

J. Phys. G49, 055001 (2022)

J. Phys. G50, 029401 (2023)

HIGHER-ORDER HADRONIC VACUUM POLARIZATION CONTRIBUTIONS TO THE MUON ANOMALOUS MAGNETIC MOMENT IN SPACELIKE AND TIMELIKE DOMAINS

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14th International Workshop on e^+e^- collisions from Phi to Psi (PhiPsi26)
Pisa, Italy, 8–11 June 2026

INTRODUCTION

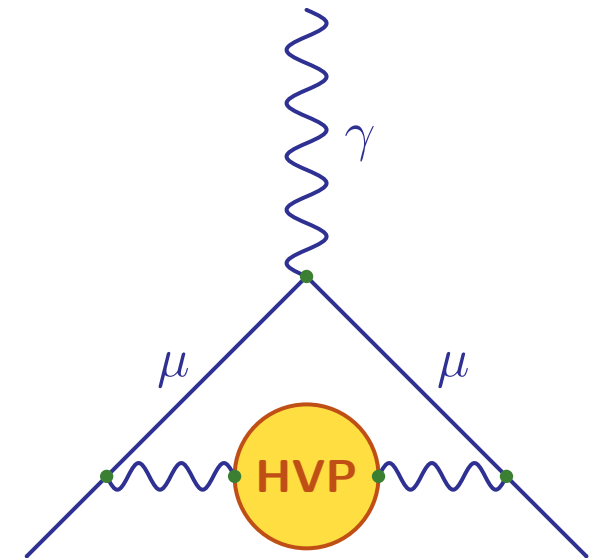
The theoretical description of $a_\mu = (g_\mu - 2)/2$ represents one of the long-standing challenging issues of contemporary elementary particle physics.

Experiment BNL E821: Phys. Rev. D **73**, 072003 (2006);
FNAL E989: Phys. Rev. Lett. **135**, 101802 (2025)

$$a_\mu^{\text{exp}} = (11659207.15 \pm 1.45) \times 10^{-10} \quad (124 \text{ ppb})$$

Theory [TI WP20] Phys. Rept. **887**, 1 (2020);
[TI WP25] Phys. Rept. **1143**, 1 (2025)

$$\begin{aligned} a_\mu^{\text{theor}} &= a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{HVP}} + a_\mu^{\text{HLbL}} \\ &= (11659203.3 \pm 6.2) \times 10^{-10} \quad (0.53 \text{ ppm}) \\ &= \text{???} \quad \text{yet unsolved puzzles} \end{aligned}$$



← $a_\mu^{\text{HVP}(2)}$: lattice
← $a_\mu^{\text{HVP}(2)}$: TL-data-driven

A novel SL-data-driven MUonE method becomes particularly significant.

SPACELIKE APPROACH

$$\begin{aligned} a_{\mu}^{\text{HVP}} &= A_0 \int_0^{\infty} K_{\Pi}(Q^2) \bar{\Pi}(Q^2) \frac{dQ^2}{4m_{\mu}^2} = A_0 \int_0^{\infty} \tilde{K}_{\Pi}(\zeta) \bar{\Pi}(4\zeta m_{\mu}^2) d\zeta = \\ &= A_0 \int_0^{\infty} K_D(Q^2) D(Q^2) \frac{dQ^2}{4m_{\mu}^2} = A_0 \int_0^{\infty} \tilde{K}_D(\zeta) D(4\zeta m_{\mu}^2) d\zeta, \quad \zeta = \frac{Q^2}{4m_{\mu}^2}. \end{aligned}$$

In this equation A_0 is a constant prefactor, $Q^2 = -q^2 \geq 0$ denotes a space-like kinematic variable, $\bar{\Pi}(Q^2) = -\Pi(-Q^2)$ stands for the subtracted at zero hadronic vacuum polarization function, $D(Q^2)$ is the Adler function, $K_{\Pi}(Q^2)$ and $K_D(Q^2)$ denote the corresponding spacelike kernel functions.

Here the perturbative results for $\bar{\Pi}(Q^2)$ and $D(Q^2)$ have to be supplemented with relevant nonperturbative inputs, that can be provided by lattice simulations, MUonE measurements, and reliable phenomenological models.

MUonE @ CERN experiment: direct extraction of SL $\bar{\Pi}(Q^2)$ from $\mu e \rightarrow \mu e$ data

■ Carloni Calame, Passera, Trentadue, Venanzoni, PLB **746**, 325 (2015); Abbiendi et al., EPJC **77**, 139 (2017).

TIMELIKE APPROACH

$$a_{\mu}^{\text{HVP}} = A_0 \int_{s_0}^{\infty} K_R(s) R(s) \frac{ds}{4m_{\mu}^2} = A_0 \int_{\chi}^{\infty} \tilde{K}_R(\eta) R(4\eta m_{\mu}^2) d\eta, \quad \eta = \frac{s}{4m_{\mu}^2}, \quad \chi = \frac{s_0}{4m_{\mu}^2}.$$

In this equation $s = q^2 \geq 0$ stands for a timelike kinematic variable, $s_0 = 4m_{\pi}^2$ denotes the hadronic threshold, $R(s)$ is the R -ratio of electron-positron annihilation into hadrons, and $K_R(s)$ stands for the respective timelike kernel function.

Here the perturbative results for $R(s)$ are usually complemented by the low-energy experimental data on the R -ratio, that constitutes the TL-data-driven method of evaluation of a_{μ}^{HVP} .

The timelike kernel functions $K_R(s)$ have been extensively studied over the past decades, whereas the corresponding spacelike kernel functions $K_{\Pi}(Q^2)$ and $K_D(Q^2)$ were largely unknown (until recently) beyond the leading-order.

GENERAL DISPERSION RELATIONS

The hadronic vacuum polarization function $\Pi(q^2)$ is defined as the scalar part of the hadronic vacuum polarization tensor

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0 | T \{ J_\mu(x) J_\nu(0) \} | 0 \rangle = \frac{i}{12\pi^2} (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2), \quad q^2 < 0.$$

The physical kinematic restrictions imply that $\Pi(q^2)$ has the only cut starting at the hadronic threshold $q^2 \geq s_0$ ■ Feynman (1972); Adler (1974).

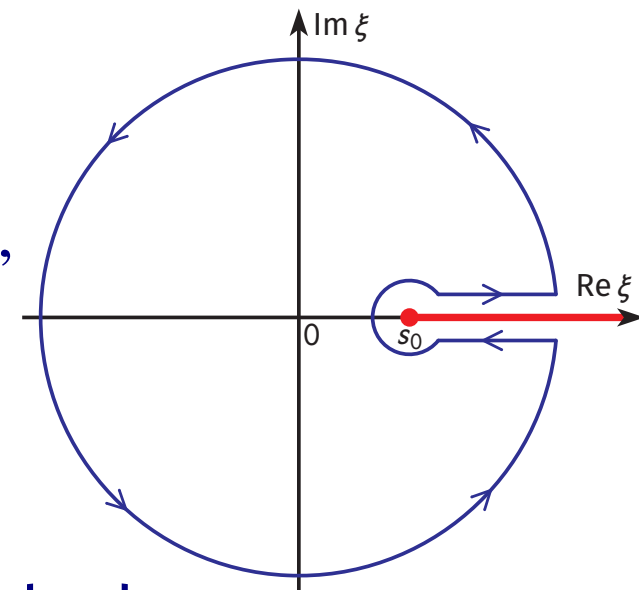
The once-subtracted Cauchy's integral formula yields

$$\Delta\Pi(q^2, q_0^2) = \Pi(q^2) - \Pi(q_0^2) = (q^2 - q_0^2) \int_{s_0}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)} d\sigma,$$

where

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \left(\Pi(s + i\varepsilon) - \Pi(s - i\varepsilon) \right) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons}; s)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-; s)}$$

denotes the R -ratio of electron-positron annihilation into hadrons.



For practical purposes it proves to be particularly convenient to deal with the Adler function

$$D(Q^2) = -\frac{d \Pi(-Q^2)}{d \ln Q^2}, \quad D(Q^2) = Q^2 \int_{s_0}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2} d\sigma, \quad Q^2 = -q^2 > 0$$

■ Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).

The dispersion relations for $\Pi(q^2)$ and $D(Q^2)$ enable one to extract their experimental predictions from respective timelike data on the R -ratio.

The inverse relations between the functions on hand read

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}, \quad \Pi(-Q^2) - \Pi(-Q_0^2) = - \int_{Q_0^2}^{Q^2} D(\xi) \frac{d\xi}{\xi}$$

■ Radyushkin (1982); Krasnikov, Pivovarov (1982) ■ Pennington, Ross (1977), (1981), (1984); Pivovarov (1992).

Additionally, the dispersion relations impose a number of stringent physical nonperturbative constraints on the functions on hand, see pages 26–29

■ Nesterenko, Phys. Rev. D **88**, 056009 (2013); J. Phys. G **42**, 085004 (2015); ISBN: 9780128034392 (2017).

RELATIONS BETWEEN SPACELIKE AND TIMELIKE KERNEL FUNCTIONS

In the ℓ -th order in the electromagnetic coupling the hadronic vacuum polarization contribution to the muon anomalous magnetic moment reads

$$\begin{aligned}
 a_{\mu}^{\text{HVP}(\ell)} &= A_0^{(\ell)} \int_0^{\infty} \overset{\text{QED}}{K_{\Pi}^{(\ell)}(Q^2)} \times \overset{\text{QCD}}{\bar{\Pi}(Q^2)} \frac{dQ^2}{4m_{\mu}^2} = \\
 &= A_0^{(\ell)} \int_0^{\infty} K_D^{(\ell)}(Q^2) \times D(Q^2) \frac{dQ^2}{4m_{\mu}^2} = \\
 &= A_0^{(\ell)} \int_{s_0}^{\infty} K_R^{(\ell)}(s) \times R(s) \frac{ds}{4m_{\mu}^2}.
 \end{aligned}
 \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{[spacelike]} \\ \text{[timelike]} \end{array}$$

The kernel functions $K_{\Pi}(Q^2)$, $K_D(Q^2)$, and $K_R(s)$ appearing in these equations can all be expressed in terms of each other:

■ Nesterenko, *J. Phys. G* **49**, 055001 (2022).

