$ is a brownian motion with variance $\mathbb{E}(B(t)) = Dt$ where

$$D = \tilde{\gamma}^{-1} \int_{\mathbb{T}} \frac{\bar{\omega}'(k)^2}{R(k)}\, dk = \tilde{\gamma}^{-1} \int_{\mathbb{T}} \frac{\sin^2(2\pi k)}{\left(\omega_0^2 + 4\sin^2(\pi k)\right) 2\sin^2(\pi k)}\, dk$$

$$= \tilde{\gamma}^{-1} \int_{\mathbb{T}} \frac{2\cos^2(\pi k)}{\left(\omega_0^2 + 4\sin^2(\pi k)\right)}\, dk.$$

Notice that $D \to +\infty$ if $\omega_0 \to 0$, that suggest a superdiffusive behaviour for the unpinned chain.

Proof of diffusion in pinned chain

Notice that $\int_{\mathbb{T}} R(k)dk = 1$, then we have an explicit solution for the equation

$$Cf(k) = \bar{\omega}'(k), \qquad k \in \mathbb{T}: \qquad f(k) = -\frac{\bar{\omega}'(k)}{\tilde{\gamma} R(k)}$$

Consequently

$$X(t) = X(0) + \int_0^t Cf(K(s))\, ds. \tag{9}$$

From the general theory of Markov processes, $\varepsilon \int_0^{\varepsilon^{-2}t} Cf(K(s))\, ds$ is a martingale with quadratic variation

$$\varepsilon^2 \int_0^{\varepsilon^{-2}t} \left[f(K(s)) Cf(K(s)) - 2Cf^2(K(s)) \right] ds \xrightarrow[\varepsilon \to 0]{a.s.} t \int_{\mathbb{T}} f(k) Cf(k)\, dk = tD$$

by the ergodicity of the process $K(t)$. This is enough to characterize the limiting process.

Diffusive equation for the phonons in Pinned chain

As consequence of the previous theorem we have the convergence of the solution of

$$\partial_t W(t,x,k) + \bar{\omega}'(k)\partial_x W(t,x,k) = \tilde{\gamma} \int_{\mathbb{T}} R(k,k') \left(W(t,x,k') - W(t,x,k) \right) dk',$$

to

$$W(\varepsilon^{-2}t, \varepsilon^{-1}x, k) \underset{\varepsilon \to 0}{\longrightarrow} T(t,x),$$

where T is the solution of the heat equation

$$\partial_t T(t,x) = D\partial_x^2 T(t,x).$$

Diffusive equation for the phonons in Pinned chain

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where T is the solution of the heat equation

$$\partial_t T(t,x) = D\partial_x^2 T(t,x).$$

Thermal boundary conditions can be included (Basile-Komorowski-Olla KRM 2016).

3/2-Levy superdiffusion for the energy in the unpinned chain

From the explicit expression of the diffusion coefficient D we see that it diverges when $\omega_0 \to 0$. In fact in the unpinned chain ($\omega_0 = 0$), the velocity

$$\bar{\omega}'(k) = \text{sign}(k) \cos(\pi k)$$

does not converges to 0 as $k \to 0$, while these longwave modes scatters very little ($R(k) \to 0$ as $k \to 0$). The phonon when is in a low mode k, move ballistically with a non-null velocity $\bar{\omega}'(k)$ scattering very rarely, then this happens it changes mode to a higher k, diffuse but sometimes scatters to a low mode and move ballistically again. This give rise to a Levy superdiffusion and it can be proven that (Jara-Komorowski-Olla 2009, Basile-Bovier 2009)

$$\left\{\varepsilon X(\varepsilon^{-3/2} t), t \geq 0\right\} \underset{\varepsilon \to 0}{\longrightarrow} \{\mathcal{L}(t), t \geq 0\} \tag{10}$$

where $\mathcal{L}(t)$ is the Markov process generated by $-|\Delta|^{3/4}$.

