$R(s)$ below charm threshold: theory and tensions with e^+e^- data

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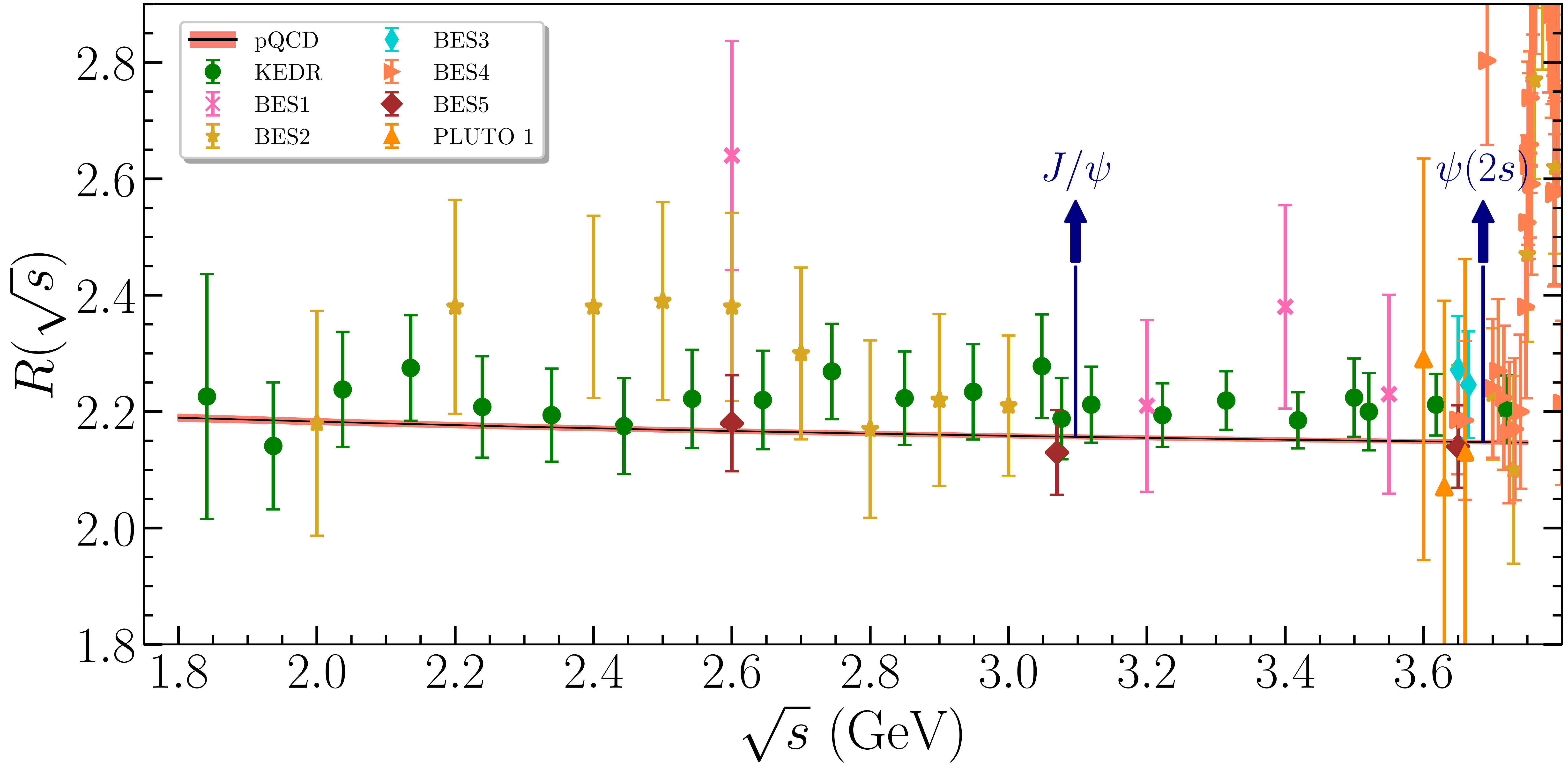
With: Marcelle Caram

2509.12956 DB, M. Caram, *Phys.Rev. D* **112** 9, 094052 (2025)

June 8, 2026

$R_{uds}(s)$ data and perturbative QCD

Inclusive data for *R(s)* below open charm (R_{uds})



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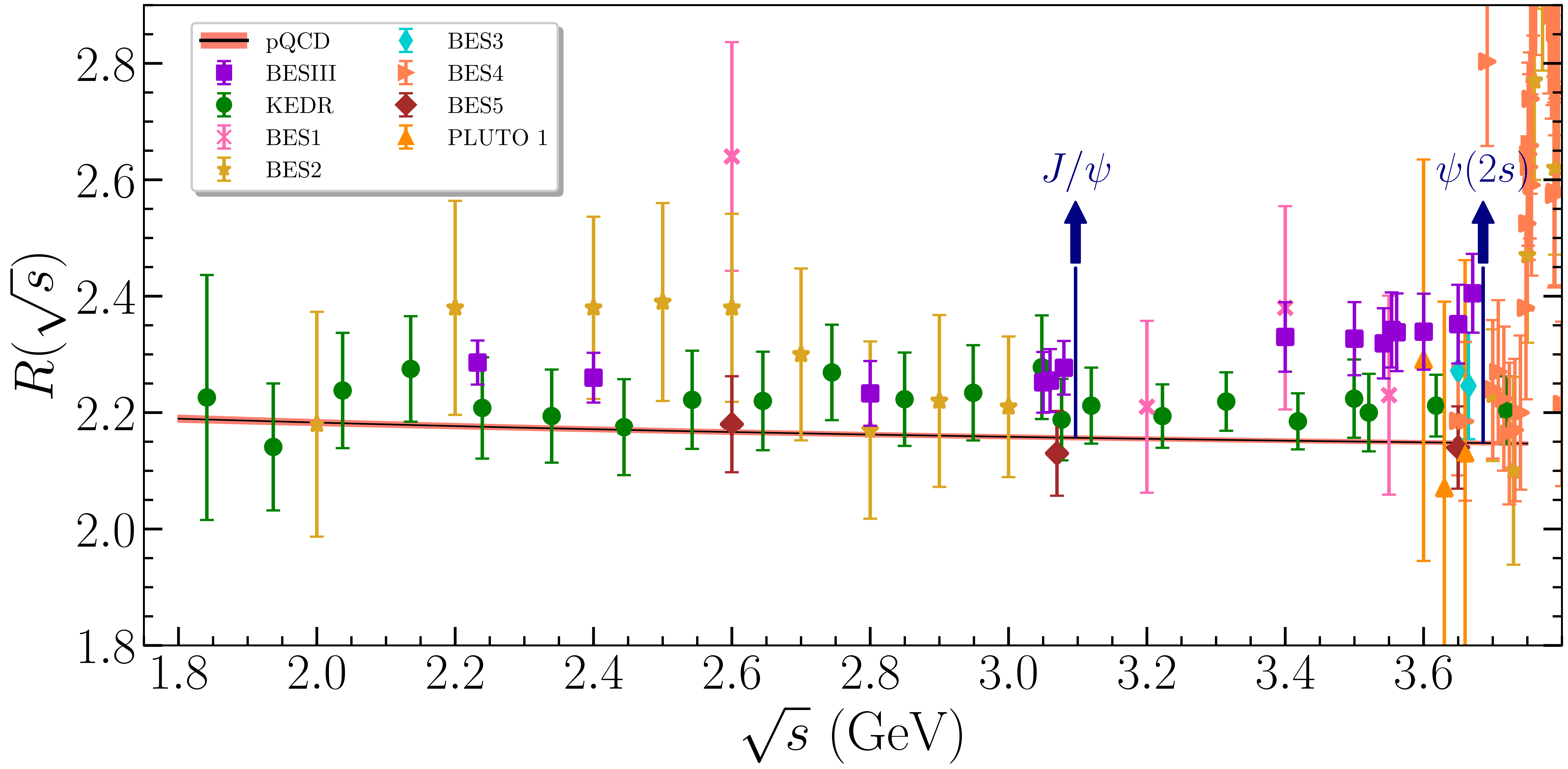

- 2021 **BES-III** results show a tension with pure pQCD. BES-III, 2112.11728, PRL (2022)

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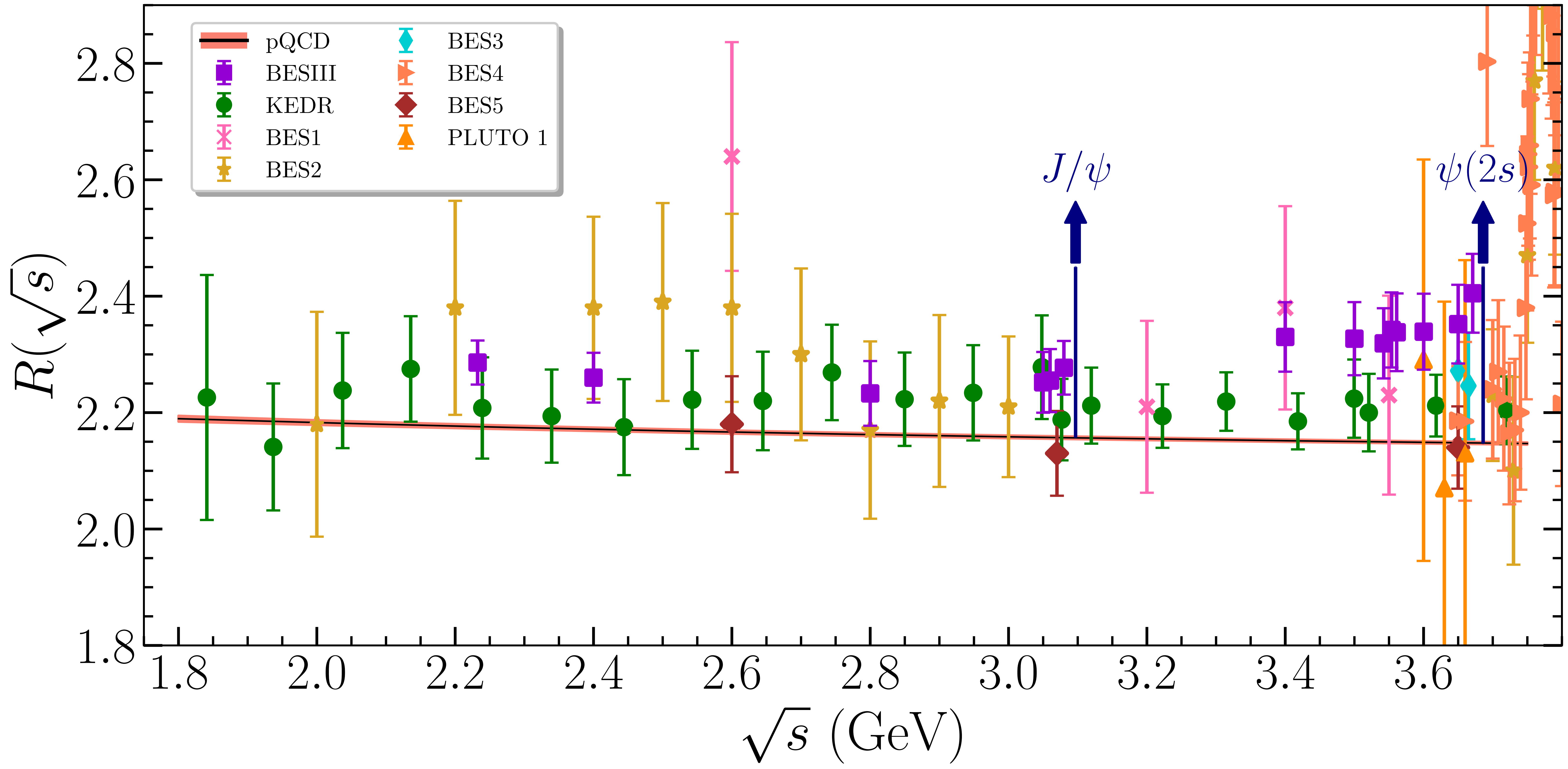