Kernel function $K_{\Pi}(Q^2)$ in terms of $K_R(s)$

$\bar{\Pi}(-q^2) = -\Pi(q^2)$: cut $q^2 \geq s_0$ ■ Feynman (1972); Adler (1974).

$K_R(q^2)$: cut $q^2 \leq 0$ ■ Barbieri, Remiddi (1975).

The contour integral of their product vanishes

$$\oint_C K_R(q^2) \bar{\Pi}(-q^2) dq^2 = 0,$$

that implies

$$-\frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \int_0^{-\infty} \bar{\Pi}(-p^2) \left(K_R(p^2 + i\varepsilon) - K_R(p^2 - i\varepsilon) \right) dp^2 = \int_{s_0}^{\infty} K_R(p^2) R(p^2) dp^2.$$

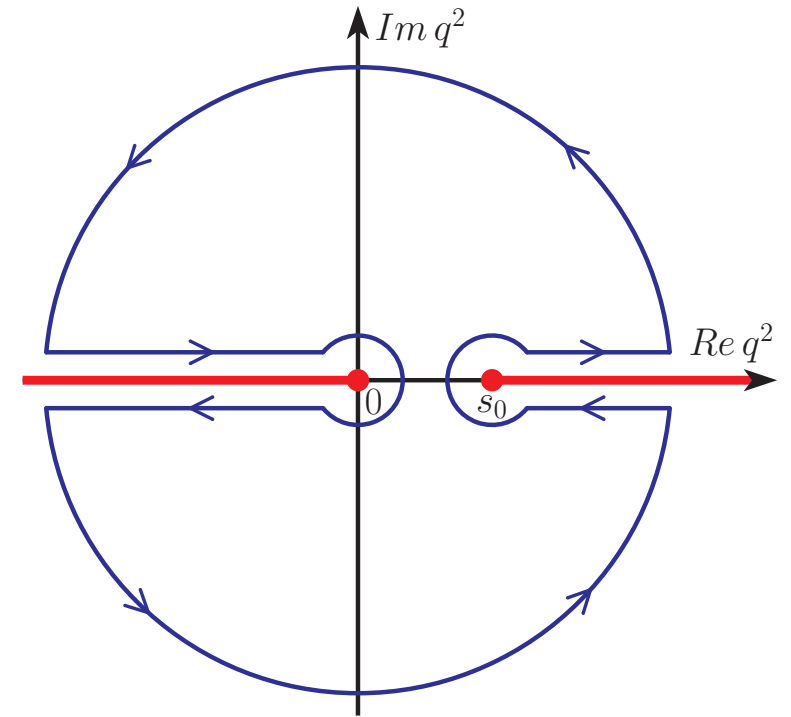
Thus, the relation, which expresses $K_{\Pi}(Q^2)$ in terms of $K_R(s)$, reads

$$K_{\Pi}(Q^2) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \left(K_R(-Q^2 + i\varepsilon) - K_R(-Q^2 - i\varepsilon) \right), \quad Q^2 \geq 0$$

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

This relation has also been independently derived in a different way in

■ Balzani, Laporta, Passera, Phys. Lett. B **834**, 137462 (2022); arXiv:2112.05704 [hep-ph].



Kernel function $K_R(s)$ in terms of $K_\Pi(Q^2)$

Dispersion relation for the hadronic vacuum polarization function leads to

$$\int_0^\infty K_\Pi(Q^2) \bar{\Pi}(Q^2) \frac{dQ^2}{4m_\mu^2} = \int_0^\infty \frac{dQ^2}{4m_\mu^2} K_\Pi(Q^2) Q^2 \int_{s_0}^\infty \frac{ds}{s} \frac{R(s)}{s+Q^2} = \int_{s_0}^\infty K_R(s) R(s) \frac{ds}{4m_\mu^2}.$$

Hence, the relation, which expresses $K_R(s)$ in terms of $K_\Pi(Q^2)$, reads

$$K_R(s) = \frac{1}{s} \int_0^\infty K_\Pi(Q^2) \frac{Q^2}{s+Q^2} dQ^2, \quad s \geq 0.$$

Kernel function $K_R(s)$ in terms of $K_D(Q^2)$

Dispersion relation for the Adler function yields

$$\int_0^\infty K_D(Q^2) D(Q^2) \frac{dQ^2}{4m_\mu^2} = \int_0^\infty \frac{dQ^2}{4m_\mu^2} K_D(Q^2) Q^2 \int_{s_0}^\infty \frac{R(s)}{(s+Q^2)^2} ds = \int_{s_0}^\infty K_R(s) R(s) \frac{ds}{4m_\mu^2}.$$

Therefore, the relation, which expresses $K_R(s)$ in terms of $K_D(Q^2)$, reads

$$K_R(s) = \int_0^\infty K_D(Q^2) \frac{Q^2}{(s+Q^2)^2} dQ^2, \quad s \geq 0.$$

Kernel function $K_{\Pi}(Q^2)$ in terms of $K_D(Q^2)$

Definition of the Adler function results in

$$\begin{aligned} \int_0^{\infty} K_D(Q^2) D(Q^2) dQ^2 &= - \int_0^{\infty} dQ^2 K_D(Q^2) Q^2 \frac{d \Pi(-Q^2)}{d Q^2} = \\ &= K_D(Q^2) Q^2 \bar{\Pi}(Q^2) \Big|_0^{\infty} - \int_0^{\infty} dQ^2 \bar{\Pi}(Q^2) \left(K_D(Q^2) + \frac{d K_D(Q^2)}{d \ln Q^2} \right), \end{aligned}$$

with the integration by parts being employed. Since the first term in the second line of this equation vanishes (see also remarks given below), the relation, which expresses $K_{\Pi}(Q^2)$ in terms of $K_D(Q^2)$, reads

$$K_{\Pi}(Q^2) = - \left(K_D(Q^2) + \frac{d K_D(Q^2)}{d \ln Q^2} \right), \quad Q^2 \geq 0$$

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

Kernel function $K_D(Q^2)$ in terms of $K_{\Pi}(Q^2)$

The solution to the differential equation derived on the previous page reads

$$K_D(Q^2) + \frac{d K_D(Q^2)}{d \ln Q^2} = -K_{\Pi}(Q^2) \quad \longrightarrow \quad K_D(Q^2) = \frac{1}{Q^2} \left(-\int K_{\Pi}(Q^2) dQ^2 + c_0 \right).$$

The constant c_0 has to be chosen in the way that makes $K_D(Q^2)$ vanishing at $Q^2 \rightarrow \infty$. The relation, which expresses $K_D(Q^2)$ in terms of $K_{\Pi}(Q^2)$, reads

$$K_D(Q^2) = \frac{1}{Q^2} \int_{Q^2}^{\infty} K_{\Pi}(\xi) d\xi = \frac{4m_{\mu}^2}{Q^2} K_0 - \frac{1}{Q^2} \int_0^{Q^2} K_{\Pi}(\xi) d\xi, \quad \xi = -p^2 \geq 0.$$

In this equation K_0 denotes the infrared limiting value of the respective spacelike and timelike (see p. 8) kernel functions, namely

$$K_0 = \lim_{Q^2 \rightarrow 0_+} \frac{Q^2}{4m_{\mu}^2} K_D(Q^2) = \lim_{s \rightarrow 0_+} \frac{s}{4m_{\mu}^2} K_R(s) = \int_0^{\infty} K_{\Pi}(\xi) \frac{d\xi}{4m_{\mu}^2},$$

which is factually identical to the corresponding QED contribution to a_{μ} of the preceding order in the electromagnetic coupling

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

Kernel function $K_D(Q^2)$ in terms of $K_R(s)$

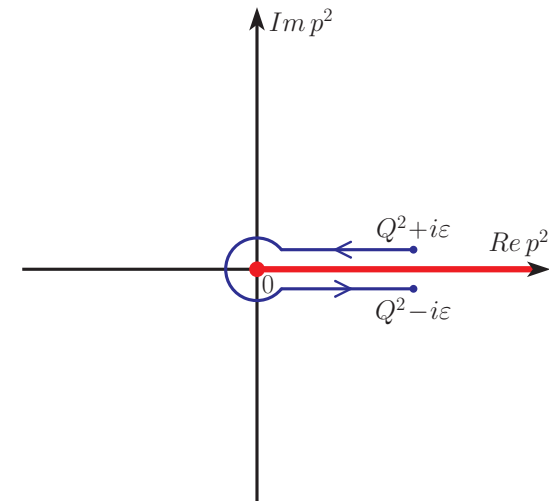
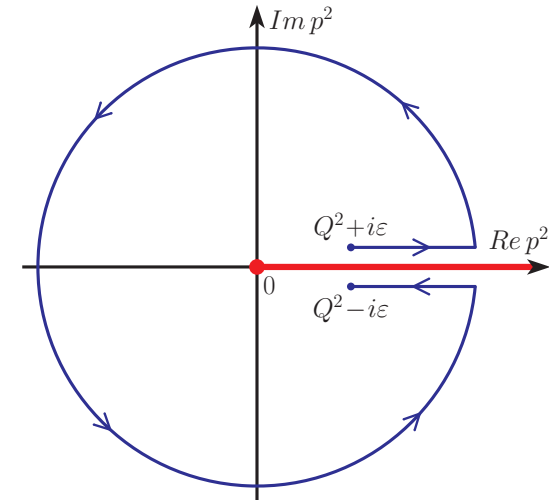
The first and the fifth derived relations between the kernel functions imply that the relation, which expresses $K_D(Q^2)$ in terms of $K_R(s)$, reads

$$\begin{aligned} K_D(Q^2) &= -\frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \frac{1}{Q^2} \int_{Q^2}^{\infty} \left(K_R(-\xi - i\varepsilon) - K_R(-\xi + i\varepsilon) \right) d\xi = \\ &= -\frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \frac{1}{Q^2} \int_{Q^2+i\varepsilon}^{Q^2-i\varepsilon} K_R(-p^2) dp^2, \end{aligned}$$

where the integration contour in the complex p^2 -plane lies in the region of analyticity of the function $K_R(-p^2)$.