3/2-Levy superdiffusion for the energy in the unpinned chain

$-|\Delta|^{3/4}$ is fractional laplacian is defined as follows. Consider the ON base for the Laplacian on $\mathbb{T}$: $\psi_k(x) = e^{i2\pi kx}, k \in \mathbb{Z}$, then

$$|\Delta|^{\alpha} f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)|2\pi k|^{2\alpha}\psi_k(x)^*.$$

For $\alpha \neq 1$ it can also be (formally) written as

$$-|\Delta|^{\alpha} f(x) = \int_{\mathbb{T}} K(x,y)\left(f(y) - f(x)\right)\, dy, \tag{11}$$

with

$$K(x,y) = \sum_{n=0}^{\infty} \frac{4^{\alpha}\Gamma\left(\frac{1}{2}+\alpha\right)}{\pi^{1/2}|\Gamma(-s)|} \frac{1}{|x-y+n|^{1+2\alpha}}. \tag{12}$$

in the sense that

$$-|\Delta|^{\alpha} f(x) = \lim_{r\to 0^+} \int_{\mathbb{T}\setminus B_r(x)} K(x,y)\left(f(y)-f(x)\right) dy.$$

Notice the non-locality of this operator if $\alpha \neq 1$.

Fractional heat equation for unpinned chain

As consequence of the previous theorem we have the convergence

$$W(\varepsilon^{-3/2}t, \varepsilon^{-1}x, k) \underset{\varepsilon\to 0}{\longrightarrow} T(t,x),$$

where T is the solution of the fractional heat equation

$$\partial_t T(t,x) = -\gamma^{-1/2}|\Delta_x|^{3/4} T(t,x).$$

Tomasz will explain in the next lectures how to obtain directly the fractional heat equation as a direct Hydrodynamic limit without passing through the kinetic limit.

Heuristic of the Levy superdiffusion

For $\varepsilon > 0$ very small (and $N \sim \varepsilon^{-3/2}$)

$$\varepsilon\left(X(\varepsilon^{-3/2}t) - X(0)\right) \sim \frac{1}{N^{2/3}}\sum_{j=0}^{[Nt]} \bar{\omega}'(\tilde{K}_j)\frac{1}{R(\tilde{K}_j)}\tau_j$$

where $\{\tilde{K}_j\}j \geq 1$ is the embedded Markov chain $(p(k',k) = R(k))$, $R(k) \sim k^2$ for $k \to 0$, $\{\tau_j\}_{j\geq 1}$ are exponential i.i.d.
The stationary measure for the embedded chain is $\tilde{\pi}(dk) = R(k)dk$.

$$\tilde{\pi}\left(\frac{1}{R(k)} > \lambda\right) = \tilde{\pi}\left(R(k) < \lambda^{-1}\right) \sim \tilde{\pi}\left(|k| < \lambda^{-1/2}\right) \sim \int_0^{\lambda^{-1/2}} k^2 dk \sim \lambda^{-3/2}$$

Harmonic Chain with Random Masses

(F. Huveneers, C. Bernardin, S.Olla, 2017)

$\{m_j\}$ i.i.d. with absolutely continuous distribution,
$0 < m_- \leq m_j \leq m_+$,
$\overline{m} = \mathbb{E}(m_x)$.

$$m_j \dot{q}_j(t) = p_j(t), \qquad \dot{p}_j(t) = \Delta q_j(t), \qquad j = 1, \dots, N$$

with $q_0 = q_1$ and $q_{N+1} = q_N$ as boundary conditions.

Gibbs States, Local Gibbs States

The Gibbs states are characterized by three parameters: $\beta > 0$ and $p, r \in \mathbb{R}$. Its probability density writes

$$\prod_{j=1}^{N} \frac{e^{-\frac{\beta m_j}{2}\left(\frac{p_j}{m_j}-\frac{p}{m}\right)^2-\frac{\beta}{2}(r_j-r)^2}}{Z(\beta, p, r, m_j)}.$$

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$$\prod_{j=1}^{N} \frac{e^{-\frac{\beta m_j}{2}\left(\frac{p_j}{m_j}-\frac{p}{\bar{m}}\right)^2 - \frac{\beta}{2}(r_j - r)^2}}{Z(\beta, p, r, m_j)}.$$

We start with a local Gibbs state:

$$\prod_{j=1}^{N} \frac{e^{-\frac{\beta(j/N)m_x}{2}\left(\frac{p_j}{m_j}-\frac{p(j/N)}{\bar{m}}\right)^2 - \frac{\beta(j/N)}{2}(r_j - r(j/N))^2}}{Z(\beta(j/N), p(j/N), r(j/N), m_j)}.$$