- 2021 **BES-III** results show a tension with pure pQCD. BES-III, 2112.11728, PRL (2022)
- Do the data agree with perturbative QCD (pQCD)? Possible duality violations (DV)?
- Are the data sets mutually compatible? Can the data be combined?
- Potential implications for g−2, strong coupling, and charm- and bottom-mass determinations.

R(s): theory

- Usual definition of *R*(*s*)

$$R(s) = \frac{3s}{4\pi\alpha_{\rm EM}^2}\sigma^{(0)}(e^+e^- \to \text{hadrons}(+\gamma))$$

- Correlator of two vector currents

$$\Pi^V_{\mu\nu}(q^2) = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2) = i\int d^4x\, e^{iq\cdot x}\langle 0|j^V_\mu(x) j^V_\nu(0)^\dagger|0\rangle$$

spectral function

$$\rho(s) = \frac{1}{\pi}\text{Im}\Pi(s+i0)$$

- EM current for the *R*(*s*):

$$j^{\rm EM}_\mu = Q_u\,\bar u\gamma_\mu u + Q_d\,\bar d\gamma_\mu d + Q_s\,\bar s\gamma_\mu s = \frac{1}{2}(\bar u\gamma_\mu u - \bar d\gamma_\mu d) + \frac{1}{6}(\bar u\gamma_\mu u + \bar d\gamma_\mu d - 2\bar s\gamma_\mu s)$$

I = 1 I = 0

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$I = 1$ $\qquad$ $I = 0$

- We will work with the Adler function when discussing pQCD

$$D(q^2) = -q^2\frac{d}{dq^2}\Pi(q^2) = \frac{N_c}{12\pi^2}\left(1+\widehat{D}(q^2)\right)$$

hatted quantities contain only α_s corrections

R(s): contributions

$$R_{uds}(s) = 12\pi^2 \rho_{\rm EM}(s) = N_c \sum_{q=u,d,s} Q_q^2 \left(1 + \delta_{\alpha_s}^{(0)} + \delta_{\rm DVs} + \delta_{m_q^2} + \delta_{\rm EM}\right)$$

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small OPE condensate contributions suppressed by α_s and 1/s not included

pQCD contribution

- pQCD computed with massless quarks.
- Singlet diagram contributions vanish because of the quark charges for $R_{\rm uds}$.

The spectral function can be written as an integral over the Adler function in the complex plane

$$\widehat{\rho}(s) = \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{dx}{x} \widehat{D}(sx)$$

Perturbative Adler function

$$\widehat{D}_{\rm pert}(q^2) = \sum_{n=1}^{\infty} a_s^n(\mu^2) \sum_{k=1}^{n+1} k\, c_{n,k} \left[\log\left(\frac{-q^2}{\mu^2}\right)\right]^{k-1}$$

$$a_s \equiv \alpha_s/\pi$$

- Coefficients $c_{n,1}$ exactly known up to five loops, $\mathcal{O}(\alpha_s^4)$.

  Baikov, Chetyrkin, and Kühn '08
- We will consider an estimate for the six-loop coefficient as well

$$c_{5,1} = 280 \pm 140$$

Beneke and Jamin '08
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This is a particular case of the integrated Adler function moments

$$\frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}W_m(x)\widehat{D}(sx)$$

- Extensively studied in the context of hadronic tau decays.
- There is a body of work about how these moments behave in perturbation theory, convergence issues related to the gluon-condensate IR renormalon, how to cure them etc.

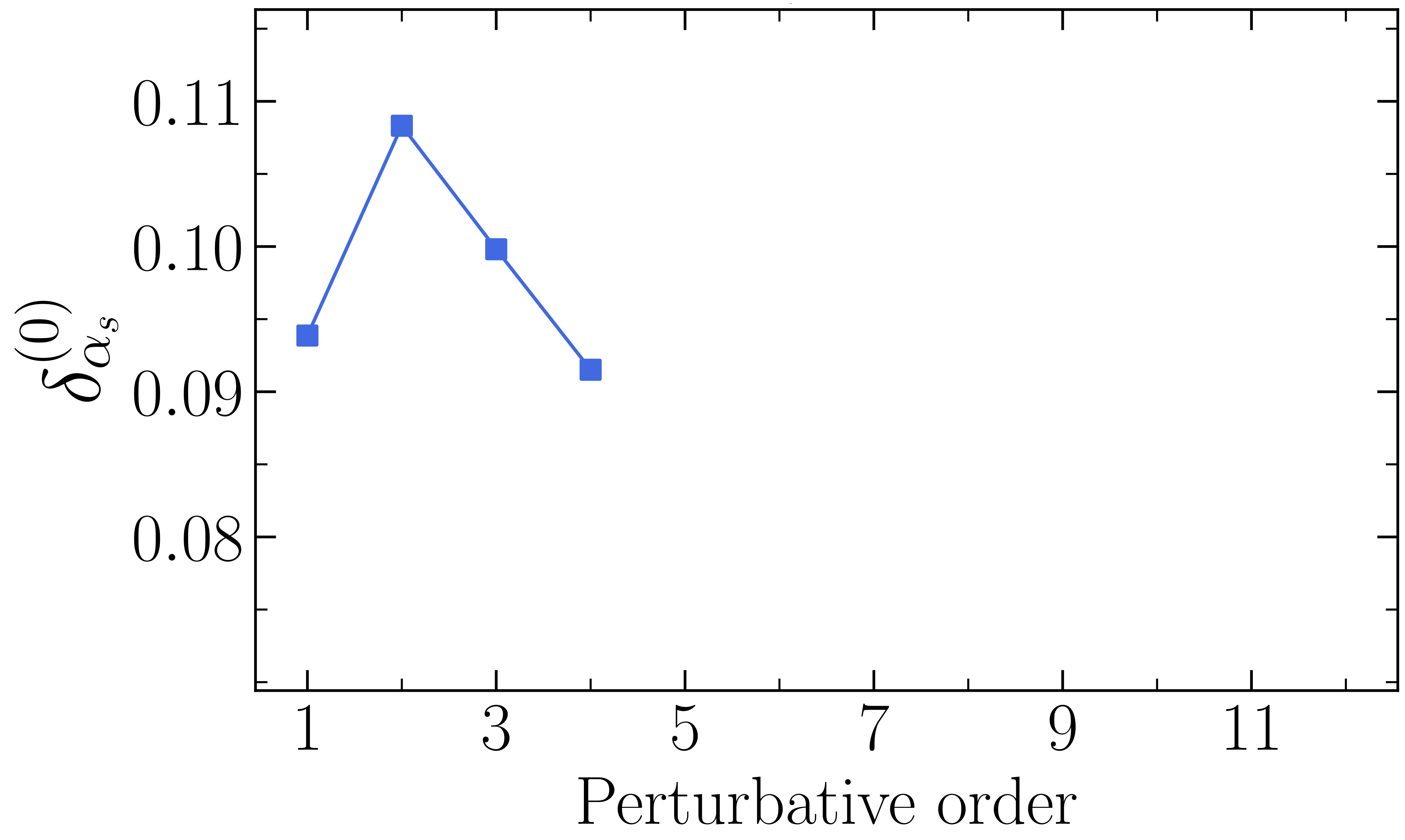
Beneke & Jamin ’08; Descotes-Genon & Malaescu ’10; DB, Beneke and Jamin ’12; DB and Oliani ’20; Hoang & Regner ’20; Benitez-Rathgeb, DB and Hoang ’22; Golterman, Maltman and Peris ’23, Gracia, Hoang and Mateu '23….

pQCD contribution

Using “standard" Fixed Order Perturbation Theory (FOPT) $(\mu^2 = s)$

$$\delta^{(0)}_{\alpha_s}(s) = a_s(s) + 1.6398\, a_s^2(s) - 10.284 a_s^3(s) - 106.88 a_s^4(s)$$

$$\delta^{(0)}_{\alpha_s}(4\ \mathrm{GeV}^2) = 0.09387 + 0.01445 - 0.008506 - 0.008298$$