The obtained six equations constitute the **complete set of relations**, which mutually express the spacelike and timelike kernel functions $K_{\Pi}(Q^2)$, $K_D(Q^2)$, and $K_R(s)$ in terms of each other. The obtained relations **enable one to directly calculate** the unknown kernels by making use of the known one

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

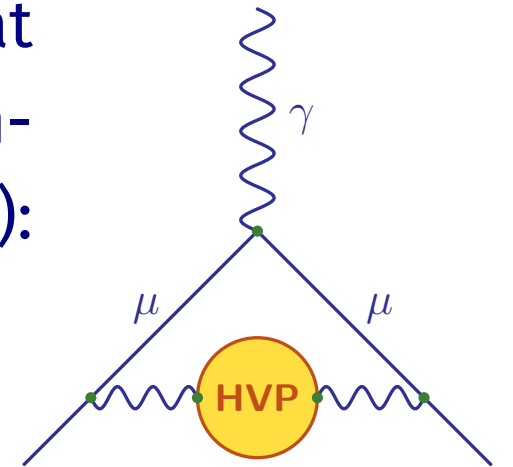


KERNEL FUNCTIONS IN THE LEADING ORDER

All three leading-order kernel functions are available, that can be used to exemplify the obtained relations. The contribution $a_\mu^{\text{HVP}(2)}$ in terms of the R -ratio (timelike approach):

$$a_\mu^{\text{HVP}(2)} = A_0^{(2)} \int_{s_0}^{\infty} K_R^{(2)}(s) R(s) \frac{ds}{4m_\mu^2}, \quad A_0^{(2)} = \frac{1}{3} \left(\frac{\alpha}{\pi} \right)^2,$$

$$K_R^{(2)}(s) = \frac{4m_\mu^2}{s} \int_0^1 \frac{x^2(1-x)}{x^2 + (1-x)s/m_\mu^2} dx, \quad s = q^2 \geq 0, \quad K_R^{(2)}(s) = \tilde{K}_R^{(2)}\left(\frac{s}{4m_\mu^2}\right), \quad \eta = \frac{s}{4m_\mu^2}$$



■ Berestetskii, Krokhin, Khlebnikov (1956); Bouchiat, Michel (1961); Kinoshita, Oakes (1967).

Explicit expression for the leading-order timelike kernel function:

$$\eta \tilde{K}_R^{(2)}(\eta) = \frac{1}{2} + 4\eta \left((2\eta - 1) \ln(4\eta) - 1 \right) - 2 \left(2(2\eta - 1)^2 - 1 \right) \operatorname{arctanh}(\psi(\eta)) \frac{\sqrt{\eta}}{\sqrt{\eta - 1}}$$

■ Berestetskii, Krokhin, Khlebnikov (1956); Durand (1962); Brodsky, de Rafael (1968); Lautrup, de Rafael (1968).

Factually, the specific form of the leading-order timelike kernel function $K_R^{(2)}(s)$ makes it possible to express $a_\mu^{\text{HVP}(2)}$ in terms of the spacelike functions $\bar{\Pi}(Q^2)$ and $D(Q^2)$, namely

$$a_\mu^{\text{HVP}(2)} = A_0^{(2)} \int_0^1 dx (1-x) \int_{s_0}^{\infty} \frac{ds R(s) m_\mu^2 x^2 (1-x)^{-1}}{s + m_\mu^2 x^2 (1-x)^{-1}} = A_0^{(2)} \int_0^1 (1-x) \bar{\Pi} \left(m_\mu^2 \frac{x^2}{1-x} \right) dx$$

■ Lautrup, Peterman, de Rafael, Phys. Rept. **3**, 193 (1972); de Rafael, Phys. Rev. D **96**, 014510 (2017).

In turn, its integration by parts eventually yields

$$a_\mu^{\text{HVP}(2)} = A_0^{(2)} \int_0^1 (1-x) \left(1 - \frac{x}{2}\right) D \left(m_\mu^2 \frac{x^2}{1-x} \right) \frac{dx}{x}$$

■ Knecht, Lect. Notes Phys. **629**, 37 (2004); de Rafael, Phys. Rev. D **96**, 014510 (2017).

It is necessary to emphasize here that this way of the derivation of the spacelike expressions for $a_\mu^{\text{HVP}(2)}$ from the timelike one **entirely relies** on the particular form of the leading-order kernel function $K_R^{(2)}(s)$.

The explicit form of the leading-order spacelike kernel functions $K_{\Pi}^{(2)}(Q^2)$ and $K_D^{(2)}(Q^2)$ can be obtained by mapping the integration range $0 \leq x < 1$ in the equations given on the previous page onto the kinematic interval $0 \leq Q^2 < \infty$. Specifically, the kernel function $K_{\Pi}^{(2)}(Q^2)$ takes the following form

$$K_{\Pi}^{(2)}(Q^2) = \tilde{K}_{\Pi}^{(2)}\left(\frac{Q^2}{4m_{\mu}^2}\right), \quad \zeta \tilde{K}_{\Pi}^{(2)}(\zeta) = \frac{1}{\zeta^2} \frac{y^5(\zeta)}{1-y(\zeta)}, \quad y(\zeta) = \zeta \left(\sqrt{1+\zeta^{-1}} - 1 \right), \quad \zeta = \frac{Q^2}{4m_{\mu}^2}$$

- Groote, Korner, Pivovarov, Eur. Phys. J. C **24**, 393 (2002); Blum, Phys. Rev. Lett. **91**, 052001 (2003); Nesterenko, J. Phys. G **42**, 085004 (2015); de Rafael, Phys. Rev. D **96**, 014510 (2017).

For the kernel function $K_D^{(2)}(Q^2)$ mapping the integration range $0 \leq x < 1$ onto the kinematic interval $0 \leq Q^2 < \infty$ yields

$$K_D^{(2)}(Q^2) = \tilde{K}_D^{(2)}\left(\frac{Q^2}{4m_{\mu}^2}\right), \quad \zeta \tilde{K}_D^{(2)}(\zeta) = (2\zeta + 1)^2 - 2(2\zeta + 1)\sqrt{\zeta(\zeta + 1)} - \frac{1}{2}, \quad \zeta = \frac{Q^2}{4m_{\mu}^2}$$

- Groote, Korner, Pivovarov, Eur. Phys. J. C **24**, 393 (2002); de Rafael, Phys. Rev. D **96**, 014510 (2017).

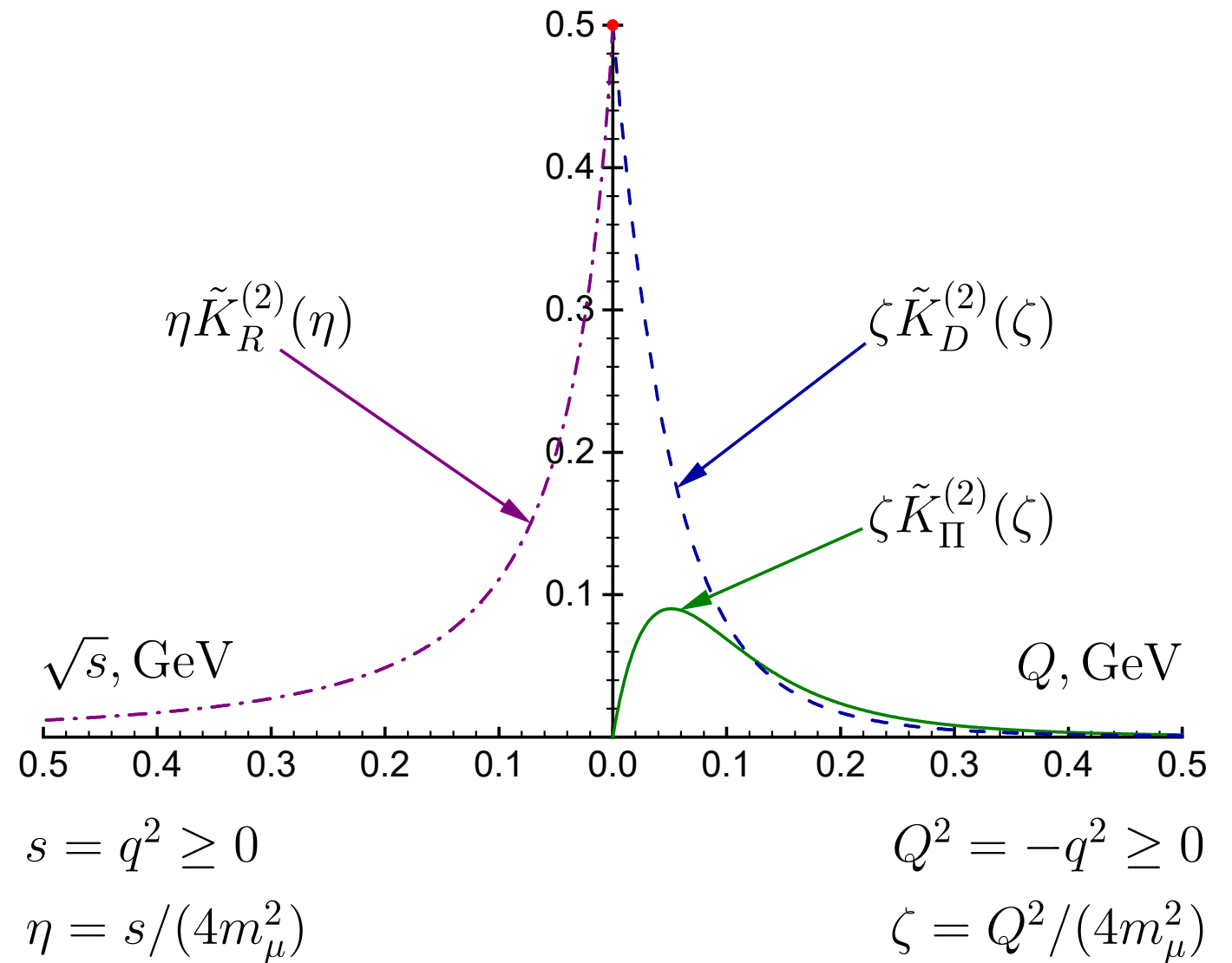
It is straightforward to verify that all six obtained relations for the spacelike and timelike kernel functions hold for $K_{\Pi}^{(2)}(Q^2)$, $K_D^{(2)}(Q^2)$, and $K_R^{(2)}(s)$