Harmonic Chain with Random Masses: hydrodynamic limit

Almost surely with respect to $\{m_x\}$:

$$< r_{[Ny]}(Nt) >, < p_{[Ny]}(Nt) >, < \mathcal{E}_{[Ny]}(Nt) > \rightharpoonup \ (\mathbf{r}(y,t), \mathbf{p}(y,t), \mathfrak{e}(y,t))$$

$$\partial_t \mathbf{r}(t,y) = \frac{1}{\overline{m}} \partial_y \mathbf{p}(t,y)$$

$$\partial_t \mathbf{p}(t,y) = \partial_y \mathbf{r}(t,y)$$

$$\partial_t \mathfrak{e}(t,y) = \frac{1}{\overline{m}} \partial_y \left(\mathbf{r}(t,y) \mathbf{p}(t,y) \right)$$

with initial conditions:

$$\mathbf{r}(y,0) = r(y), \qquad \mathbf{p}(y,0) = p(y), \qquad \mathfrak{e}(y,0) = \frac{1}{\beta(y)} + \frac{p^2(y)}{2\overline{m}} + \frac{r^2(y)}{2}.$$

Random Masses: Localization of Thermal Modes

Equation of motion can be written as

$$\ddot{r}_j = -(\nabla^* M^{-1} \nabla r)_j \quad (1 \le j \le N-1), \qquad \ddot{p}_j = \left(\Delta M^{-1} p\right)_j \quad (1 \le j \le N),$$

where $M_{j,j'} = \delta_{j,j'} m_j$.

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where $M_{j,j'} = \delta_{j,j'} m_j$.

$$M^{-1/2}(-\Delta) M^{1/2} \varphi^k = \omega_k^2 \, \varphi^k, \qquad k = 0, \dots, N-1.$$

$$\psi^k = M^{-1/2} \varphi^k, \qquad M^{-1} \Delta \psi^k = \omega_k^2 \psi_k$$

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$$r(t) = \sum_{k=1}^{N-1} \left(\frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \cos \omega_k t + \langle \psi^k, p(0) \rangle \sin \omega_k t \right) \frac{\nabla \psi^k}{\omega_k},$$

$$p(t) = \sum_{k=0}^{N-1} \left(\langle \psi^k, p(0) \rangle \cos \omega_k t - \frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \sin \omega_k t \right) M \psi^k.$$

Localization of Thermal Modes

Localization length ξ_k diverges with N:

$$\xi_k^{-1} \sim \omega_k^2 \sim \left(\frac{k}{N}\right)^2,$$

only the modes $k > \sqrt{N}$ are localized.

Localization of Thermal Modes

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$$\xi_k^{-1} \sim \omega_k^2 \sim \left(\frac{k}{N}\right)^2,$$

only the modes $k > \sqrt{N}$ are localized.
More precisely: for $0 < \alpha < \frac{1}{2}$

$$\mathbb{E}\left(\sum_{k=N^{1-\alpha}}^{N-1} |\psi_j^k \psi_{j'}^k|\right) \leq Ce^{-cN^{-2\alpha}|j-j'|}.$$

This estimate is enough to prove that thermal modes remains localized and do not *move* macroscopically.

Random masses: Larger time scales

Assume for simplicity that we are in a *mechanical equilibrium*:

$$\langle\, r_j(0)\,\rangle = 0, \qquad \langle\, p_j(0)\,\rangle = 0,$$

(only thermal energy present)
but not in thermal equilibrium, then, for any $\alpha \geq 1$

$$< \mathcal{E}_{[Ny]}(N^\alpha t) > \underset{N\ to \infty}{\longrightarrow} \mathbf{e}(0,y) = \bar{C}(\beta(y))$$

NO evolution for the temperature profile at any scale!

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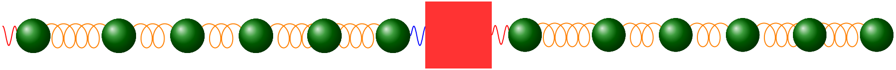
All this results can be extended to the quantum harmonic chain with random masses (Hannani CMP 2021)

Thermal boundaries: Langevin Heat Bath



$$
\begin{aligned}
\dot{q}_j(t) &= p_j(t), \\
dp_j(t) &= \Delta q_j(t)dt \\
&\quad + \left(p_{j+1}(t^-) - p_j(t^-)\right) dN_{j,j+1}(\tilde{\gamma} t/n) + \left(p_{j-1}(t^-) - p_j(t^-)\right) dN_{j-1,j}(\tilde{\gamma} t/n) \\
&\quad - \delta_{j,0} p_0(t) dt + \sqrt{2T} dW_0(t)
\end{aligned}
$$

