Precise predictions for radiative return at flavour factories

14th edition of the International Workshop on e+e− collisions from Phi to Psi
(PhiPsi 2026)

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On behalf of the BabaYaga team

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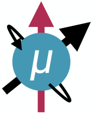
INFN National Institute for Nuclear Physics
UNIVERSITÀ DI PAVIA

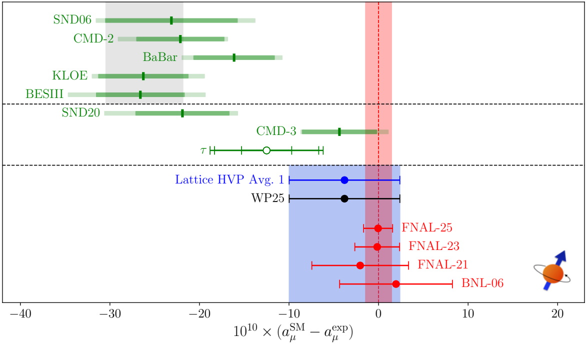
Data-theory comparison of $g-2$

The Muon anomaly

$$\vec{\mu} = -g\mu_B\vec{S} \qquad a_\mu = \frac{(g-2)_\mu}{2}$$



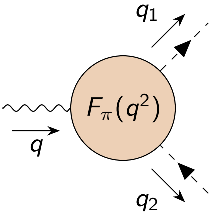
The current status of the theory–experiment comparison is the following:

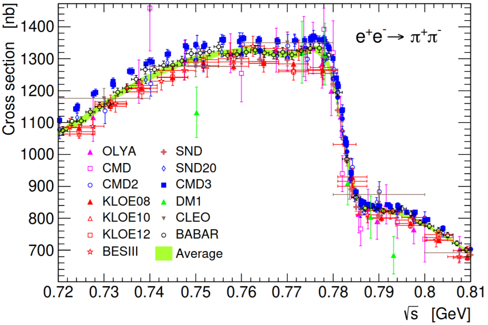


https://arxiv.org/abs/2505.21476

The Pion Form Factor

$$\langle \pi^+(q_2)\pi^-(q_1)|J_\pi^\mu(0)|0\rangle = e(q_1 - q_2)^\mu F_\pi\,(q^2)$$





https://arxiv.org/pdf/2312.02053

The radiative return

At fixed collider energy $\sqrt{s}$, lower-energy two pion pairs can be studied via initial-state radiation:

$$e^+e^- \to \pi^+\pi^-\gamma, \quad M^2_{\pi\pi} = (p_{\pi^+} + p_{\pi^-})^2 < s$$

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The measured differential cross section relates to the pion production cross section:

$$\frac{\mathrm{d}\sigma_{\pi^+\pi^-\gamma}}{dM_{\pi\pi}} = H(s, M_{\pi\pi}^2)\,\sigma_{\pi^+\pi^-}(M_{\pi\pi}^2),$$

where $H(s, M_{\pi\pi}^2)$ is the radiator function.

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A precise theoretical treatment of radiative corrections is required to extract the pion form factor at the sub-percent level.

The perturbative accuracy

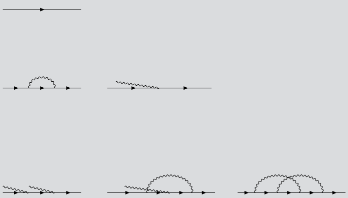
LL resummation

Approximate calculation of soft-virtual corrections to any order.

1

$\mathcal{C}_\alpha^{\mathrm{LL}}$

$\frac{\left(\mathcal{C}_\alpha^{\mathrm{LL}}\right)^2}{2}$



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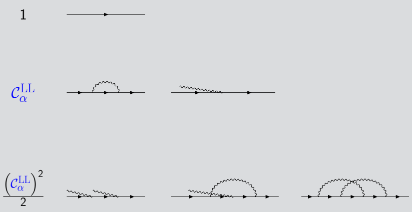
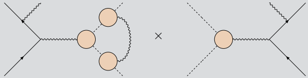


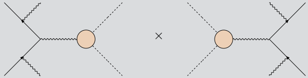
NLO

Exact calculation up to $\mathcal{O}(\alpha)$ w.r.t. the born scattering amplitude.