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

The aforementioned infrared limiting value of the spacelike and timelike kernel functions

$$K_0^{(2)} = \lim_{Q^2 \rightarrow 0_+} \frac{Q^2}{4m_\mu^2} K_D^{(2)}(Q^2) = \lim_{s \rightarrow 0_+} \frac{s}{4m_\mu^2} K_R^{(2)}(s) = \int_0^\infty K_{\Pi}^{(2)}(\xi) \frac{d\xi}{4m_\mu^2} = \frac{1}{2}$$

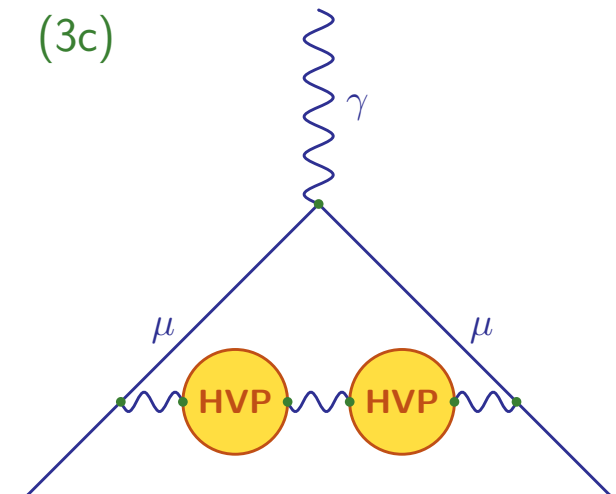
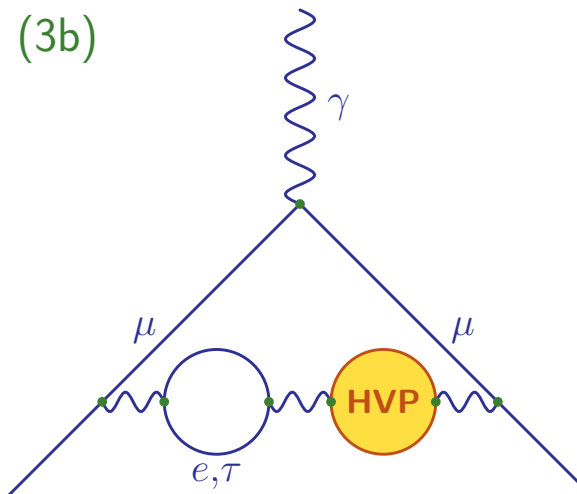
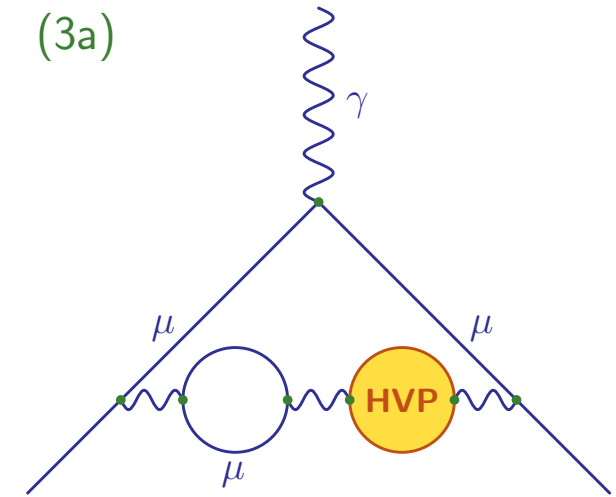
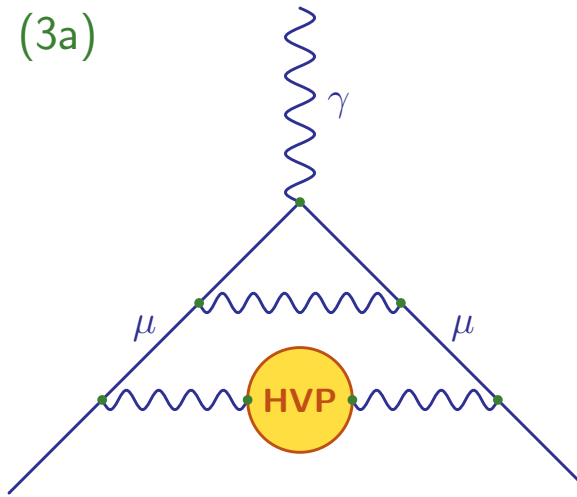
corresponds to the leading Schwinger contribution ■ Schwinger, Phys. Rev. **73**, 416 (1948).



KERNEL FUNCTIONS IN THE NEXT-TO-LEADING ORDER

In the next-to-leading order of perturbation theory (i.e., in the third order in the electromagnetic coupling) the hadronic vacuum polarization contribution to the muon anomalous magnetic moment consists of three parts, namely

$$a_{\mu}^{\text{HVP}(3)} = a_{\mu}^{\text{HVP}(3a)} + a_{\mu}^{\text{HVP}(3b)} + a_{\mu}^{\text{HVP}(3c)}.$$



Kernel functions (3a)

Here the explicit expression for the timelike kernel function $K_R^{(3a)}(s)$ is available, whereas the spacelike kernel functions $K_{\square}^{(3a)}(Q^2)$ and $K_D^{(3a)}(Q^2)$ can be calculated by making use of the relations obtained above. Namely,

$$\begin{aligned} \eta \tilde{K}_R^{(3a)}(\eta) = & -\frac{139}{144} + \frac{115}{18}\eta + \left(\frac{19}{12} - \frac{7}{9}\eta + \frac{23}{9}\eta^2 + \frac{1}{4(\eta-1)} \right) \ln(4\eta) + \\ & + \left(\frac{2}{3\eta} - \frac{127}{18} + \frac{115}{9}\eta - \frac{46}{9}\eta^2 \right) \frac{A(\eta)}{\psi(\eta)} + \left(\frac{9}{4} + \frac{5}{6}\eta - 8\eta^2 - \frac{1}{2\eta} \right) \zeta_2 + \frac{5}{6}\eta^2 \ln^2(4\eta) + \\ & + \left(\frac{14}{3}\eta - 1 \right) (\eta-1) \frac{1}{\psi(\eta)} T_1(\eta) + \left(\frac{19}{6} + \frac{53}{3}\eta - \frac{58}{3}\eta^2 - \frac{1}{3\eta} + \frac{2}{\eta-1} \right) A^2(\eta) + \\ & + \left(\frac{13}{12\eta} - \frac{7}{6} + \eta - \frac{8}{3}\eta^2 - \frac{1}{4\eta(\eta-1)} \right) \frac{T_2(\eta)}{\psi(\eta)} + \left(\frac{1}{2} - \frac{14}{3}\eta + 8\eta^2 \right) T_3(\eta), \quad \eta = \frac{s}{4m_\mu^2}, \end{aligned}$$

with $s = q^2 \geq 0$ being the timelike kinematic variable, $A_0^{(3a)} = (2/3)(\alpha/\pi)^3$,

$$T_1(\eta) = A(\eta) \ln(4\eta) + 2 \left\{ \text{Li}_2(1 - B(\eta)) + A^2(\eta) \right\}, \quad T_2(\eta) = \text{Li}_2(-B(\eta)) + A^2(\eta) + \frac{1}{2} \zeta_2,$$

$$T_3(\eta) = -6\text{Li}_3(B(\eta)) - 3\text{Li}_3(-B(\eta)) + 4 \ln(1 - B(\eta)) A^2(\eta) + \\ + \left(2A^2(\eta) + 3\zeta_2 \right) \ln(1 + B(\eta)) - 4 \left\{ \text{Li}_2(-B(\eta)) + 2\text{Li}_2(-B(\eta)) \right\} A(\eta),$$

$$A(\eta) = \text{arctanh}(\psi(\eta)), \quad B(\eta) = \frac{1 - \psi(\eta)}{1 + \psi(\eta)}, \quad \psi(\eta) = \frac{\sqrt{\eta - 1}}{\sqrt{\eta}},$$

$$\text{Li}_2(y) = - \int_0^y \ln(1 - t) \frac{dt}{t}, \quad \text{Li}_3(y) = \int_0^y \text{Li}_2(t) \frac{dt}{t}, \quad \zeta_t = \sum_{n=1}^{\infty} \frac{1}{n^t}$$

■ Barbieri, Remiddi, Nucl. Phys. B **90**, 233 (1975).

Spacelike kernel functions in terms of the timelike one (see p. 7 and p. 10):

$$K_{\square}^{(3a)}(Q^2) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \left(K_R^{(3a)}(-Q^2 + i\varepsilon) - K_R^{(3a)}(-Q^2 - i\varepsilon) \right), \quad Q^2 \geq 0,$$

$$K_D^{(3a)}(Q^2) = \frac{1}{Q^2} \int_{Q^2}^{\infty} K_{\square}^{(3a)}(\xi) d\xi = \frac{4m_{\mu}^2}{Q^2} K_0^{(3a)} - \frac{1}{Q^2} \int_0^{Q^2} K_{\square}^{(3a)}(\xi) d\xi, \quad \xi = -p^2 \geq 0.$$