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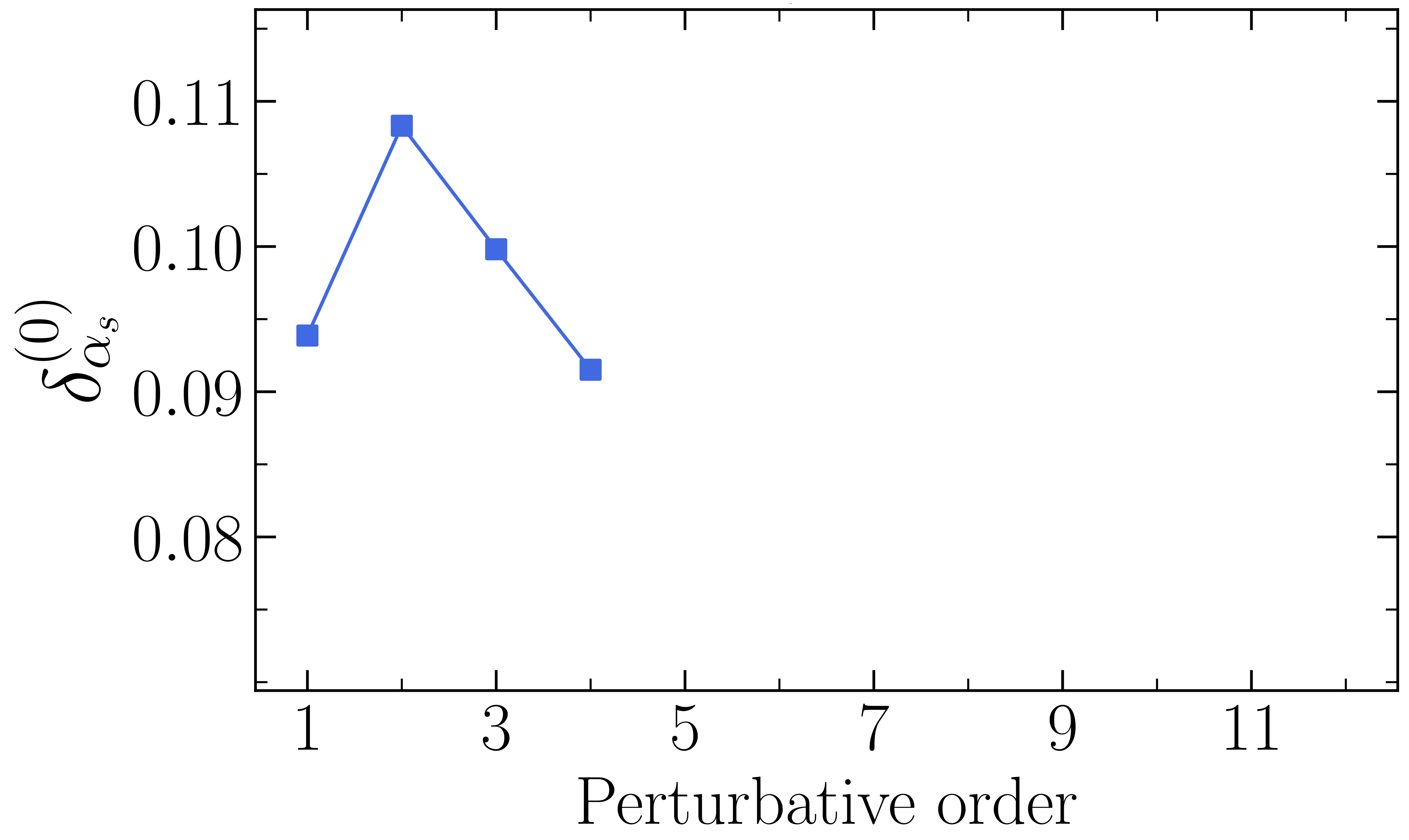

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$$\delta^{(0)}_{\alpha_s}(4\ \mathrm{GeV}^2) = 0.09387 + 0.01445 - 0.008506 - 0.008298 - 0.0036(10)_{c_{51}} + \cdots = 0.0879(21)$$

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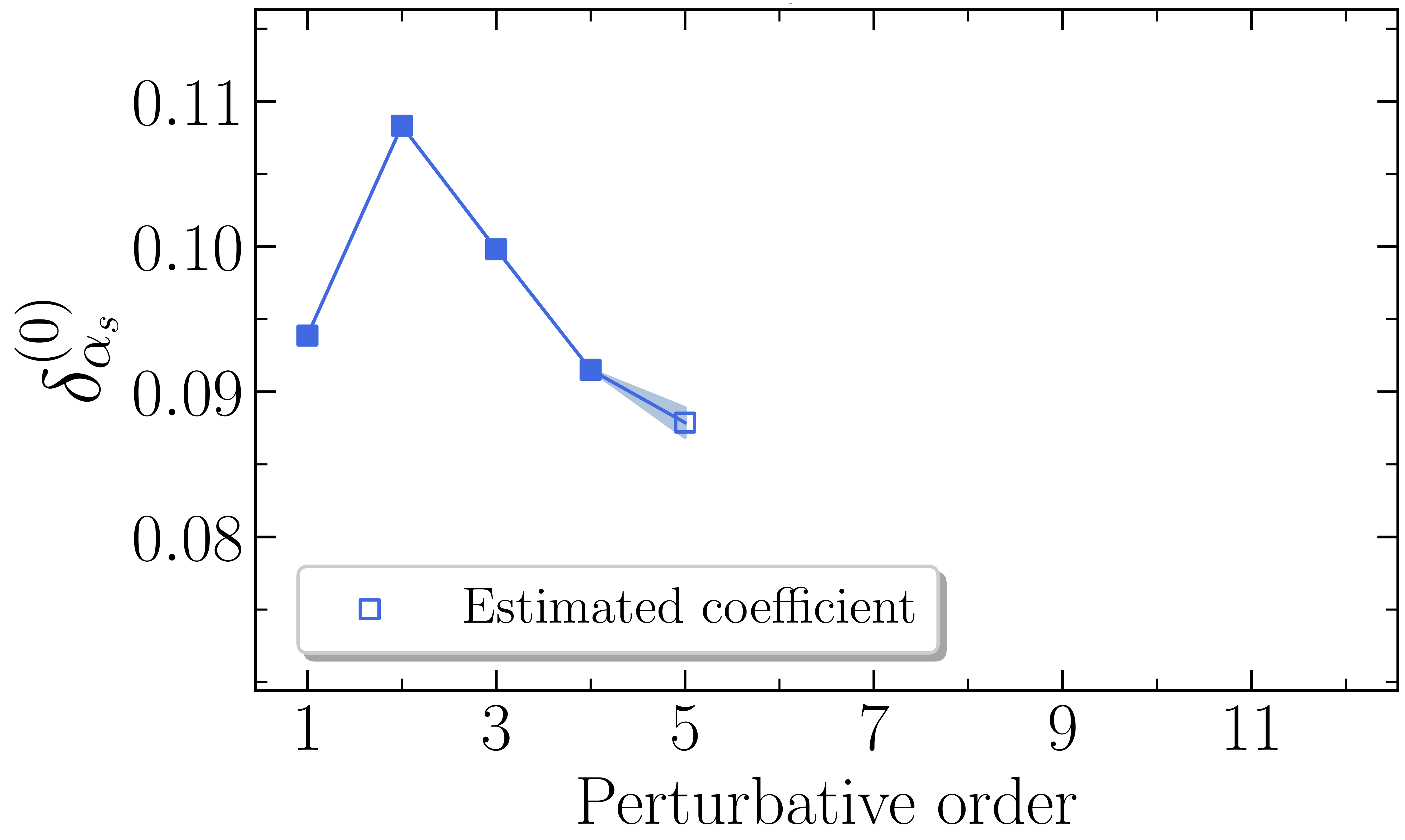

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see also Kataev & Todyshev 2603.39803

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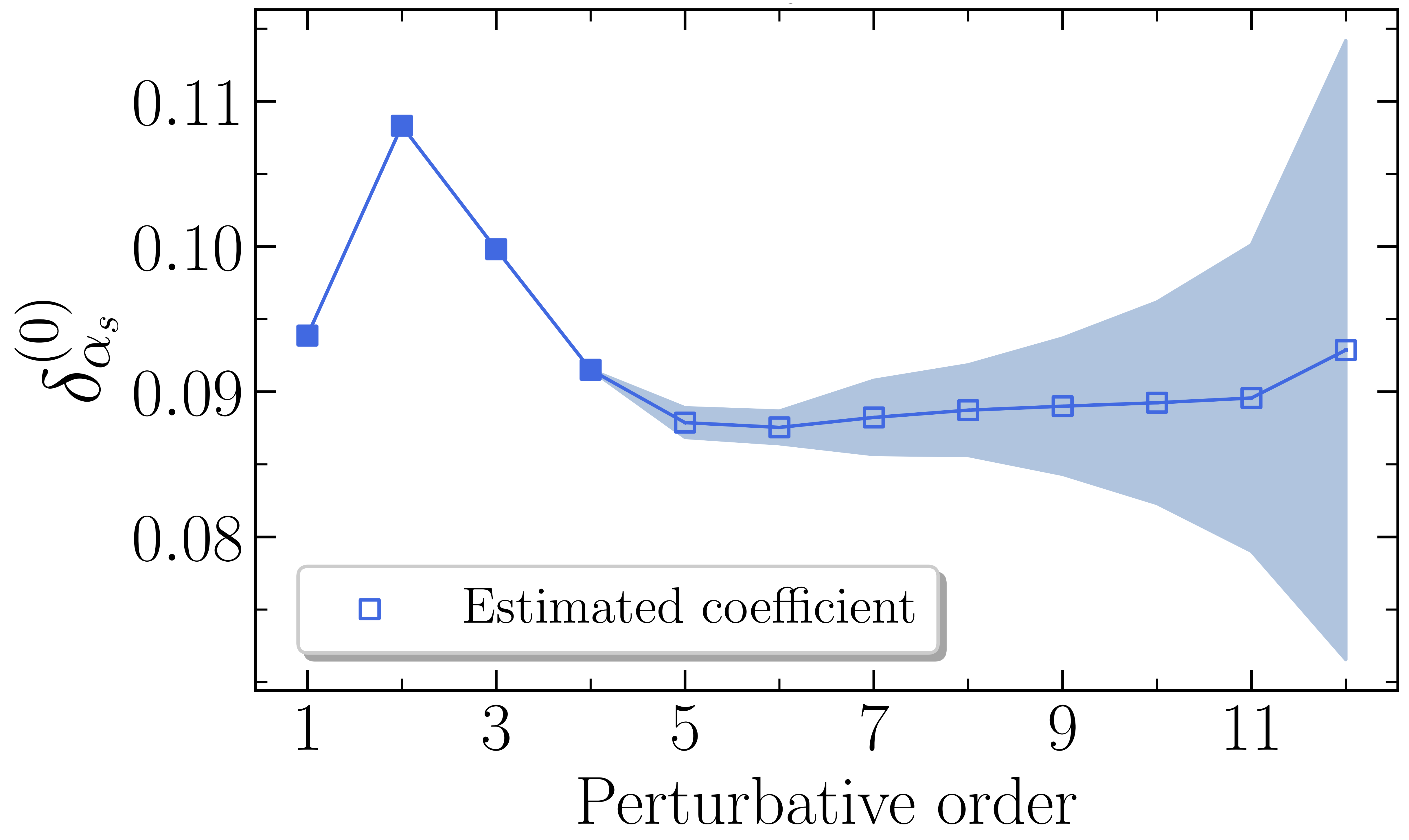
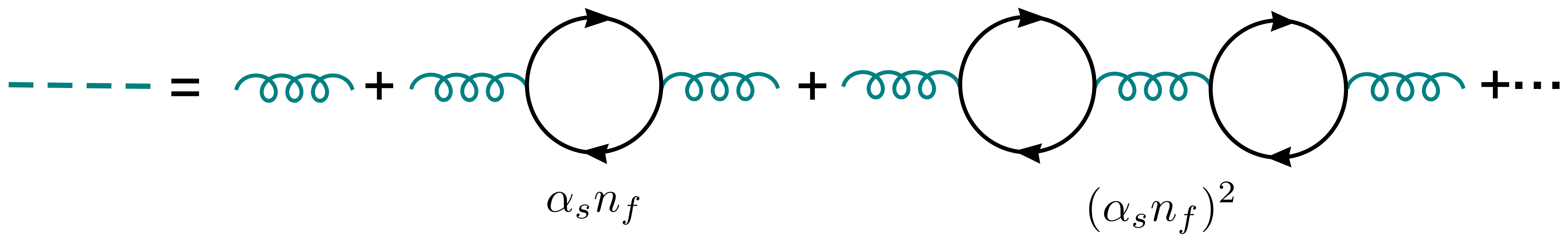