- Virtual corrections



- Real corrections



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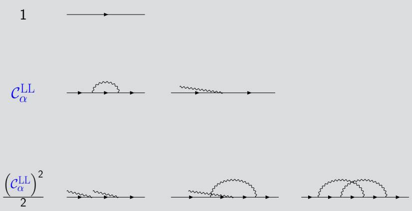

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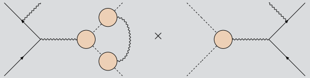

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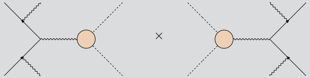

The perturbative accuracy

LL resummation

Approximate calculation of soft-virtual corrections to any order.

α^0		
αL	α	
$\frac{1}{2}\alpha^2 L^2$	$\frac{1}{2}\alpha^2 L$	$\frac{1}{2}\alpha^2$
$\sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^n$	$\sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^{n-1}$	$\dots$

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The perturbative accuracy

NLO+LL (NLOPS)

LO	α^0		
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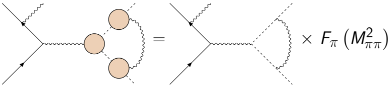
h.o.	$\sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^n$	$\sum_{n=3}^{\infty} \frac{\alpha^n}{n!} L^{n-1}$	$\ldots$

NLOPS master formula

$$d\sigma_{\mathrm{NLOPS}} = \sum_{n=1}^{\infty} \exp\{-\mathcal{C}_{\alpha}^{\mathrm{LL}}\}\, \mathrm{F}_{\mathrm{SV}}\, \frac{1}{n!} \left|\mathcal{M}_{\mathrm{n}}^{\mathrm{X}}\right|^2 d\Phi_{\mathrm{n}}(\{p\},\{k\})\,.$$

The Pion internal structure

The standard way which respects IR cancellation is F×sQED:



The Pion internal structure

F×sQED

2601.19530

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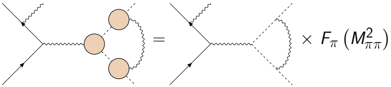

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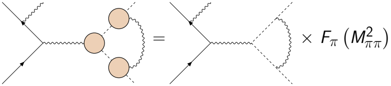
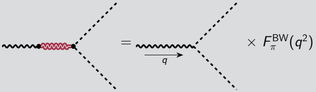


To include the pion form factor into loop integration:

GVMD

$$F_\pi^{\mathrm{BW}}(q^2) = \sum_{v=1}^{n_r} F_{\pi,v}^{\mathrm{BW}}(q^2)$$

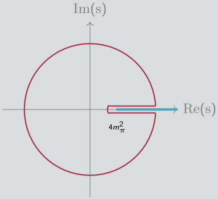
$$F_{\pi,v}^{\mathrm{BW}}(q^2) = \frac{c_v}{c_t}\frac{\Lambda_v^2}{\Lambda_v^2 - q^2}$$



FsQED

$$F_\pi(q^2) = 1 - \frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\frac{\mathrm{Im}F_\pi(s')}{q^2 - s'}$$

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\mathrm{Im}F_\pi(s') = 1$$



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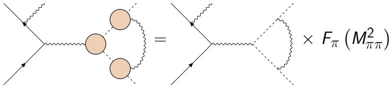

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2603.28621

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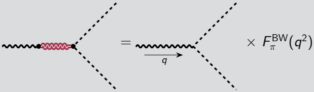

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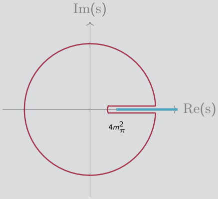

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2601.19530

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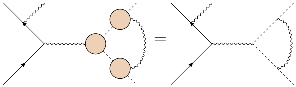

$\times\ F_\pi\left(M^2_{\pi\pi}\right)$

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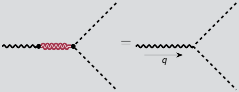

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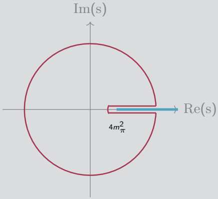