Kinetic equation with thermal boundary

$$\nu(k) = \left(1 + \gamma \tilde{J}(-i\omega(k))\right)^{-1}, \qquad \tilde{J}(\lambda) = \int_{\mathbb{T}} \frac{\lambda}{\lambda^2 + \omega^2(k)} dk.$$

$$\mathrm{Re}\, \nu(k) = \left(1 + \frac{\gamma\pi}{|\omega'(k)|}\right) |\nu(k)|^2$$

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$$\mathrm{Re}\, \nu(k) = \left(1 + \frac{\gamma\pi}{|\omega'(k)|}\right) |\nu(k)|^2$$

$$\mathfrak{g}(k) = \frac{2\pi\gamma|\nu(k)|^2}{|\omega'(k)|} \qquad \text{absorbtion probability}$$

$$p_+(k) = \left|1 - \frac{\gamma\pi\nu(k)}{|\omega'(k)|}\right|^2 \qquad \text{transmission probability}$$

$$p_-(k) = \left|\frac{\gamma\pi\nu(k)}{|\omega'(k)|}\right|^2 \qquad \text{reflection probability}$$

$$\mathfrak{g}(k) + p_+(k) + p_-(k) = 1$$

Kinetic equation with thermal boundary

T.Komorowski, L.Ryzhik, S.O., H.Spohn, ARMA (2020)

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = CW(t,y,k), \qquad y \in \mathbb{R} \setminus \{0\}$$

with boundary conditions:

$$W(t,0^+,k) = p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + \mathfrak{g}(k)T, \quad 0 < k < 1/2$$
$$W(t,0^-,k) = p_-(k)W(t,0^-,-k) + p_+(k)W(t,0^+,k) + \mathfrak{g}(k)T, \quad -1/2 < k < 0$$

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$$W(t,0^-,k) = p_-(k)W(t,0^-,-k) + p_+(k)W(t,0^+,k) + \mathfrak{g}(k)T, \quad -1/2 < k < 0$$

Since $\mathfrak{g}(k) + p_+(k) + p_-(k) = 1$, we have that

$$W(t,y,k) = T \qquad \text{is a stationary solution.}$$

Without the thermostat: Basile, O., Spohn, ARMA 2009.

Diffusive behaviour with thermal BC: $\gamma' > 0$, $\gamma > 0$

Giada Basile, Tomasz Komorowski, S.O.,
Kinetic and Related Models, AIMS, (2019)
In the cases of bulk diffusive behavior: $\omega'(k) \sim k$ (optical chain)

$$W(\lambda^2 t, \lambda y, k) \underset{\lambda \to 0}{\longrightarrow} e(t, y)$$

$$\partial_t e = D \partial_{yy} e, \qquad y \neq 0, \qquad D = \frac{1}{4\pi^2 \gamma'} \int \frac{\omega'(k)^2}{R(k)} dk$$

$$e(t, 0^+) = T = e(t, 0^-)$$

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$$e(t, 0^+) = T = e(t, 0^-)$$

Reflection and transmission of phonons are irrelevant in this time scale: phonons gets absorbed and created such that energy density is T at $y = 0$.

Superdiffusive behaviour with thermal boundary

Tomasz Komorowki, S.O., Lenya Rhyzik, Ann. of Prob. (2020)

Situation is different in the super-diffusive case.

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = \int \mathfrak{R}(k,k')\big(W(k') - W(k)\big)$$

$$R(k,k') = R(K)R(k'), \qquad R(k) \sim |k|^2, \qquad |\omega'(k)| \sim 2, \qquad k \sim 0,$$

$$\begin{aligned} W(t,0^+,k) &= p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + \mathfrak{g}(k)T, \quad 0 < k < 1/2 \\ W(t,0^-,k) &= p_-(k)W(t,0^-,-k) + p_+(k)W(t,0^+,k) + \mathfrak{g}(k)T, \quad -1/2 < k < 0 \end{aligned}$$

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$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = \int \mathfrak{R}(k,k')\left(W(k') - W(k)\right)$$

$$R(k,k') = R(K)R(k'), \qquad R(k) \sim |k|^2, \qquad |\omega'(k)| \sim 2, \qquad k \sim 0,$$

$$W(t,0^+,k) = p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + \mathfrak{g}(k)T, \quad 0 < k < 1/2$$
$$W(t,0^-,k) = p_-(k)W(t,0^-,-k) + p_+(k)W(t,0^+,k) + \mathfrak{g}(k)T, \quad -1/2 < k < 0$$

$$R(k,k') \sim 0, \qquad k \sim 0,$$

the phonon is crossing the thermostat when $k \sim 0$.