- Higher-order coefficients from a reconstruction using Padé approximants DB, Masjuan, Oliani ’18
- Other higher-order models give similar results Beneke & Jamin ’08
- Indication that the series stabilizes not too far from the result at $\mathcal{O}(\alpha_s^4)$
- Leading IR renormalon strongly suppressed (gluon condensate). No indication of a renormalon-related issue DB, Caram ’25

pQCD contribution: large-β_0

Gluon propagator with insertions of $q\bar{q}$ loops



$\alpha_s n_f \sim \mathcal{O}(1)$ $\beta_{0,f} = \frac{n_f}{6\pi}$

"Non-abelianization" of the result

$$\beta_{0,f} \rightarrow \beta_0 = \beta_{0,f} + \beta_{0,\mathrm{nA}}$$

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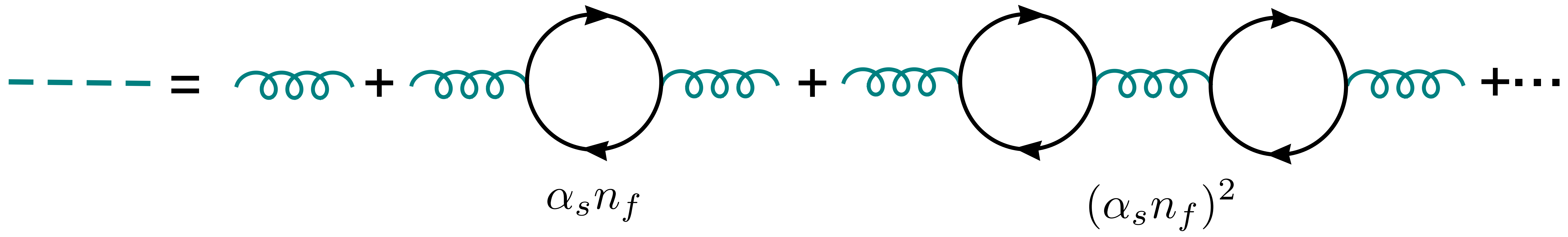

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Model for the series to all orders in pQCD

$$\widehat{D}(Q^2) = \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(Q^2) \quad \Rightarrow \quad B[\widehat{D}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}$$

Borel transform

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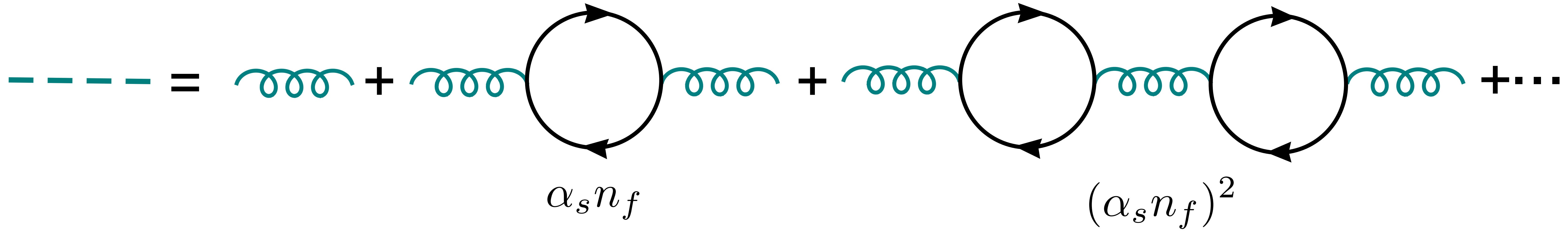

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Borel transform

Borel transform exactly known (here we know everything)

$$B[\widehat{D}_{L\beta_0}](u) = \frac{32}{3\pi}\left(\frac{Q^2}{\mu^2}\right)^{-u} \frac{\mathrm{e}^{-Cu}}{(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k k}{[k^2-(1-u)^2]^2}$$

Beneke '93
Broadhurst '93

$$\widehat{D}(\alpha) = \int_0^{\infty} dt\, e^{-t/\alpha} B[\widehat{D}](t)$$

True value of the series can be calculated from the Borel integral

pQCD contribution: large-β_0

large-β_0: $\delta^{(0)}_{\alpha_s,L\beta_0} = a_s(s) + 1.56\, a_s^2(s) - 0.944 a_s^3(s) - 52.9 a_s^4(s) - 283 a_s^5(s) + \cdots$

QCD: $\delta^{(0)}_{\alpha_s}(s) = a_s(s) + 1.6398\, a_s^2(s) - 10.284 a_s^3(s) - 106.88 a_s^4(s) - (499.6 \pm 140) a_s^5(s) + \cdots$

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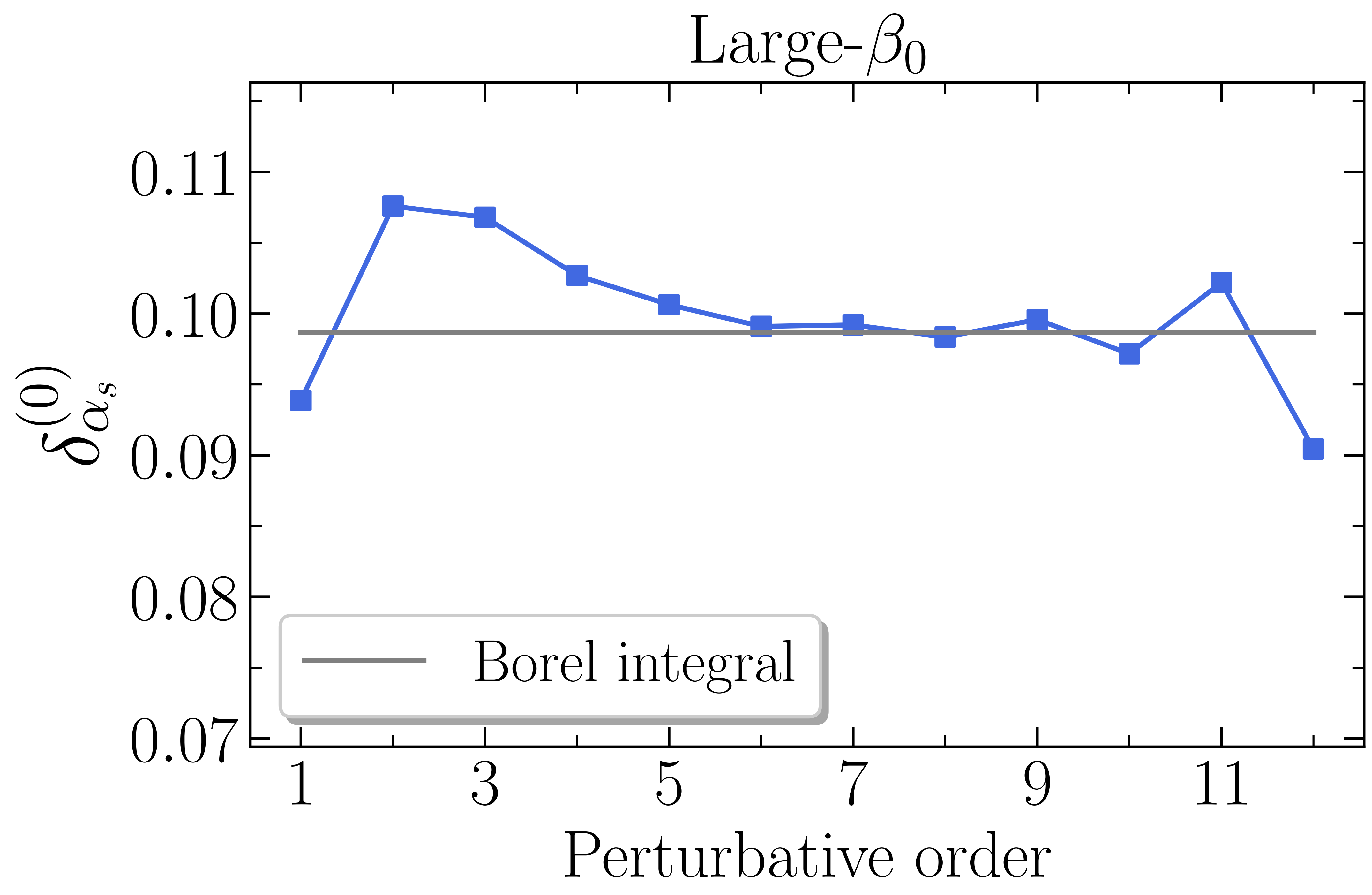