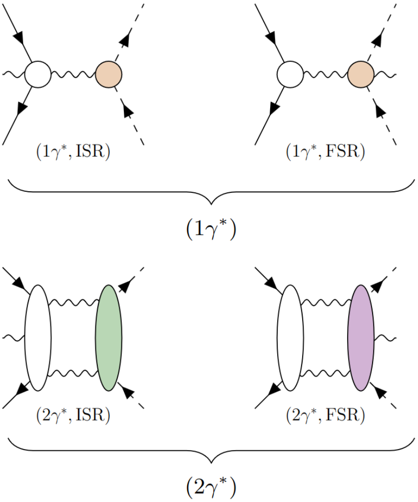
GVMD

2603.28621

FsQED

W.I.P.

Subsets of GVMD corrections



Scenarios of arxiv:2410.22882

(a) KLOE I Large-Angle (LA)	(b) KLOE II Small-Angle (SA)
$\sqrt{s} = 1.02$ GeV	$\sqrt{s} = 1.02$ GeV
$50^\circ \leq \theta^\pm \leq 130^\circ$	$50^\circ \leq \theta^\pm \leq 130^\circ$
$p_z^\pm \geq 90$ MeV $\vee$ $p_\perp^\pm \geq 160$ MeV	$p_z^\pm \geq 90$ MeV $\vee$ $p_\perp^\pm \geq 160$ MeV
$50^\circ \leq \theta_\gamma \leq 130^\circ \wedge E_\gamma \geq 20$ MeV	$\theta_{\tilde{\gamma}} \leq 15^\circ \vee \theta_{\tilde{\gamma}} \geq 165^\circ$
0.1 GeV$^2 \leq M_{XX}^2 \leq 0.85$ GeV^2	0.35 GeV$^2 \leq M_{XX}^2 \leq 0.95$ GeV^2
(c) BES III	**(d) B**
$\sqrt{s} = 4$ GeV	$\sqrt{s} = 10$ GeV
$\lvert\cos\theta^\pm\rvert \leq 0.93 \wedge p_\perp^\pm \geq 300$ MeV	0.65 rad $\leq \theta^\pm \leq 2.75$ rad $\wedge$ $p^\pm \geq 1$ GeV
$\left[\lvert\cos\theta_\gamma\rvert \leq 0.8 \wedge E_\gamma \geq 25 \text{ MeV}\right]$	0.6 rad $\leq \theta_\gamma \leq 2.7$ rad $\wedge$ $E_\gamma \geq 3$ GeV
$\vee \left[0.86 \leq \lvert\cos\theta_\gamma\rvert \leq 0.92 \wedge E_\gamma \geq 50 \text{ MeV}\right]$	$\theta_{\tilde{\gamma},\gamma^{(h)}} \leq 0.3$ rad
$\exists!\,\gamma : \ E_\gamma \geq 400$ MeV	$M_{XX\gamma} \geq 8$ GeV

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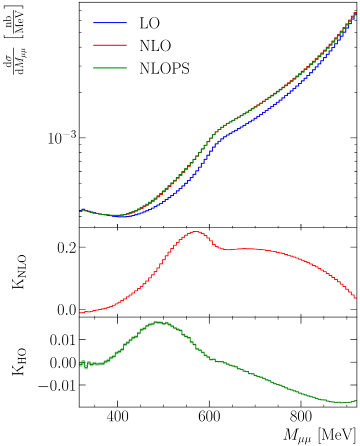
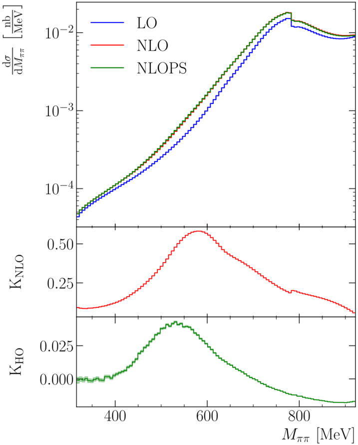
$M_{\mu\mu}$, $M_{\pi\pi}$ - F×sQED