Superdiffusive behaviour with thermal boundary

Tomasz Komorowski, S.O., Lenya Rhyzik, Ann. of Prob. (2020)

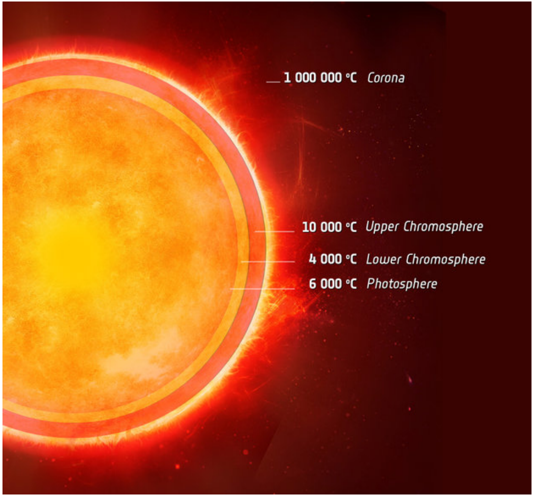
For $k \to 0$, $\mathfrak{g}(k) \to \mathfrak{g}_0 > 0$, $\quad p_-(k) \to p_-(0) > 0$.

$$W(\lambda^{3/2} t, \lambda y, k) \underset{\lambda \to 0}{\longrightarrow} e(t, y)$$

$$\begin{aligned} \partial_t e(t,y) = &- \hat{c}|\Delta|^{3/4} e(t,y) \\ &+ \hat{c}\mathfrak{g}_0 \int_{yy'<0} q(y-y')(T - e(t,y'))dy' \\ &+ \hat{c}p_-(0) \int_{yy'<0} q(y-y')(e(t,-y') - e(t,y))dy'. \end{aligned}$$

$$q(y) = \frac{c}{|y|^{5/4}} \qquad \text{kernel of } |\Delta|^{3/4}.$$

How hot is the sun?



Why? It has been a mistery for a while

nature communications

Article https://doi.org/10.1038/s41467-023-41029-8

Polarisation of decayless kink oscillations of solar coronal loops

Received: 2 March 2023

Accepted: 18 August 2023

Published online: 12 September 2023

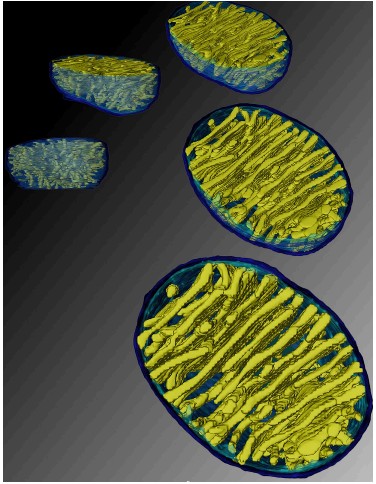
Check for updates

Sihui Zhong [1], Valery M. Nakariakov [1,2], Dmitrii Y. Kolotkov [1,3], Lakshmi Pradeep Chitta [4], Patrick Antolin [5], Cis Verbeeck [6] & David Berghmans [6]

Decayless kink oscillations of plasma loops in the solar corona may contain an answer to the enigmatic problem of solar and stellar coronal heating. The polarisation of the oscillations gives us a unique information about their excitation mechanisms and energy supply. However, unambiguous determination of the polarisation has remained elusive. Here, we show simultaneous detection of a 4-min decayless kink oscillation from two non-parallel lines-of-sights, separated by about 104°, provided by unique combination of the High Resolution Imager on Solar Orbiter and the Atmospheric Imaging Assembly on Solar Dynamics Observatory. The observations reveal a horizontal or weakly oblique linear polarisation of the oscillation. This conclusion is based on the comparison of observational results with forward modelling of the observational manifestation of various kinds of polarisation of kink oscillations. The revealed polarisation favours the sustainability of these oscillations by quasi-steady flows which may hence supply the energy for coronal heating.