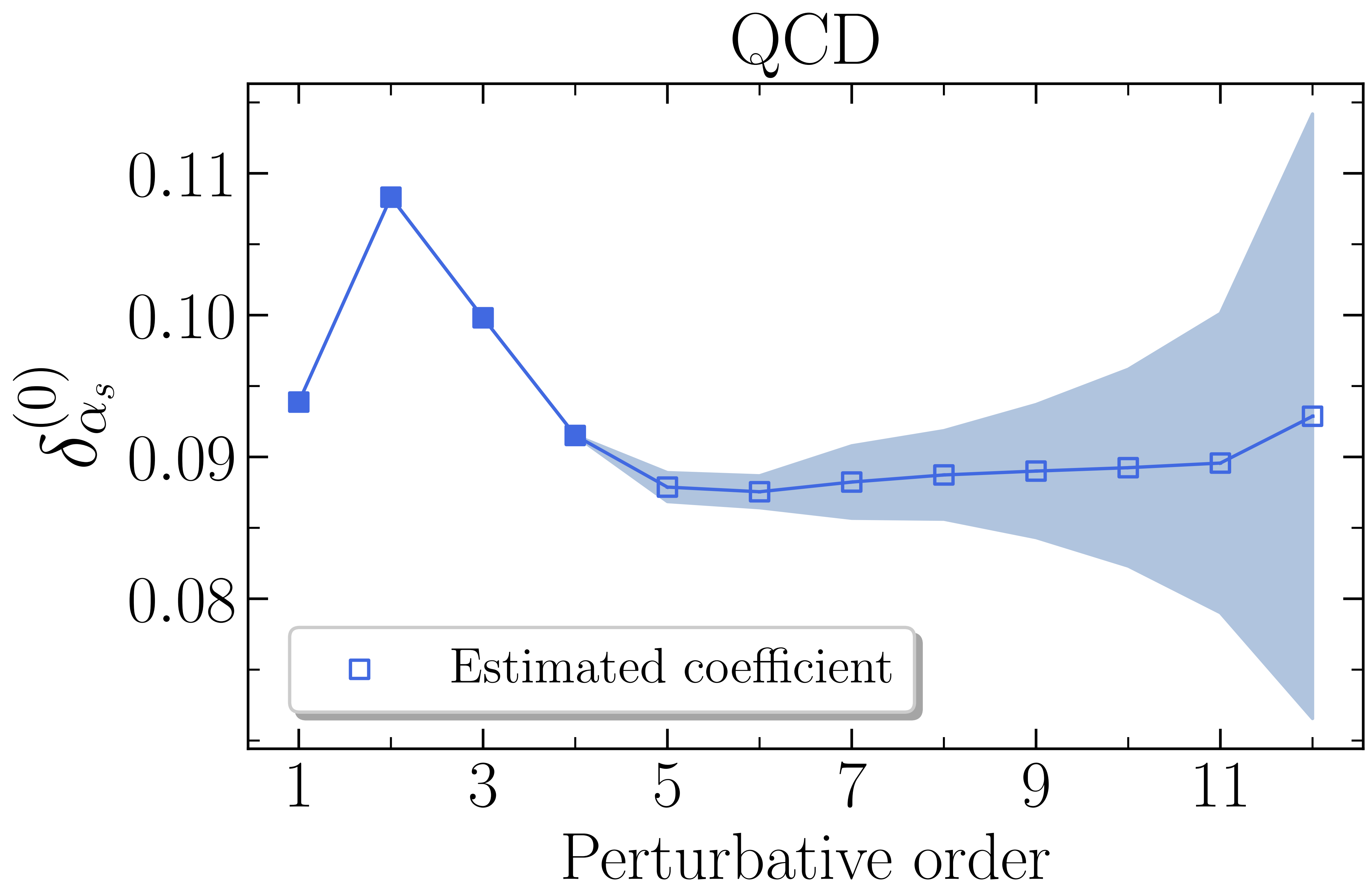

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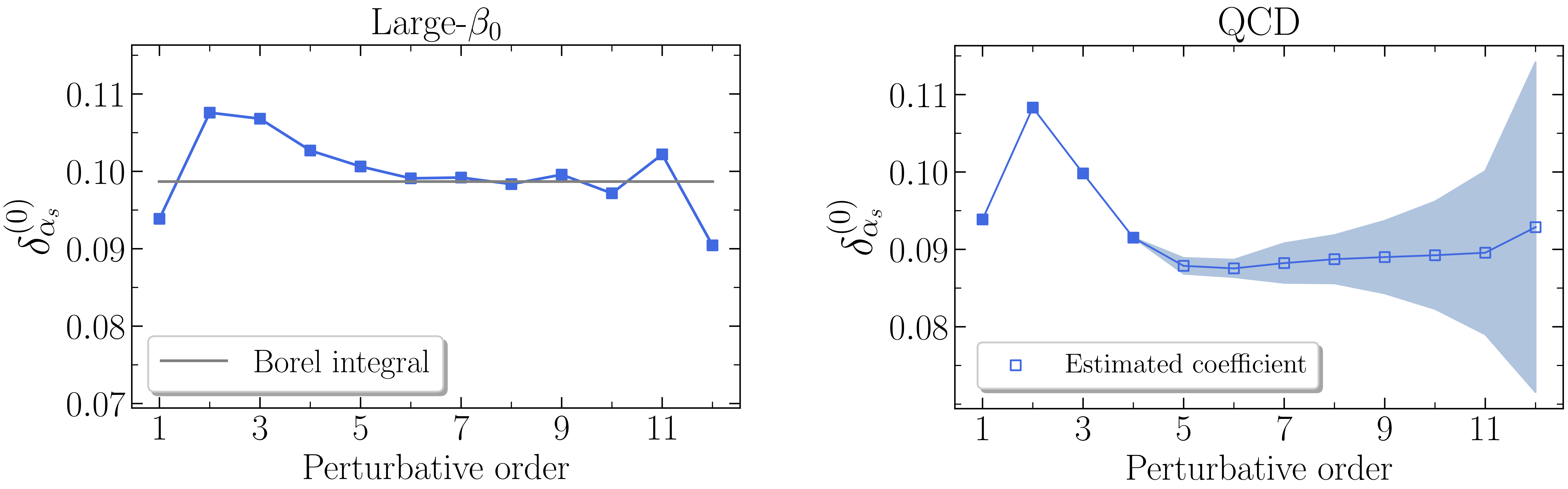

Another indication that the series is not a bad approximation provided we have at least 4 or 5 terms

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pQCD: renormalon-free gluon-condensate scheme

- Another prescription for pQCD is Contour Improved Perturbation Theory (CIPT). It is strongly affected by the leading IR renormalon: **can only be applied in schemes that consistently removes the GC renormalon**. Here, we use CIPT in the renormalon free scheme of Benitez-Rathgeb, DB, Hoang '22 (based on the conceptual work of Hoang & Regner '20)
- Results for FOPT and RF-CIPT with $s = 4$ GeV^2:

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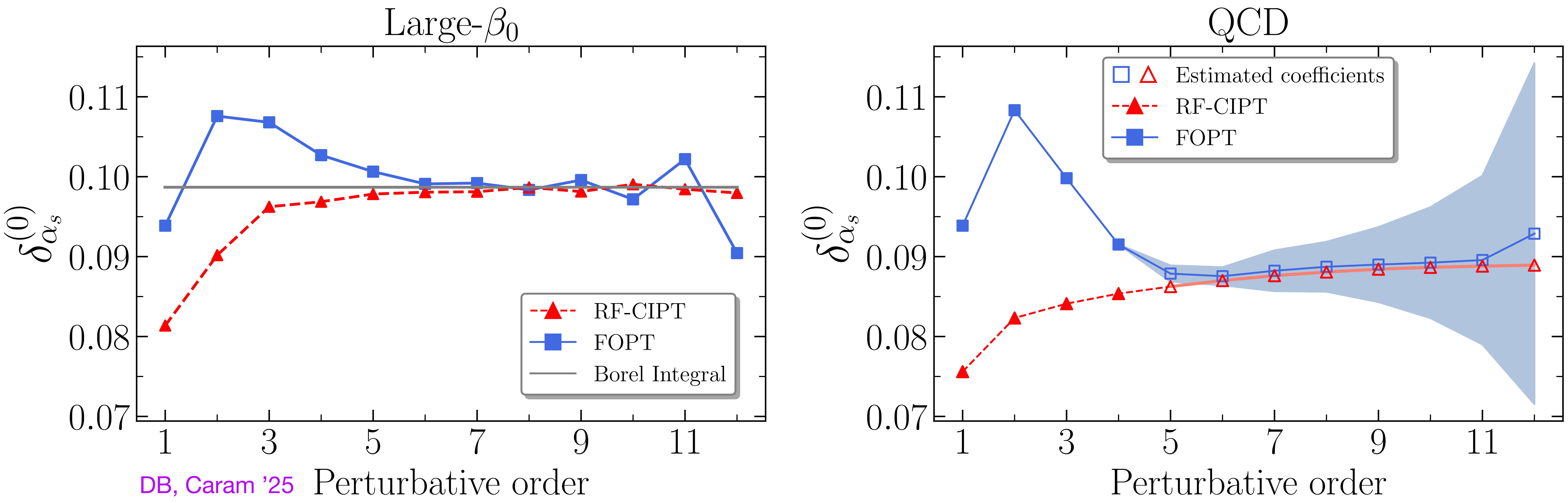


DB, Caram '25

- RF-CIPT leads to a fixed sign series that approaches the true value from below, smoothly. Good agreement with FOPT at higher orders.

3rd indication that the series is not a bad approximation provided we have at least 4 or 5 terms

Duality Violations

- pQCD is not all: beyond pQCD + OPE condensates there are potential quark-hadron duality violations (DVs)

- DVs cannot be obtained from first principles but can be parametrized based on reasonable assumptions about the QCD spectrum (asymptotic Regge trajectories and large N_c arguments)

$$\Pi(z) = \Pi_{\rm OPE}(z) + [\Pi(z) - \Pi_{\rm OPE}(z)] = \Pi_{\rm OPE}(z) + \Delta(z) \qquad \rho^{\rm DVs}_{\rm EM}(s) = \frac{1}{\pi}{\rm Im}\Delta(s)$$

- Asymptotically (large energies) one can show, given the assumptions above that (EM current, $I = 0$ and $I = 1$)

$$\rho^{\rm DVs}_{\rm EM}(s) = \underbrace{\frac{5}{9}e^{-\delta_1-\gamma_1 s}\sin(\alpha_1+\beta_1 s)}_{I\,=\,1} + \underbrace{\frac{1}{9}e^{-\delta_0-\gamma_0 s}\sin(\alpha_0+\beta_0 s)}_{I\,=\,0}$$

Catà, Golterman and Peris ’05, ’08
DB, Caprini, Golterman, Maltman and Peris ’17
DB, Golterman, Maltman and Peris ’18
also used in, e.g,
Benton, DB, Keshavarzi, Maltman and Peris ’23, ’24, ’25