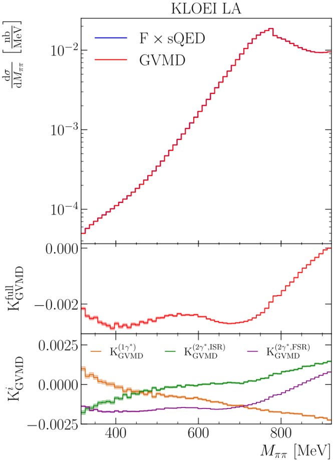
$$K_{\rm NLO} = \frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{d\sigma_{\rm LO}} \qquad K_{\rm HO} = \frac{d\sigma_{\rm NLOPS} - d\sigma_{\rm NLO}}{d\sigma_{\rm NLO}}$$

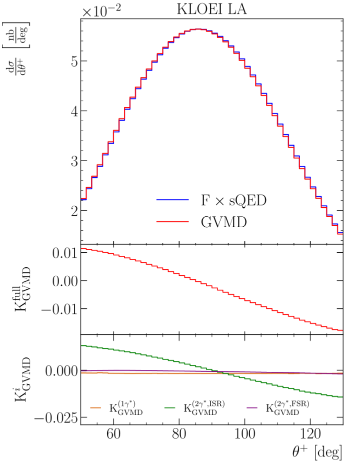




$M_{\pi\pi}, \theta_{\pi^+}$ - GVMD

$$K_{\mathrm{GVMD}} = \frac{d\sigma_{\mathrm{GVMD}} - d\sigma_{F \times sQED}}{d\sigma_{F \times sQED}}$$





KLOE-LA asymmetry

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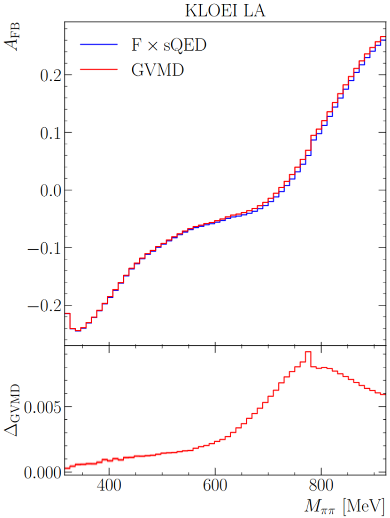
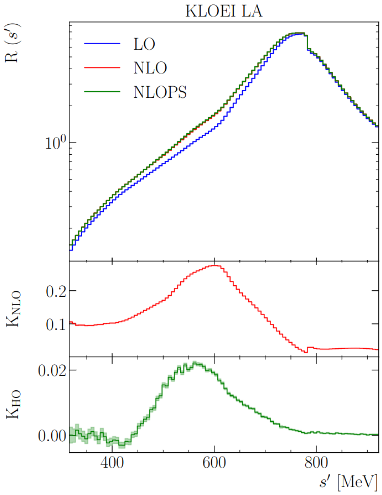


$$A_{FB}(M_{\pi\pi}) = \frac{d\sigma_F - d\sigma_B}{d\sigma_F + d\sigma_B},$$

where

$$d\sigma_F = \int_0^1 \frac{d\sigma}{dM_{\pi\pi}\ d\cos\theta^+} d\cos\theta^+,$$
$$d\sigma_B = \int_{-1}^0 \frac{d\sigma}{dM_{\pi\pi}\ d\cos\theta^+} d\cos\theta^+.$$

KLOE-LA R-ratio



$$R\left(s'\right) = \frac{d\sigma(e^+e^- \to \pi^+\pi^-\gamma)/ds'}{d\sigma(e^+e^- \to \mu^+\mu^-\gamma)/ds'}$$