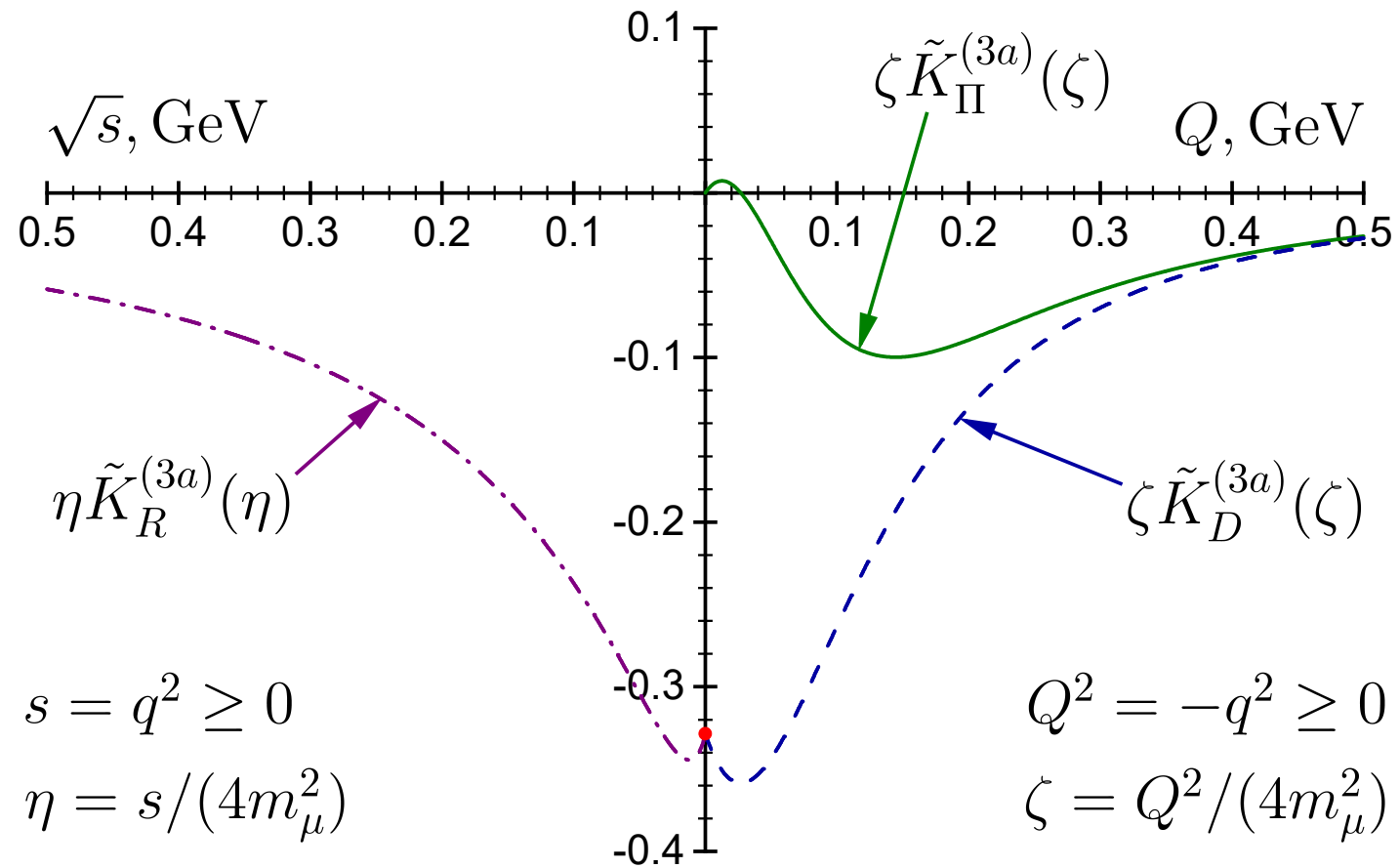
The explicit expression for the spacelike kernel function $K_{\square}^{(3a)}(Q^2)$ reads

$$\begin{aligned} \zeta \tilde{K}_{\square}^{(3a)}(\zeta) = & -\left(\frac{19}{12} + \frac{7}{9}\zeta + \frac{23}{9}\zeta^2 - \frac{1}{4(\zeta+1)}\right) + \left(\frac{1}{3\zeta} + \frac{127}{36} + \frac{115}{18}\zeta + \frac{23}{9}\zeta^2\right)\psi(\zeta+1) - \\ & - \left(\frac{14}{3}\zeta + 1\right)(\zeta+1)\psi(\zeta+1) \left\{ \frac{1}{2} \ln(4\zeta) + 3A(\zeta+1) + 2 \ln(1+B(\zeta+1)) \right\} + \\ & + \left(-\frac{19}{6} + \frac{53}{3}\zeta + \frac{58}{3}\zeta^2 - \frac{1}{3\zeta} + \frac{2}{\zeta+1}\right)A(\zeta+1) - \frac{5}{3}\zeta^2 \ln(4\zeta) + \\ & + \left(\frac{13}{12\zeta} + \frac{7}{6} + \zeta + \frac{8}{3}\zeta^2 + \frac{1}{4\zeta(\zeta+1)}\right)\psi(\zeta+1)A(\zeta+1) - \\ & - \left(\frac{1}{2} + \frac{14}{3}\zeta + 8\zeta^2\right) \left\{ 2A(\zeta+1) \left\{ 2 \ln(1+B(\zeta+1)) + \ln(1-B(\zeta+1)) \right\} - \right. \\ & \left. - 2 \left\{ \text{Li}_2(B(\zeta+1)) + 2\text{Li}_2(-B(\zeta+1)) \right\} \right\}, \quad \zeta = \frac{Q^2}{4m_{\mu}^2} \end{aligned}$$

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

An equivalent form of this equation has been independently derived in

■ Balzani, Laporta, Passera, Phys. Lett. B **834**, 137462 (2022); arXiv:2112.05704 [hep-ph].



The infrared limiting value of the spacelike and timelike kernel functions

$$\begin{aligned}
 K_0^{(3a)} &= \lim_{Q^2 \rightarrow 0_+} \frac{Q^2}{4m_\mu^2} K_D^{(3a)}(Q^2) = \lim_{s \rightarrow 0_+} \frac{s}{4m_\mu^2} K_R^{(3a)}(s) = \int_0^\infty K_\Pi^{(3a)}(\xi) \frac{d\xi}{4m_\mu^2} = \\
 &= \frac{197}{144} + \frac{1}{2}\zeta_2 - 3\zeta_2 \ln(2) + \frac{3}{4}\zeta_3 \simeq -0.328479
 \end{aligned}$$

corresponds to the QED contribution ■ Sommerfield (1957), (1958); Petermann (1957), (1958).

Since the diagram on hand factually constitutes an additional lepton loop insertion into the only internal photon line of the leading-order diagram (see p. 12), the spacelike kernel function $K_{\Pi}^{(3b)}(Q^2)$ is the product of the kernel function of the preceding order $K_{\Pi}^{(2)}(Q^2)$ (see p. 14) and the leptonic vacuum polarization function $\bar{\Pi}_{\ell}(Q^2)$ (see p. 21), namely

$$K_{\Pi}^{(3b)}(Q^2) = K_{\Pi}^{(2)}(Q^2)\bar{\Pi}_{\ell}(Q^2).$$

Note that the spacelike kernel function $K_{\Pi}^{(3b)}(Q^2)$ has also been derived from the timelike one $K_R^{(3b)}(s)$ by making use of the relevant dispersion relation

■ Chakraborty, Davies, Koponen, Lepage, de Water (2018); Balzani, Laporta, Passera (2022).

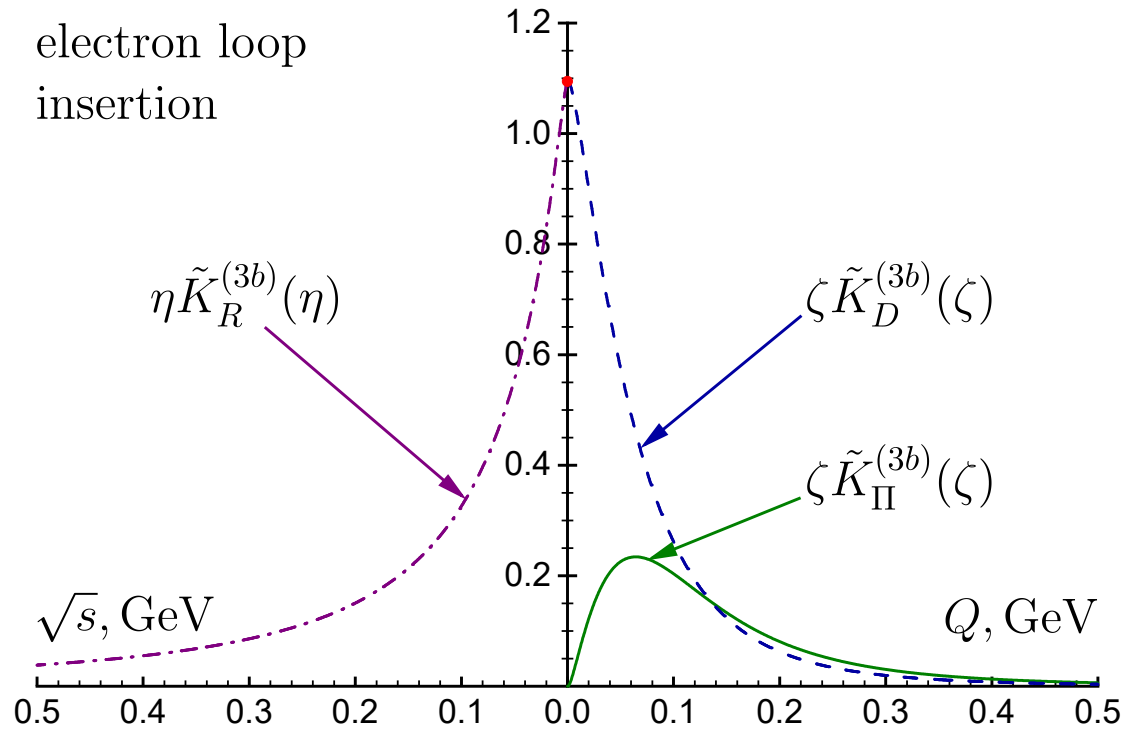
In turn, the spacelike kernel function $K_D^{(3b)}(Q^2)$ can be calculated by making use of the relation obtained earlier (see p. 10):

$$K_D^{(3b)}(Q^2) = \frac{1}{Q^2} \int_{Q^2}^{\infty} K_{\Pi}^{(3b)}(\xi) d\xi = \frac{4m_{\mu}^2}{Q^2} K_0^{(3b)} - \frac{1}{Q^2} \int_0^{Q^2} K_{\Pi}^{(3b)}(\xi) d\xi, \quad \xi = -p^2 \geq 0.$$

All the relevant details can be found in

■ Nesterenko, *J. Phys. G* **49**, 055001 (2022); **50**, 029401 (2023).