The outermost, fully-ionised and magnetically-dominated part of the atmosphere of the Sun, the corona, attracts our attention as the birthplace of extreme events of space weather, such as flares and coronal mass ejections[1]. The corona is a natural plasma physics laboratory allowing for high-resolution study of basic phenomena which are of interest for various astrophysical, geophysical and laboratory plasma physics applications. The corona offers us several enigmatic puzzles, such as the rapid release of the magnetic energy, and heating of the plasma to temperatures about three orders of magnitude higher than the surface temperature. One of the most rapidly developing avenues of modern coronal physics is the study of oscillations of coronal plasma loops[9], which appear in two different regimes. Large-amplitude rapidly-decaying kink oscillations are normally excited by displacements of the loops from the equilibrium, caused by low-coronal eruptions[10], and decay by resonant absorption (e.g., ref. 11) and/or by Kelvin–Helmholtz instability (KHI, e.g., refs. 12,13). The nature of another type, low-amplitude decayless kink oscillations[14–17] is subject to intensive ongoing studies. The mechanisms which could counteract the oscillation damping are random footpoint movements, e.g., refs. 18–20 and quasi-stationary coronal, chromospheric or photospheric flows, e.g., refs. 21–23, or their combination[24]. Alternative mechanisms for the apparent decayless

Another example (in a different scale): the mitocondria



Another example (in a different scale): the mitocondria

PLOS | BIOLOGY

PRIMER

Hot mitochondria?

Nick Lane*

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Abstract

Mitochondria generate most of the heat in endotherms. Given some impedance of heat transfer across protein-rich bioenergetic membranes, mitochondria must operate at a higher temperature than body temperature in mammals and birds. But exactly how much hotter has been controversial, with physical calculations suggesting that maximal heat gradients across cells could not be greater than 10^{-5} K. Using the thermosensitive mitochondrial-targeted fluorescent dye Mito Thermo Yellow (MTY), Chrétien and colleagues suggest that mitochondria are optimised to nearly 50 °C, 10 °C hotter than body temperature. This extreme value questions what temperature really means in confined far-from-equilibrium systems but encourages a reconsideration of thermal biology.

Check for updates

OPEN ACCESS

Citation: Lane N (2018) Hot mitochondria? PLoS Biol 16(1): e2005113. https://doi.org/10.1371/journal.pbio.2005113

Published: January 25, 2018

Self-respecting scientists might be inclined to back away from the claims made by Chrétien and colleagues in *PLOS Biology* this week: mitochondria in endothermic mammals operate optimally at close to 50 °C [1]. That is a radical claim, and if it is true, how come we didn't know something so important long ago? If it isn't true, then why is it figuring so prominently in the pages of a respected journal and how did it get through peer review? The answer is that this finding challenges many of our cherished beliefs, whether it is right or wrong, or perhaps more probably somewhere in between.

Molecular probes

Stationary non-equilibrium states

These are stationary states of the system (in the time scale we are interested they do not change in time) where there is a constant flux of energy (and eventually mass or other quantities).

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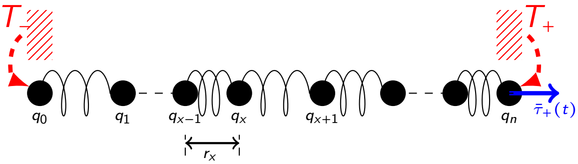
The fluxes are of quantities that are conserved by the dynamics in the bulk of the system, and are caused by difference in the boundary conditions.

Harmonic chain with velocity random flip

Joint works with *Tomasz Komorowski* and *Marielle Simon*.



Chain of n harmonic springs,

- with random velocity sign flip,
- in contact with two Langevin thermostats,
- pulled on one side by a force $\bar{\tau}$.

$r_i = q_i - q_{i-1}, i = 1, \dots, N$.

$$(\mathbf{r}, \mathbf{p}) = (r_1, \dots, r_N, p_0, \dots, p_N) \in \mathbb{R}^N \times \mathbb{R}^{N+1}.$$

$$\mathcal{H}_n := \sum_i \left\{ \frac{p_i^2}{2} + \frac{r_i^2}{2} \right\} + \frac{p_0^2}{2}.$$

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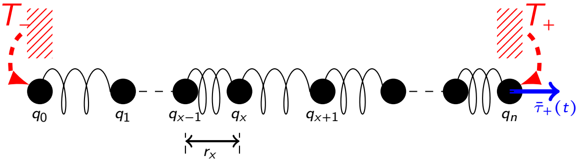