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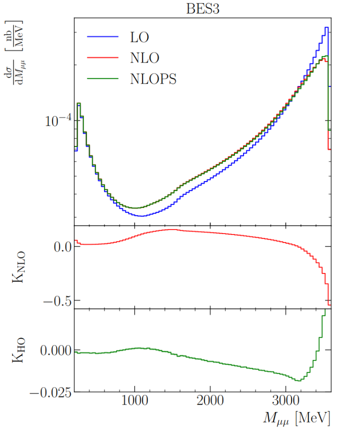
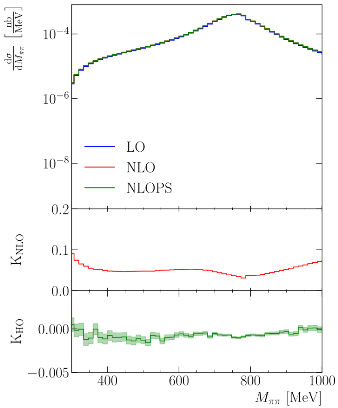
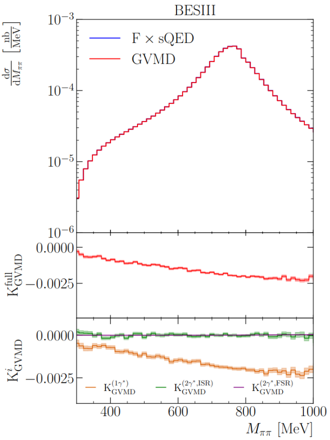
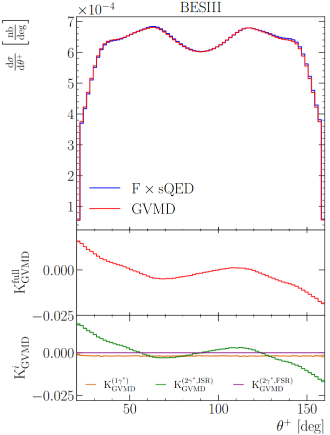
BESIII $M_{\pi\pi}$ - F×sQED



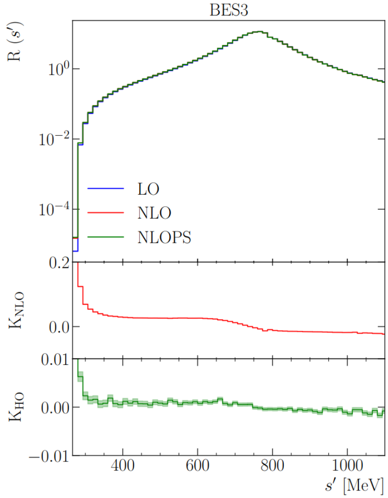


BESIII $M_{\pi\pi}, \theta_{\pi^+}$ - GVMD





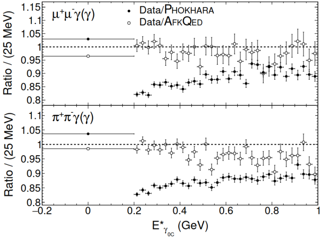
BESIII R-ratio



$$\mathrm{R}\left(s'\right) = \frac{\mathrm{d}\sigma(e^+e^- \to \pi^+\pi^-\gamma)/\mathrm{d}s'}{\mathrm{d}\sigma(e^+e^- \to \mu^+\mu^-\gamma)/\mathrm{d}s'}$$

BaBar test (arXiv:2308.05233)

- $E^*_\gamma > 4\,\mathrm{GeV}$ and $0.35 < \theta_\gamma < 2.4$ rad
- $p_\perp > 0.1\,\mathrm{GeV}$ and $0.4 < \theta_\pm < 2.45$ rad
- $M_{\mu\mu} < 1.4\,\mathrm{GeV}$ and $0.6 < M_{\pi\pi} < 0.9$ GeV
- p_{0c} missing momentum with $E^*_{0c} > 200$ MeV



BaBar test (arXiv:2308.05233)

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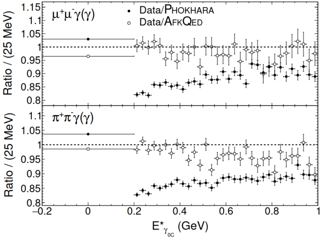
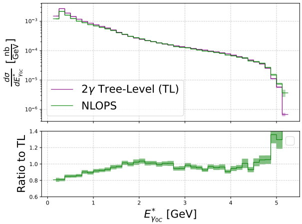
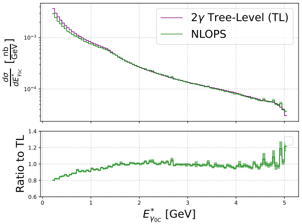


$\pi\pi\gamma$



$\mu\mu\gamma$



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- Future developments
  - Complete $\pi\pi\gamma$ with FsQED.
  - $ee\gamma$ at NLOPS accuracy.
  - Extension to other hadronic channels ($K^+K^- \dots$)

Thank you !