electron loop
insertion



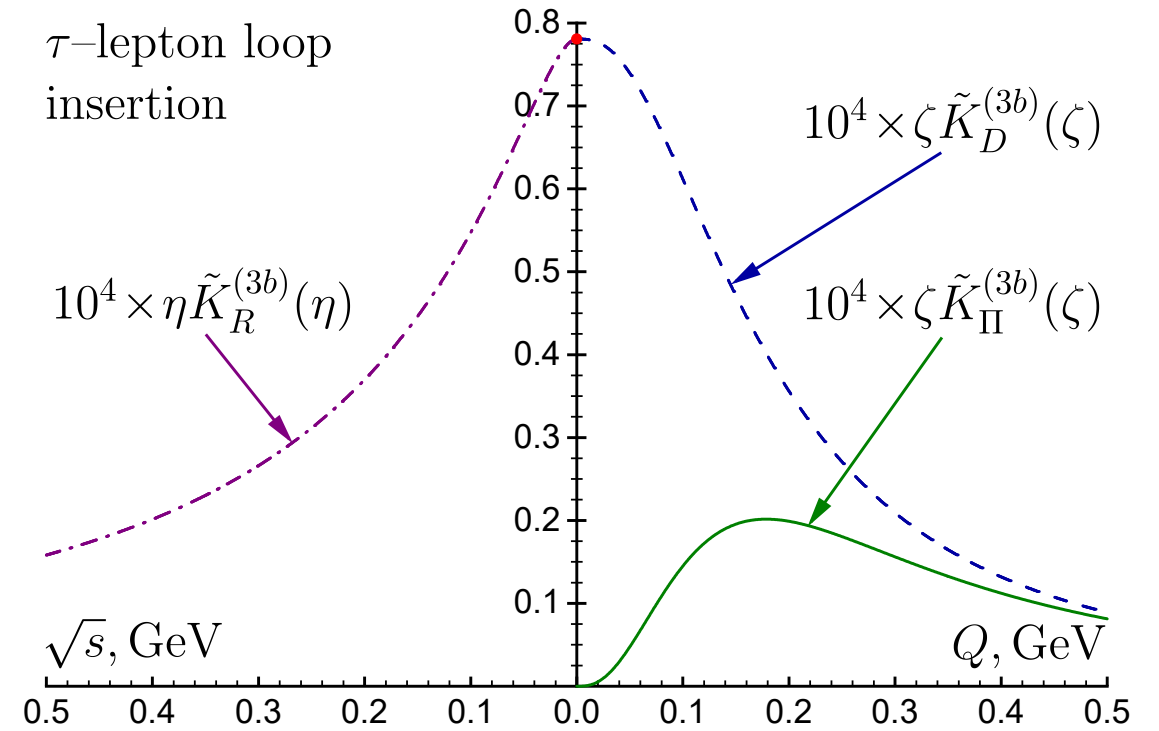
$$s = q^2 \geq 0$$

$$\eta = s/(4m_\mu^2)$$

$$Q^2 = -q^2 \geq 0$$

$$\zeta = Q^2/(4m_\mu^2)$$

τ -lepton loop
insertion



$$s = q^2 \geq 0$$

$$\eta = s/(4m_\mu^2)$$

$$Q^2 = -q^2 \geq 0$$

$$\zeta = Q^2/(4m_\mu^2)$$

The infrared limiting value of the spacelike and timelike kernel functions

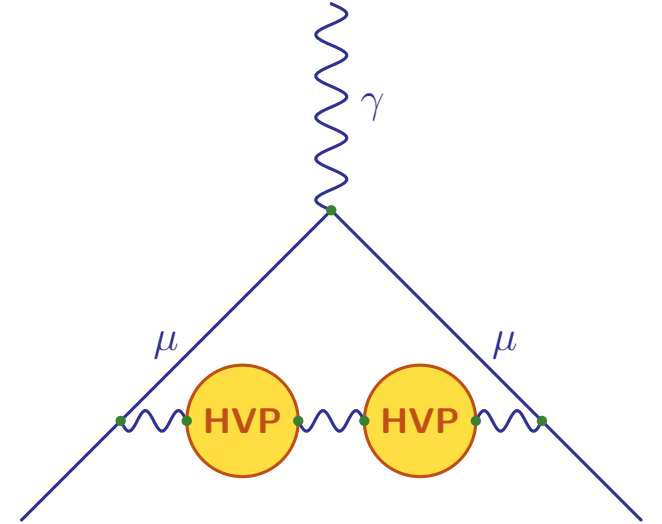
$$K_0^{(3b)} = \lim_{Q^2 \rightarrow 0_+} \frac{Q^2}{4m_\mu^2} K_D^{(3b)}(Q^2) = \lim_{s \rightarrow 0_+} \frac{s}{4m_\mu^2} K_R^{(3b)}(s) = \int_0^\infty K_\Pi^{(3b)}(\xi) \frac{d\xi}{4m_\mu^2} \simeq \begin{cases} 1.094258, & [\text{electron}] \\ 0.780758 \times 10^{-4}, & [\tau\text{-lepton}] \end{cases}$$

corresponds to the QED contribution ■ Elend, Phys. Lett. **20**, 682 (1966); **21**, 720(E) (1966).

Kernel functions (3c)

The contribution $a_\mu^{\text{HVP}(3c)}$ takes a particularly simple form in terms of the hadronic vacuum polarization function $\bar{\Pi}(Q^2)$

$$a_\mu^{\text{HVP}(3c)} = A_0^{(3c)} \int_0^\infty K_\Pi^{(2)}(Q^2) \left(\bar{\Pi}(Q^2) \right)^2 \frac{dQ^2}{4m_\mu^2}, \quad A_0^{(3c)} = \frac{1}{9} \left(\frac{\alpha}{\pi} \right)^3,$$



where $K_\Pi^{(2)}(Q^2)$ stands for the spacelike kernel function of the preceding order (see page 14). The contribution $a_\mu^{\text{HVP}(3c)}$ can also be represented in terms of the Adler function $D(Q^2)$ by making use of the relevant dispersion relation (see page 5):

$$a_\mu^{\text{HVP}(3c)} = A_0^{(3c)} \int_0^\infty \frac{dQ^2}{4m_\mu^2} K_\Pi^{(2)}(Q^2) \left(\int_0^{Q^2} \frac{d\xi}{\xi} D(\xi) \right)^2,$$

with $\xi = -p^2 \geq 0$ being a spacelike kinematic variable.

In turn, the contribution $a_\mu^{\text{HVP}(3c)}$ can be expressed in terms of the R -ratio of electron–positron annihilation into hadrons by making use of the pertinent dispersion relation (see page 4):

$$a_\mu^{\text{HVP}(3c)} = A_0^{(3c)} \int_{s_0}^{\infty} \frac{ds_1}{s_1} \int_{s_0}^{\infty} \frac{ds_2}{s_2} K_R^{(3c)}(s_1, s_2) R(s_1) R(s_2),$$

where

$$K_R^{(3c)}(s_1, s_2) = \int_0^\infty \frac{K_\pi^{(2)}(Q^2) Q^4}{(s_1 + Q^2)(s_2 + Q^2)} \frac{dQ^2}{4m_\mu^2}$$

■ Calmet, Narison, Perrottet, de Rafael, Phys. Lett. B **61**, 283 (1976).

The explicit form of the timelike kernel function $K_R^{(3c)}(s_1, s_2)$ was given in

■ B. Krause, Phys. Lett. B **390**, 392 (1997).

Note that the NNLO spacelike kernels $K_\pi^{(4)}(Q^2)$ have also been addressed in

■ Balzani, Laporta, Passera, Phys. Lett. B **834**, 137462 (2022).

DISPERSIVELY IMPROVED PERTURBATION THEORY (DPT)

The DPT merges the nonperturbative constraints, which relevant dispersion relations impose on $\Pi(q^2)$, $R(s)$, $D(Q^2)$, with corresponding perturbative input in a self-consistent way, thereby overcoming some intrinsic difficulties of the perturbative approach to QCD and extending its applicability range towards the IR domain

■ Nesterenko, Phys. Rev. D **88**, 056009 (2013); J. Phys. G **42**, 085004 (2015)

$$\Delta\Pi(q^2, q_0^2) = N_c \sum_{f=1}^{n_f} Q_f^2 \left\{ \Delta\Pi^{(0)}(q^2, q_0^2) + \int_{s_0}^{\infty} \rho(\sigma) \ln \left(\frac{\sigma - q^2}{\sigma - q_0^2} \frac{s_0 - q_0^2}{s_0 - q^2} \right) \frac{d\sigma}{\sigma} \right\},$$

$$R(s) = N_c \sum_{f=1}^{n_f} Q_f^2 \left\{ R^{(0)}(s) + \theta(s - s_0) \int_s^{\infty} \rho(\sigma) \frac{d\sigma}{\sigma} \right\},$$

$$D(Q^2) = N_c \sum_{f=1}^{n_f} Q_f^2 \left\{ D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + s_0} \int_{s_0}^{\infty} \rho(\sigma) \frac{\sigma - s_0}{\sigma + Q^2} \frac{d\sigma}{\sigma} \right\},$$

$$\rho(\sigma) = \frac{1}{\pi} \frac{d}{d \ln \sigma} \text{Im} \lim_{\varepsilon \rightarrow 0_+} p(\sigma - i\varepsilon) = -\frac{d r(\sigma)}{d \ln \sigma} = \frac{1}{\pi} \text{Im} \lim_{\varepsilon \rightarrow 0_+} d(-\sigma - i\varepsilon).$$

Derivation of the DPT integral representations for $\Pi(q^2)$, $R(s)$, $D(Q^2)$ involves neither additional approximations nor model-dependent assumptions, with all the nonperturbative constraints being embodied.

The leading-order terms of the functions on hand read

$$\Delta\Pi^{(0)}(q^2, q_0^2) = 2\left(1 + \frac{1}{\xi}\right)\left(1 - \sqrt{1 + \xi^{-1}} \operatorname{arcsinh}\sqrt{\xi}\right) - 2\left(1 + \frac{1}{\xi_0}\right)\left(1 - \sqrt{1 + \xi_0^{-1}} \operatorname{arcsinh}\sqrt{\xi_0}\right),$$

$$R^{(0)}(s) = \theta(s - s_0) \left(1 - \frac{s_0}{s}\right)^{3/2}, \quad \xi = \frac{-q^2}{s_0} = \frac{Q^2}{s_0}, \quad \xi_0 = \frac{-q_0^2}{s_0} = \frac{Q_0^2}{s_0},$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi} \left(1 - \sqrt{1 + \xi^{-1}} \operatorname{arcsinh}\sqrt{\xi}\right)$$

■ Feynman (1972); Akhiezer, Berestetsky (1965).