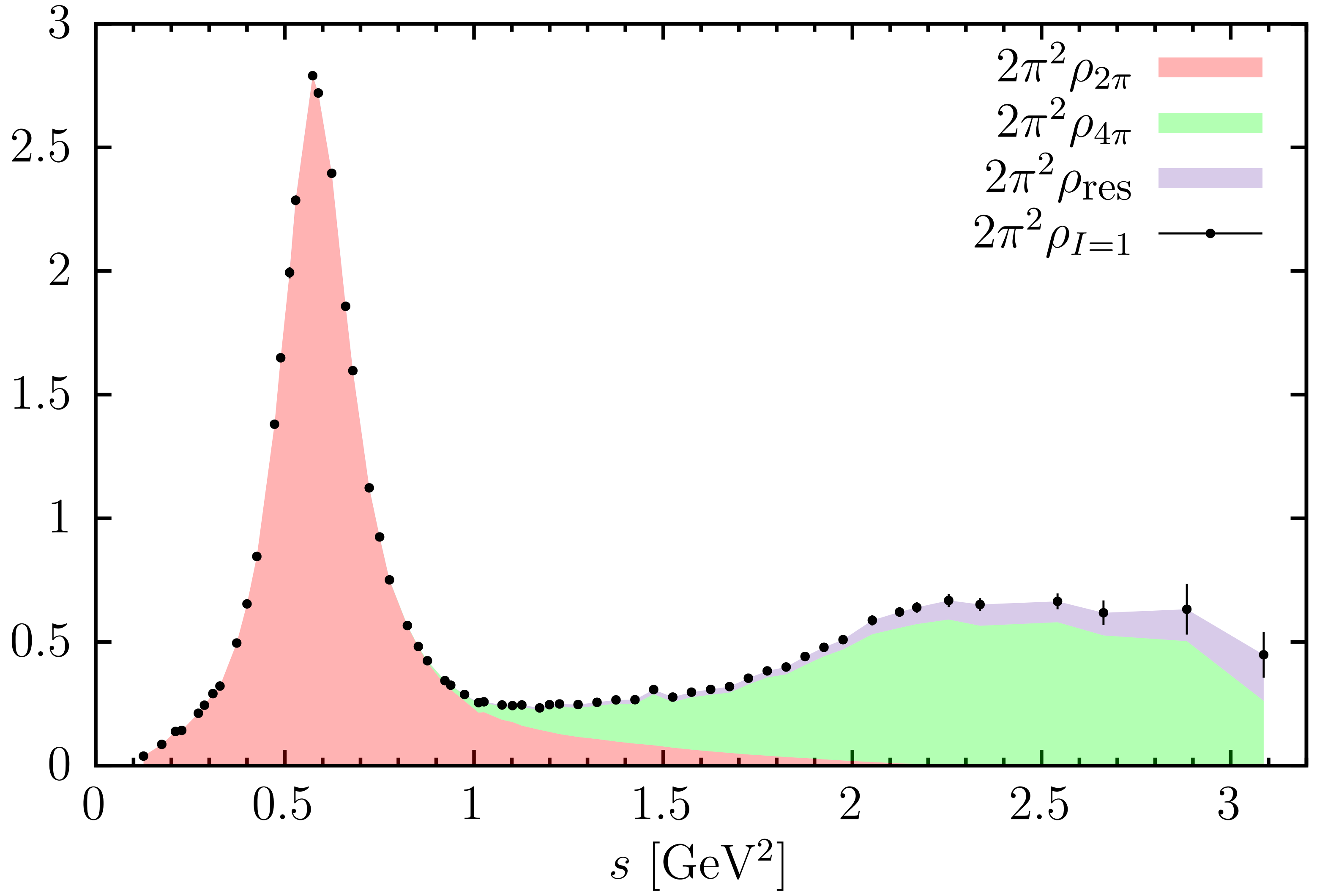
- There are four parameters *per channel*, because DVs are associated with resonances.

- Additional assumption: $\gamma_0 = \gamma_1$ and $\beta_0 = \beta_1$. All 6 parameters fixed with external information.

- $I = 1$ parameters can be fixed from fits to tau decay data while $I = 0$ can be fixed from fits to exclusive *R*(s) data.

Duality Violations: parameters

Isospin 1



I=1 parameters

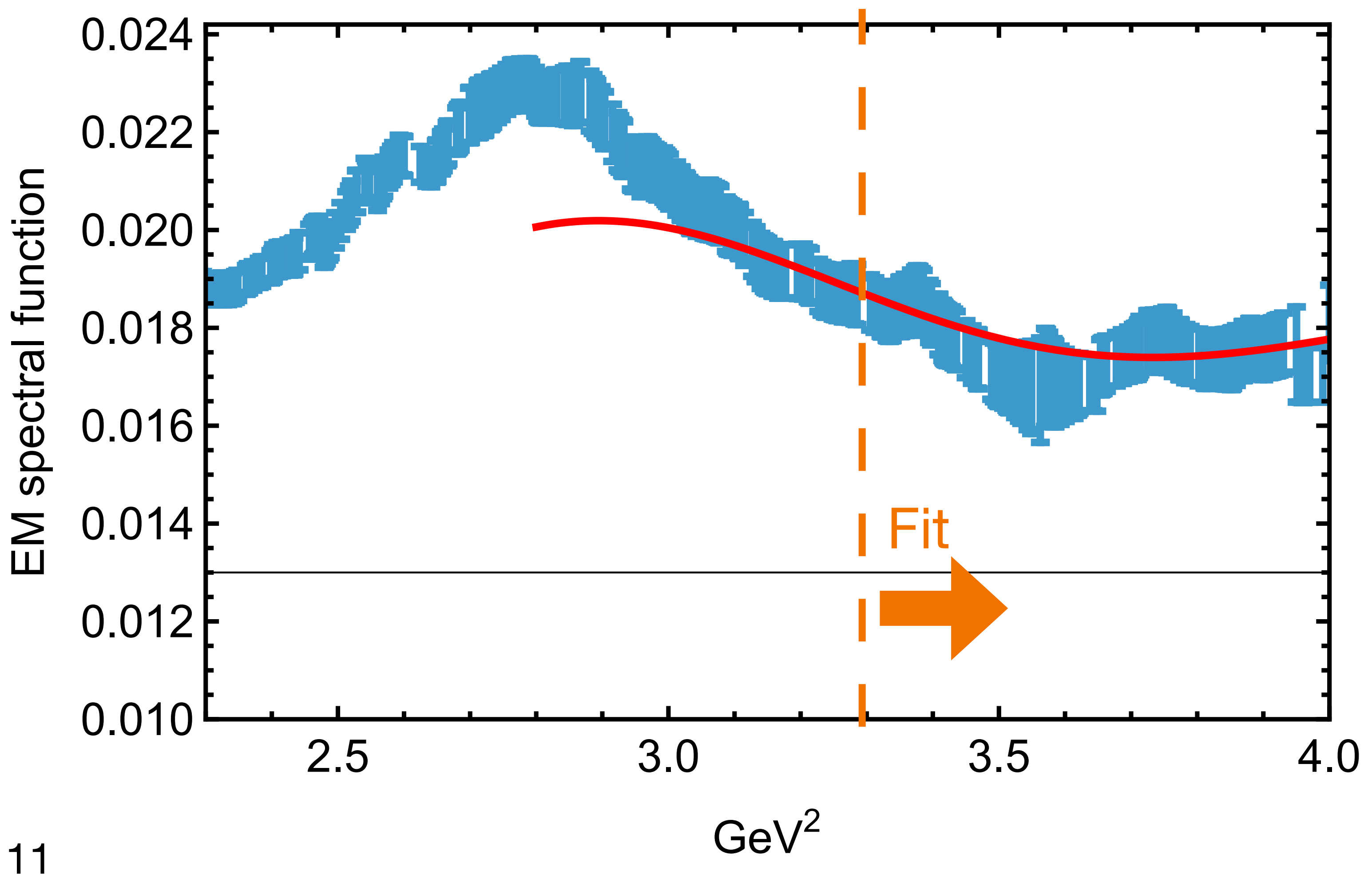
$$\delta_1 = 3.01(39)$$
$$\gamma_1 = 0.87(24)\ \text{GeV}^{-2}$$
$$\alpha_1 = -1.34(73)$$
$$\beta_1 = 3.78(38)\ \text{GeV}^{-2}$$

- Fits to sum rules from our most recent inclusive vector-isovector spectral function

DB, A. Eiben, M. Golterman, K. Maltman, L. Mansur, and S. Peris, 2502.08147, Phys. Rev D (2025)

(see talk by Lucas Mansur in the next session)

Isospin 0



I=0 parameters

$$\delta_0 = 0.96(22)$$
$$\alpha_0 = 0.80(27)$$

- Updated fit to the *exclusive* $R_{uds}(s)$ data from the KNT compilation

DB, Golterman, Keshavarzi, Maltman, Nomura, Peris and Teubner, '18

Full $R_{uds}(s)$

$$R_{uds}(s) = 12\pi^2 \rho_{\rm EM}(s) = N_c \sum_{q=u,d,s} Q_q^2 \left(1 + \delta^{(0)}_{\alpha_s} + \delta_{\rm DVs} + \delta_{m_q^2} + \delta_{\rm EM}\right)$$

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$\delta_{\rm DVs}$:

$$\delta_{\rm DVs} = 6\pi^2 \left(\frac{5}{9} e^{-\delta_1 - \gamma_1 s} \sin(\alpha_1 + \beta_1 s) + \frac{1}{9} e^{-\delta_0 - \gamma_0 s} \sin(\alpha_0 + \beta_0 s)\right)$$

$\delta_{m_q^2}$:

$$\delta_{m_q^2} = \frac{m_s^2(s)}{s} \left(1 + 2 a_s + \frac{227}{12} a_s^2 + \cdots\right)$$

$\delta_{\rm EM}$:

$$\delta_{\rm EM} = \frac{\alpha_{\rm EM}}{4\pi}$$

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Full $R_{uds}(s)$

$\sqrt{s}$ (GeV)	R_{uds}	$\delta^{(0)}_{\alpha_s}$	$\delta_{\rm DVs}$	$\delta_{m_q^2}$	$\delta_{\rm EM}$
2.0	2.12(14)	0.0879(21)	−0.030(69)	0.002960(56)	0.00058
2.5	2.181(19)	0.0822(15)	0.0060(93)	0.001621(28)	0.00058
3.0	2.1576(46)	0.0776(12)	−0.0004(19)	0.001009(17)	0.00058
3.5	2.1502(22)	0.0739(11)	−0.00003(13)	0.000683(12)	0.00058