$M_{\pi\pi}$

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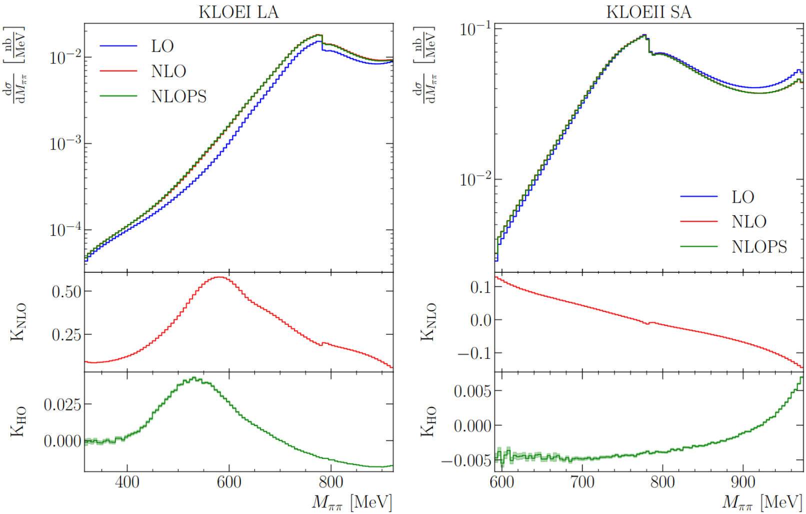
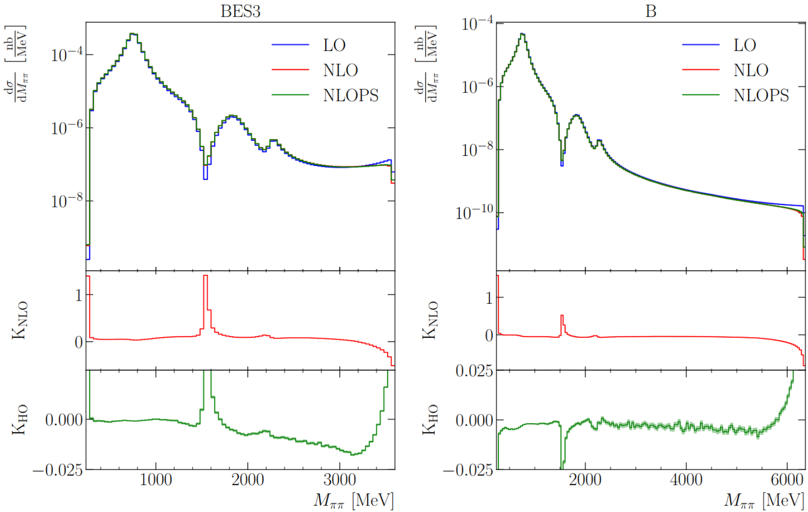
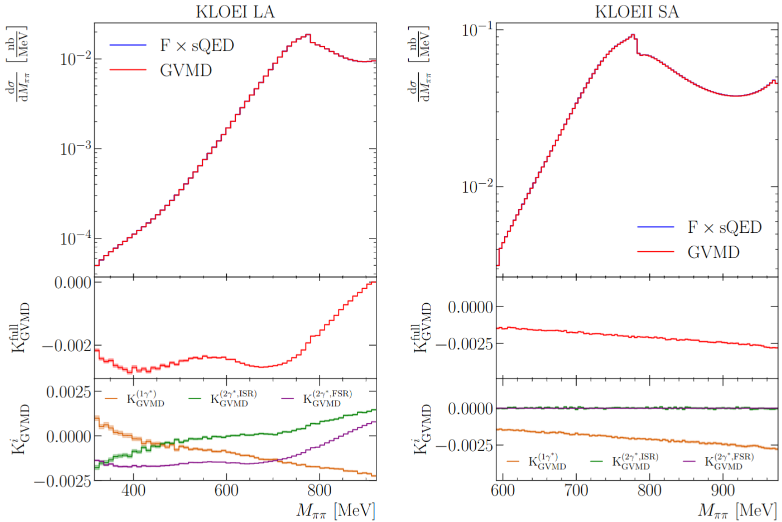