The spectral density $\rho(\sigma)$ brings in the perturbative input:

$$\rho_{\text{pert}}(\sigma) = \frac{1}{\pi} \frac{d}{d \ln \sigma} \operatorname{Im} p_{\text{pert}}(\sigma - i0_+) = -\frac{d}{d \ln \sigma} r_{\text{pert}}(\sigma) = \frac{1}{\pi} \operatorname{Im} d_{\text{pert}}(-\sigma - i0_+).$$

At the one-loop level it takes a simple form: $\rho_{\text{pert}}^{(1)}(\sigma) = 4 / [\beta_0 (\ln^2(\sigma/\Lambda^2) + \pi^2)]$.

First few loop levels in α_s : ■ Nesterenko, Simolo (2010, 2011); Bakulev (2013); Cvetič (2015).

Perturbative spectral function at the ℓ -loop level [$a_s(Q^2) = \alpha_s(Q^2)\beta_0/(4\pi)$]:

$$\rho_{\text{pert}}^{(\ell)}(\sigma) = \sum_{j=1}^{\ell} d_j \bar{\rho}_j^{(\ell)}(\sigma), \quad \bar{\rho}_j^{(\ell)}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} \left\{ [a_s^{(\ell)}(-\sigma - i\varepsilon)]^j - [a_s^{(\ell)}(-\sigma + i\varepsilon)]^j \right\}.$$

Explicit expression for $\rho_{\text{pert}}^{(\ell)}(\sigma)$ valid at an arbitrary loop level:

$$\rho_{\text{pert}}^{(\ell)}(\sigma) = \sum_{j=1}^{\ell} d_j \sum_{k=0}^{K(j)} \binom{j}{2k+1} (-1)^k \pi^{2k} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b_n^m u_n^m(\sigma) \right]^{j-2k-1} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b_n^m v_n^m(\sigma) \right]^{2k+1}$$

■ Nesterenko, *Eur. Phys. J. C* **77**, 844 (2017).

The obtained expression for $\rho_{\text{pert}}^{(\ell)}(\sigma)$ makes the higher-loop practical calculations within DPT easily accessible. Here ℓ denotes the loop level in α_s ,

$$u_n^m(\sigma) = \begin{cases} u_n^0(\sigma), & \text{if } m = 0, \\ u_n^0(\sigma)u_0^m(\sigma) - \pi^2 v_n^0(\sigma)v_0^m(\sigma), & \text{if } m \geq 1, \end{cases}$$

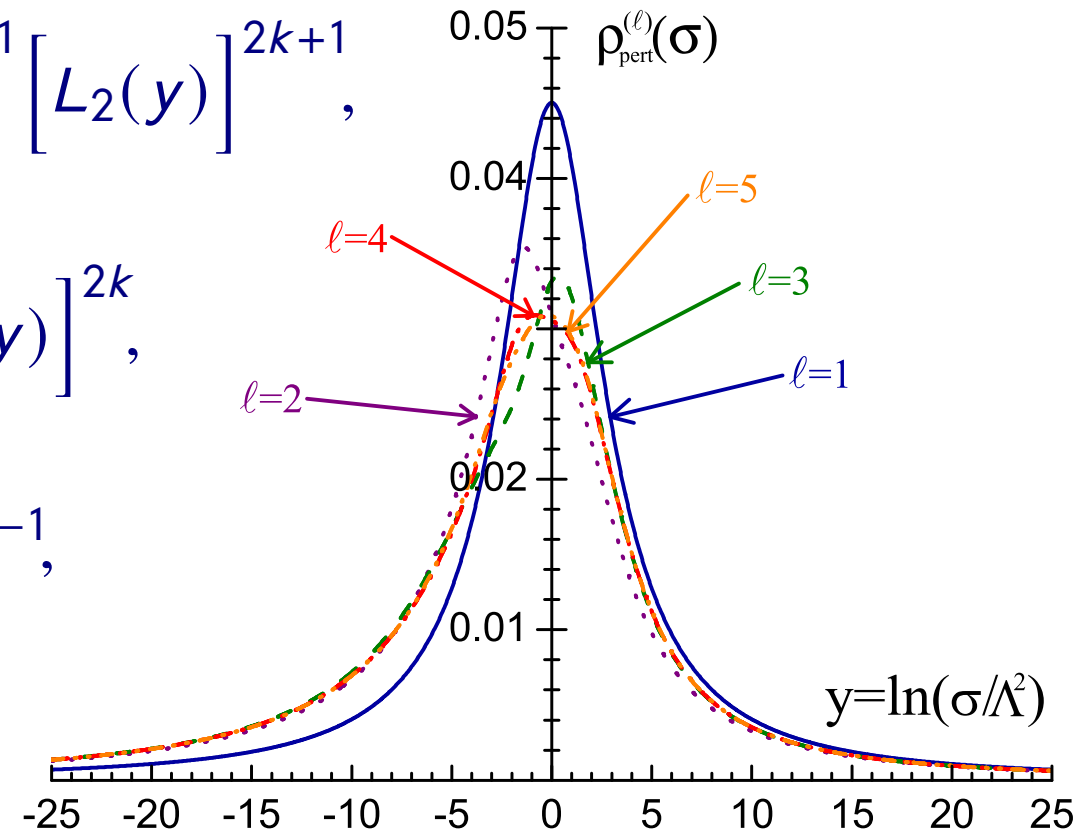
$$v_n^m(\sigma) = \begin{cases} v_n^0(\sigma), & \text{if } m = 0, \\ v_n^0(\sigma)u_0^m(\sigma) + u_n^0(\sigma)v_0^m(\sigma), & \text{if } m \geq 1, \end{cases}$$

$$v_0^m(\sigma) = \sum_{k=0}^{K(m)} \binom{m}{2k+1} (-1)^{k+1} \pi^{2k} [L_1(y)]^{m-2k-1} [L_2(y)]^{2k+1},$$

$$u_0^m(\sigma) = \sum_{k=0}^{K(m+1)} \binom{m}{2k} (-1)^k \pi^{2k} [L_1(y)]^{m-2k} [L_2(y)]^{2k},$$

$$v_n^0(\sigma) = \frac{1}{(y^2 + \pi^2)^n} \sum_{k=0}^{K(n)} \binom{n}{2k+1} (-1)^k \pi^{2k} y^{n-2k-1},$$

$$u_n^0(\sigma) = \frac{1}{(y^2 + \pi^2)^n} \sum_{k=0}^{K(n+1)} \binom{n}{2k} (-1)^k \pi^{2k} y^{n-2k},$$



$$K(n) = \frac{n-2}{2} + \frac{n \bmod 2}{2}, \quad \binom{n}{m} = \frac{n!}{m!(n-m)!}, \quad y = \ln\left(\frac{\sigma}{\Lambda^2}\right),$$

$L_1(y) = \ln\sqrt{y^2 + \pi^2}$, $L_2(y) = (1/2) - \arctan(y/\pi)/\pi$, b_n^m is a combination of the perturbative β function coefficients ($b_1^0 = 1$, $b_2^0 = 0$, $b_2^1 = -\beta_1/\beta_0^2$, etc.), and d_j is a perturbative Adler function coefficient ■ Nesterenko, Eur. Phys. J. C **77**, 844 (2017).

Application of DPT to the muon $g - 2$

The DPT $\bar{\Pi}(Q^2)$, being applicable in the entire energy range, has proved to be capable of describing a_μ^{HVP} .

The DPT assessment of a_μ^{HVP} involves:

- direct integration in SL domain
- four-loop expression for $\bar{\Pi}(Q^2)$
- PDG24 as input
- no TL data on R -ratio

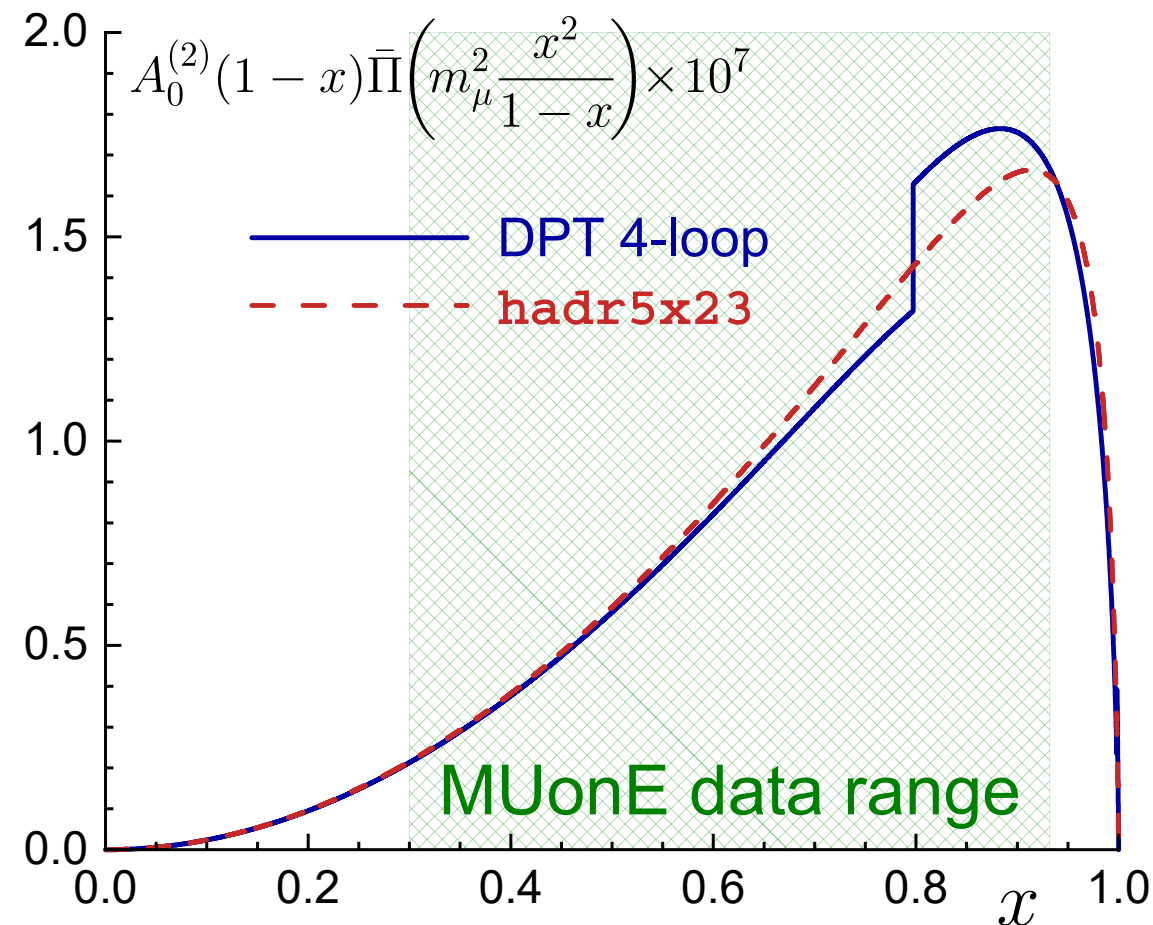
LO ($\ell = 2$): $a_\mu^{\text{HVP}(2)} = (695.95 \pm 2.83) \times 10^{-10}$