DB, Caram ’25

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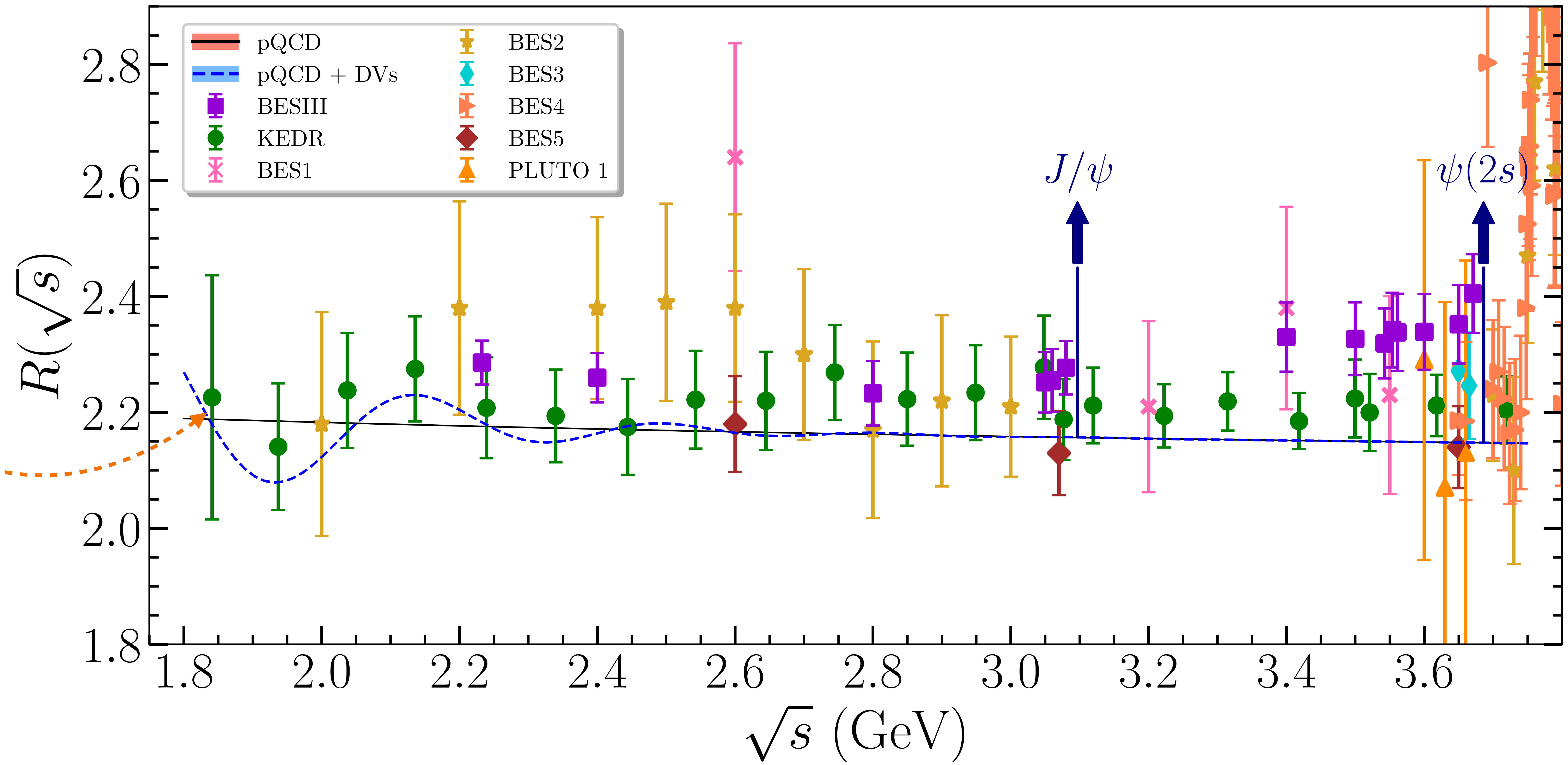
DB, Caram '25

The description with DVs (not a fit!) follows the oscillations observed in the KEDR data for lower energies



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DB, Caram '25

But with very larger errors…

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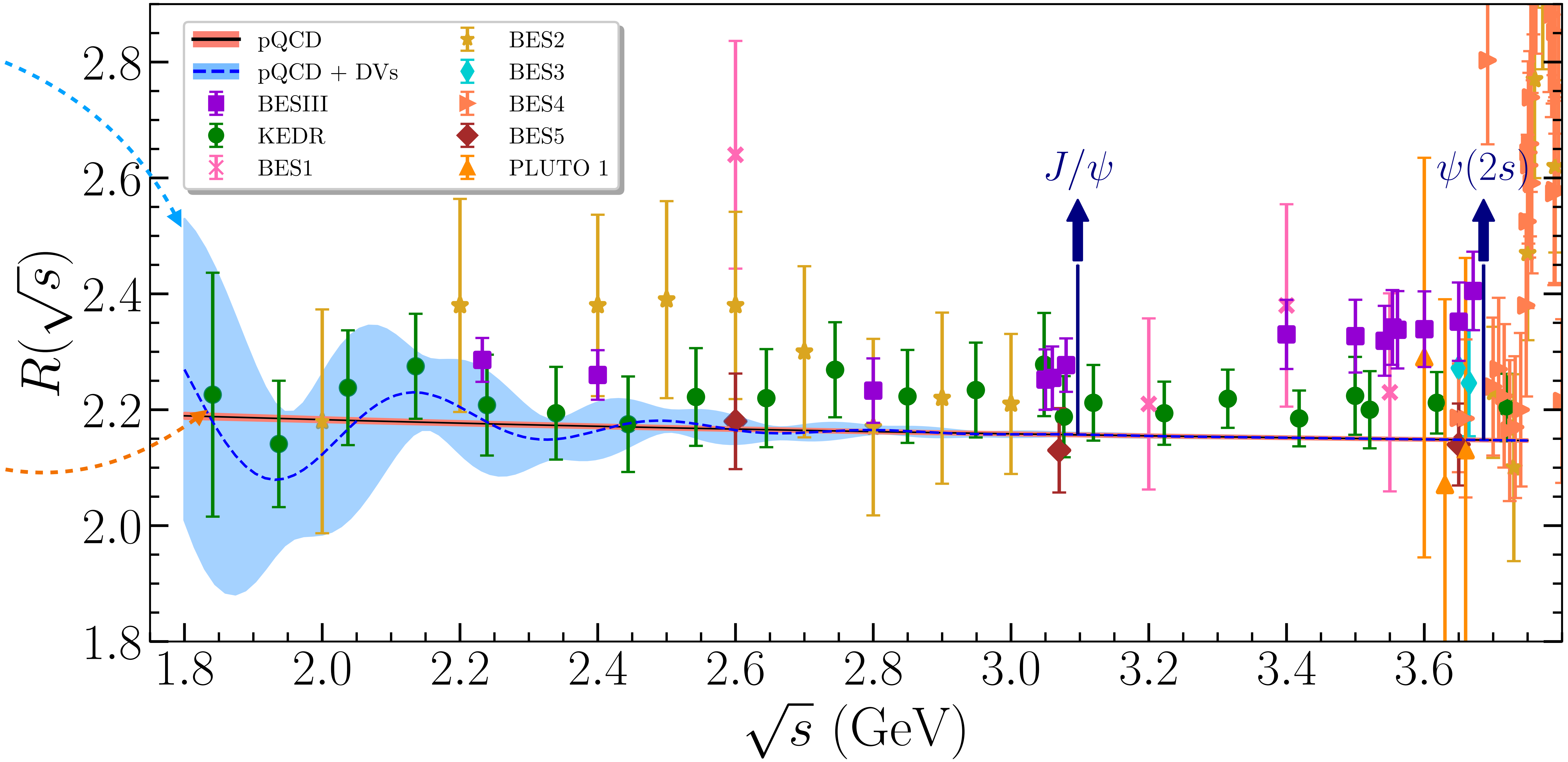