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$r_x = q_x - q_{x-1}, x = 1, \dots, n,$ $\mathcal{H}_n := \sum_{i=1}^n \left\{ \frac{p_i^2}{2} + \frac{r_i^2}{2} \right\} + \frac{p_0^2}{2}.$

$$\dot{q}_x(t) = p_x(t), \qquad x \in \{0, \dots, n\},$$
$$\dot{p}_x(t) = \Delta q_x - 2p_x(t-)dN_x(\gamma t), \quad x \in \{1, \dots, n-1\},$$

Harmonic chain with velocity random flip

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\end{aligned}
$$

Two (locally) conserved quantities:

$$
\sum_i r_i \quad \text{volume}, \qquad \sum_i e_i = \sum_i \left(\frac{p_i^2}{2} + \frac{r_i^2}{2} \right)
$$

Unpinned harmonic chain with random velocity flip

Macroscopic behaviour

$$r_{[ny]}(n^2 t) \underset{n\to\infty}{\rightharpoonup} r(t,y)$$

$$\mathcal{E}_{[ny]}(n^2 t) \underset{n\to\infty}{\rightharpoonup} e(t,y) = \frac{1}{2}\left(T(t,y) + r^2(t,y)\right)$$

Unpinned harmonic chain with random velocity flip

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$$\partial_t r = \frac{1}{2\gamma}\partial_y^2 r, \qquad r(t,0) = 0, \quad r(t,1) = \bar{\tau}_+$$

$$\partial_t T = \frac{1}{4\gamma}\partial_y^2 T + \frac{1}{2\gamma}(\partial_y r)^2, \qquad T(t,0) = T_-, \quad T(t,1) = T_+.$$

Stationary Non-equilibrium state

$$r_{ss}(x) = \bar{\tau}_+ x$$
$$\partial_x^2 T(x) + 2\,\bar{\tau}_+^2 = 0,$$
$$T(0) = T_-, \qquad T(1) = T_+.$$

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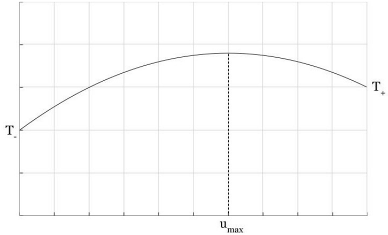
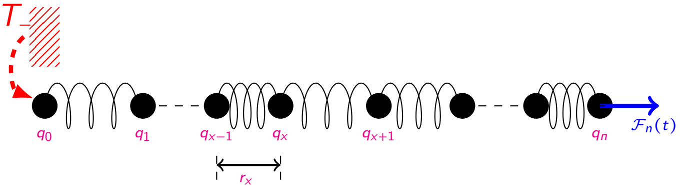
Explicit solution

$$T(x) = \bar{\tau}_+^2\, x(1-x) + (T_+ - T_-)x + T_-, \qquad x \in [0,1].$$



Periodic forcing: work and heat



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\begin{aligned}
\dot{q}_x(t) &= p_x(t), & x \in \{0, \dots, n\}, \\
\dot{p}_x(t) &= V'(q_{x+1} - q_x) - V'(q_x - q_{x-1}) - 2p_x(t-)dN_x(\gamma t), & x \in \{1, \dots, n-1\},
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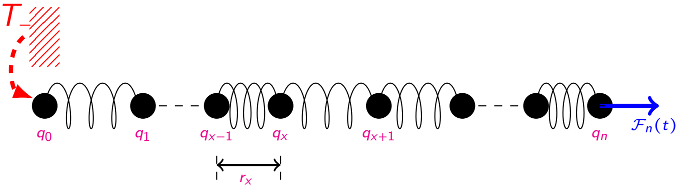
$$

Periodic forcing: work and heat



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$$dp_0(t) = V'(q_1 - q_0)dt - 2\gamma p_0(t)dt + \sqrt{4\gamma T_-}dw_-(t)$$
$$dp_n(t) = -V'(q_n(t) - q_{n-1})dt + \mathcal{F}_n(t)dt$$
$$\mathcal{F}_n(t) = \cos(wt)$$

Work and Neumann boundary condition

$$W_n(t) = \frac{1}{n} \int_0^{n^2 t} \mathcal{F}_n(s) p_n(s) ds \qquad \text{macroscopic work}$$

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$$W_n(t) \underset{n\to\infty}{\longrightarrow} W^Q t$$

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