■ Nesterenko, J. Phys. G **42**, 085004 (2015); **51**, 015005 (2024); **53**, 025003 (2026).

At low energies ($x \lesssim 0.3$) the DPT $\bar{\Pi}(Q^2)$ can be employed as a supplementing infrared input for the lattice studies, MUonE @ CERN project, etc.

$$a_\mu^{\text{HVP}(2)} = \frac{1}{3} \left(\frac{\alpha}{\pi} \right)^2 \int_0^1 (1-x) \bar{\Pi} \left(m_\mu^2 \frac{x^2}{1-x} \right) dx$$

■ Lautrup, Peterman, de Rafael, Phys. Rept. **3**, 193 (1972).



hadr5x23 routine: Jegerlehner (2023)

In the next-to-leading order ($\ell = 3$) the DPT assessment of a_μ^{HVP} yields

$$a_\mu^{\text{HVP}(3a)} = (-218.08 \pm 0.96) \times 10^{-11},$$

$$a_{\mu,e}^{\text{HVP}(3b)} = (106.53 \pm 0.44) \times 10^{-11},$$

$$a_{\mu,\tau}^{\text{HVP}(3b)} = (559.67 \pm 2.08) \times 10^{-15},$$

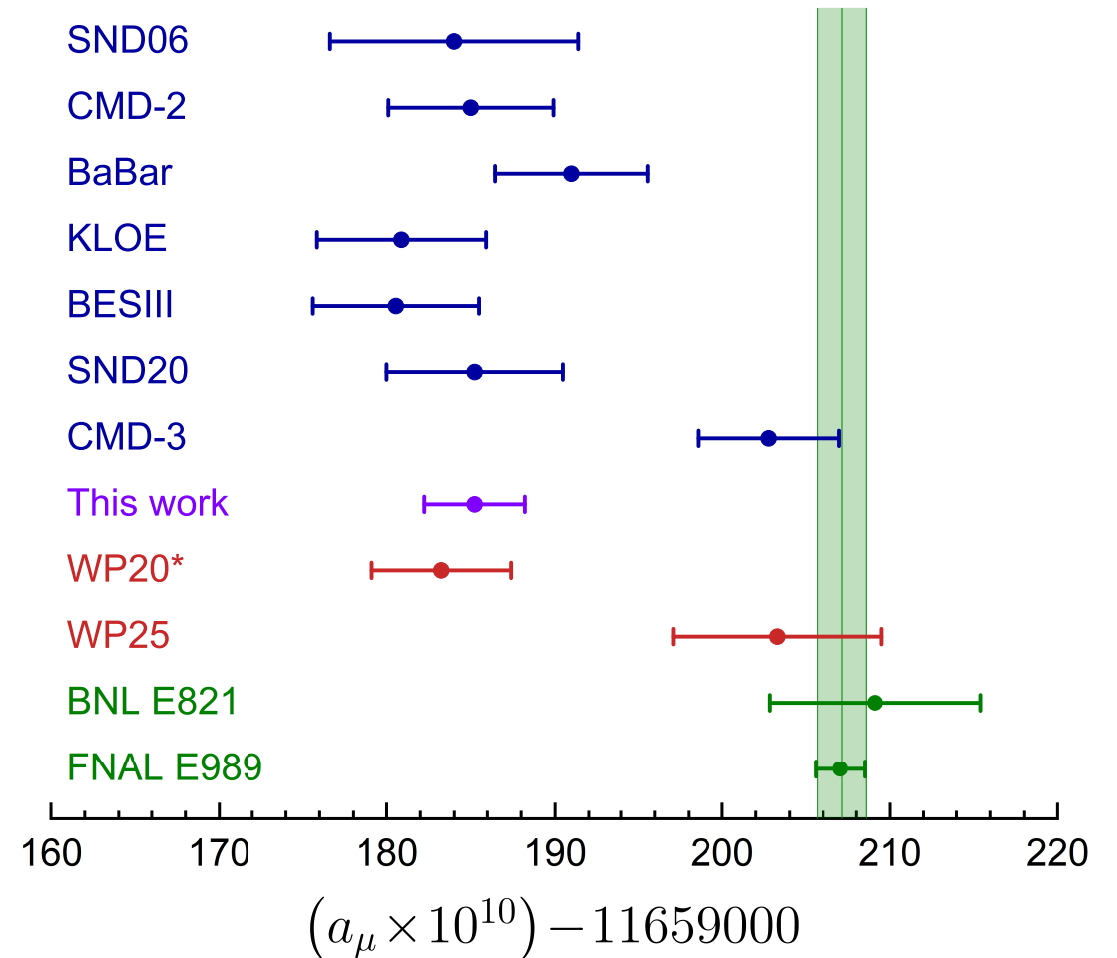
$$a_\mu^{\text{HVP}(3c)} = (334.92 \pm 2.66) \times 10^{-13},$$

that, being complemented with the remaining contributions to a_μ , results in

$$a_\mu^{\text{SM}} = (11659185.25 \pm 3.00) \times 10^{-10}.$$

The obtained value agrees with most of the TL-data-driven evaluations of a_μ

- Nesterenko, J. Phys. G **42**, 085004 (2015); **51**, 015005 (2024); **53**, 025003 (2026).



All details: [TI WP25] Phys. Rept. **1143**, 1 (2025).

Combined result of measurements:

$$a_\mu^{\text{exp}} = (11659207.15 \pm 1.45) \times 10^{-10}$$

- BNL E821: Phys. Rev. D **73**, 072003 (2006);
FNAL E989: Phys. Rev. Lett. **135**, 101802 (2025).

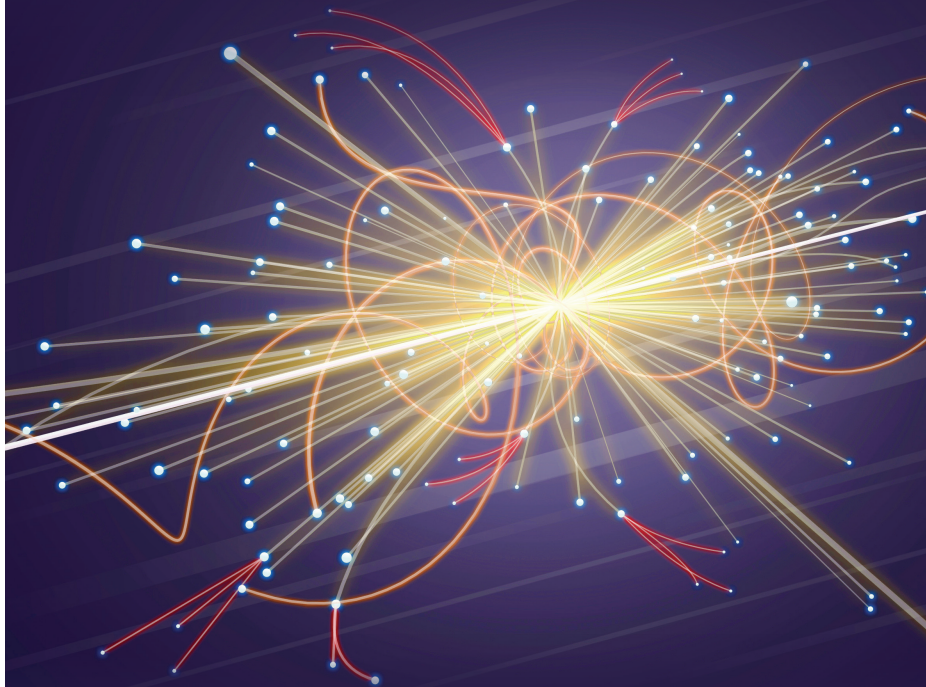
SUMMARY

- The complete set of relations, which mutually express the spacelike and timelike kernel functions for a_μ^{HVP} in terms of each other, is obtained.
- By making use of the derived relations the explicit expression for the NLO spacelike kernel function $K_\Pi^{(3a)}(Q^2)$ is obtained and the kernel functions $K_D^{(3a)}(Q^2)$ and $K_D^{(3b)}(Q^2)$ are calculated numerically.
- It is shown that the IR limiting value of the spacelike $Q^2 K_D(Q^2)/(4m_\mu^2)$ and timelike $s K_R(s)/(4m_\mu^2)$ kernels is identical to the corresponding QED contribution to a_μ of the preceding order in the electromagnetic coupling.
- The DPT function $\bar{\Pi}(Q^2)$, being capable of describing a_μ^{HVP} , provides a supplementing infrared input for the spacelike methods.
- The obtained results can be employed in the assessments of a_μ^{HVP} within lattice studies, MUonE @ CERN project etc.



Alexander V. Nesterenko

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Timelike Domains
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