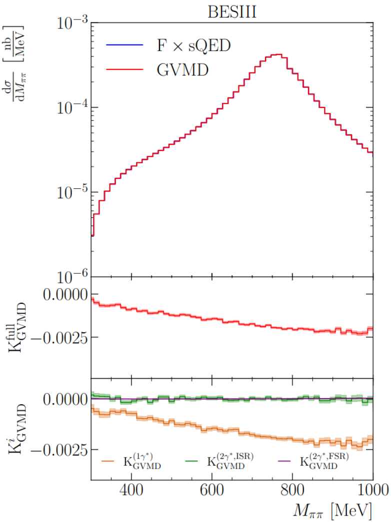
$M_{\pi\pi}$

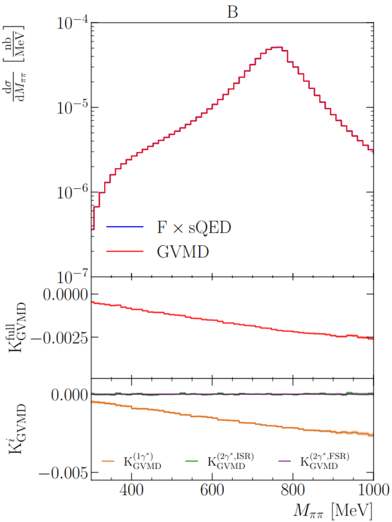

BES3
B
LO
NLO
NLOPS
$\frac{d\sigma}{dM_{\pi\pi}}\left[\frac{nb}{MeV}\right]$
K_{NLO}
K_{HO}
$M_{\pi\pi}$ [MeV]

$M_{\pi\pi}$

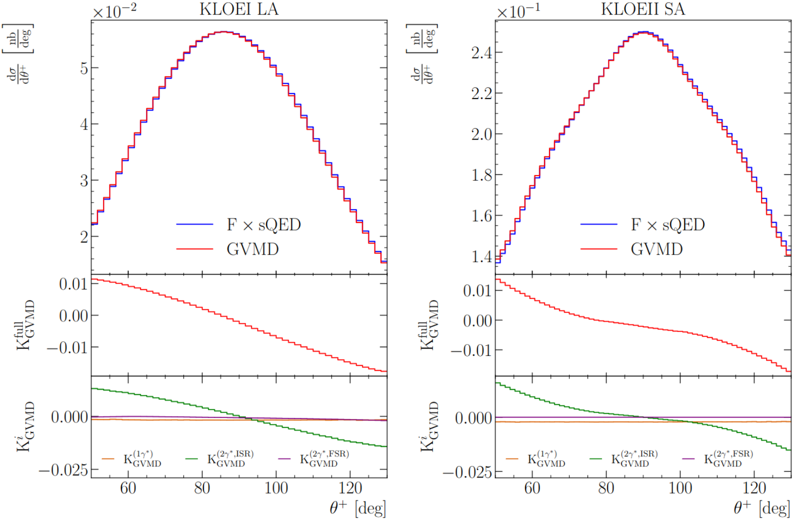

KLOEI LA
F × sQED
GVMD
10−2
10−3
10−4
0.000
−0.002
0.0025
0.0000
−0.0025
400
600
800
M ππ [MeV]
KLOEII SA
10−1
10−2
0.0000
−0.0025
600
700
800
900
M ππ [MeV]

$M_{\pi\pi}$


BESIII
B
F × sQED
GVMD
$\frac{d\sigma}{dM_{\pi\pi}}$ [nb/MeV]
K^{full}_{GVMD}
K^{i}_{GVMD}
$K^{(1\gamma^*)}_{GVMD}$
$K^{(2\gamma^*,ISR)}_{GVMD}$
$K^{(2\gamma^*,FSR)}_{GVMD}$
$M_{\pi\pi}$ [MeV]



θ_{π^+}



θ_{π^+}