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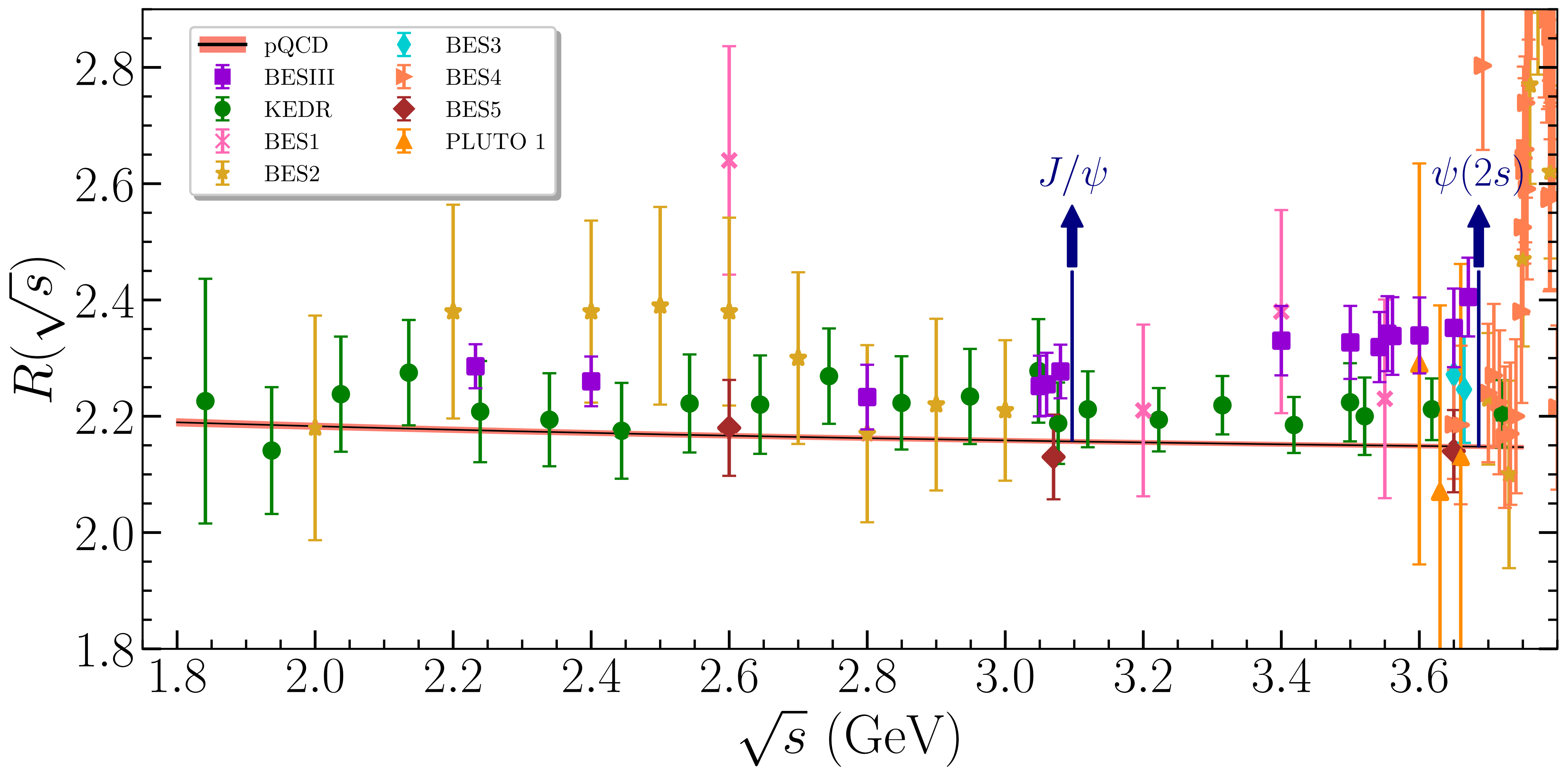
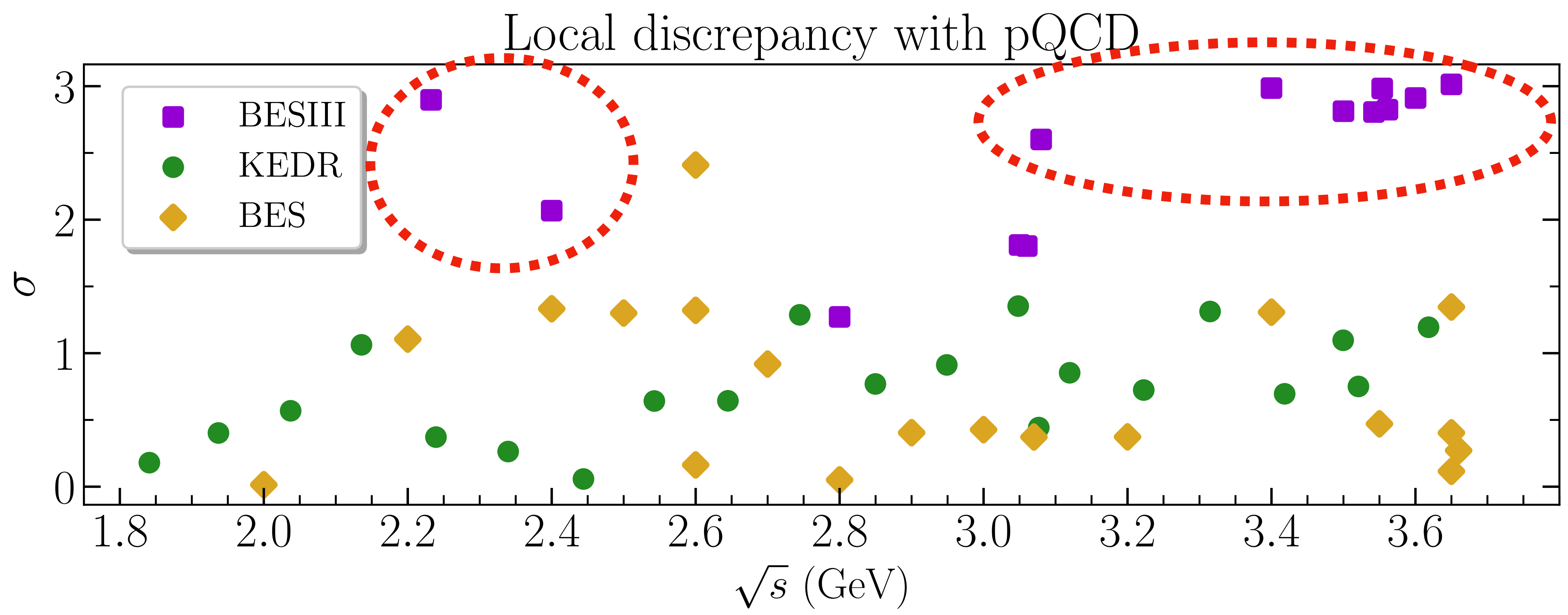
Local discrepancies

Discrepancies with respect to pQCD





DB, Caram '25

(all results include m_s and EM corrections)

Local discrepancies

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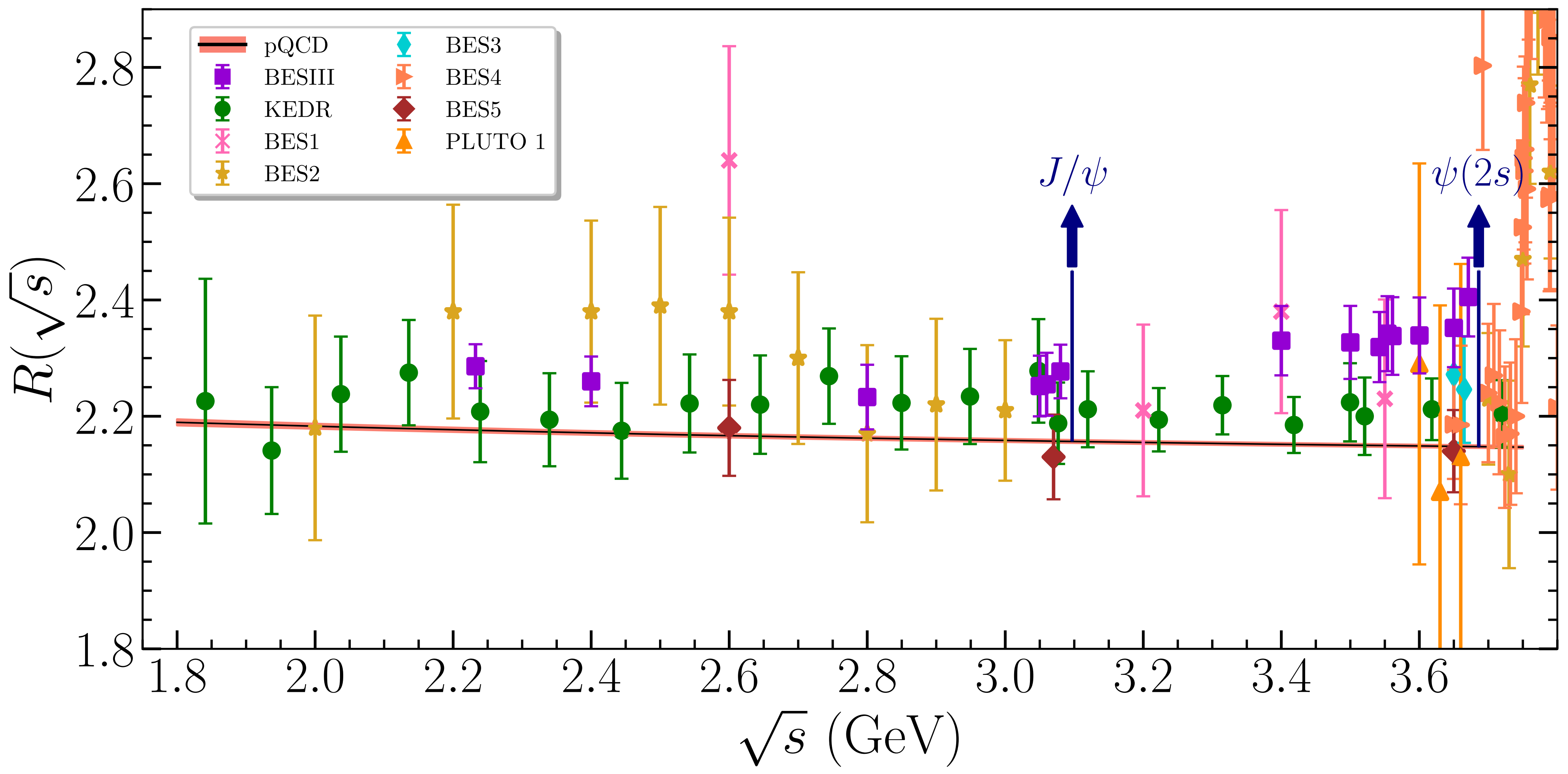
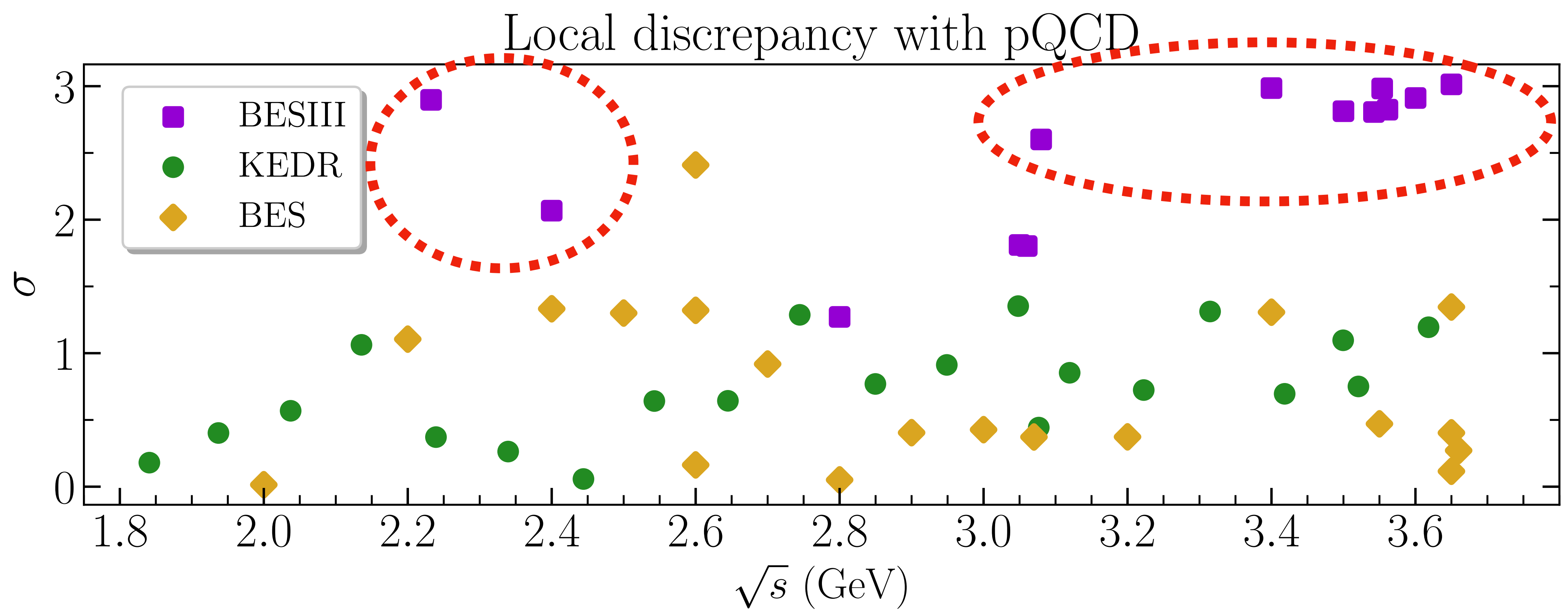
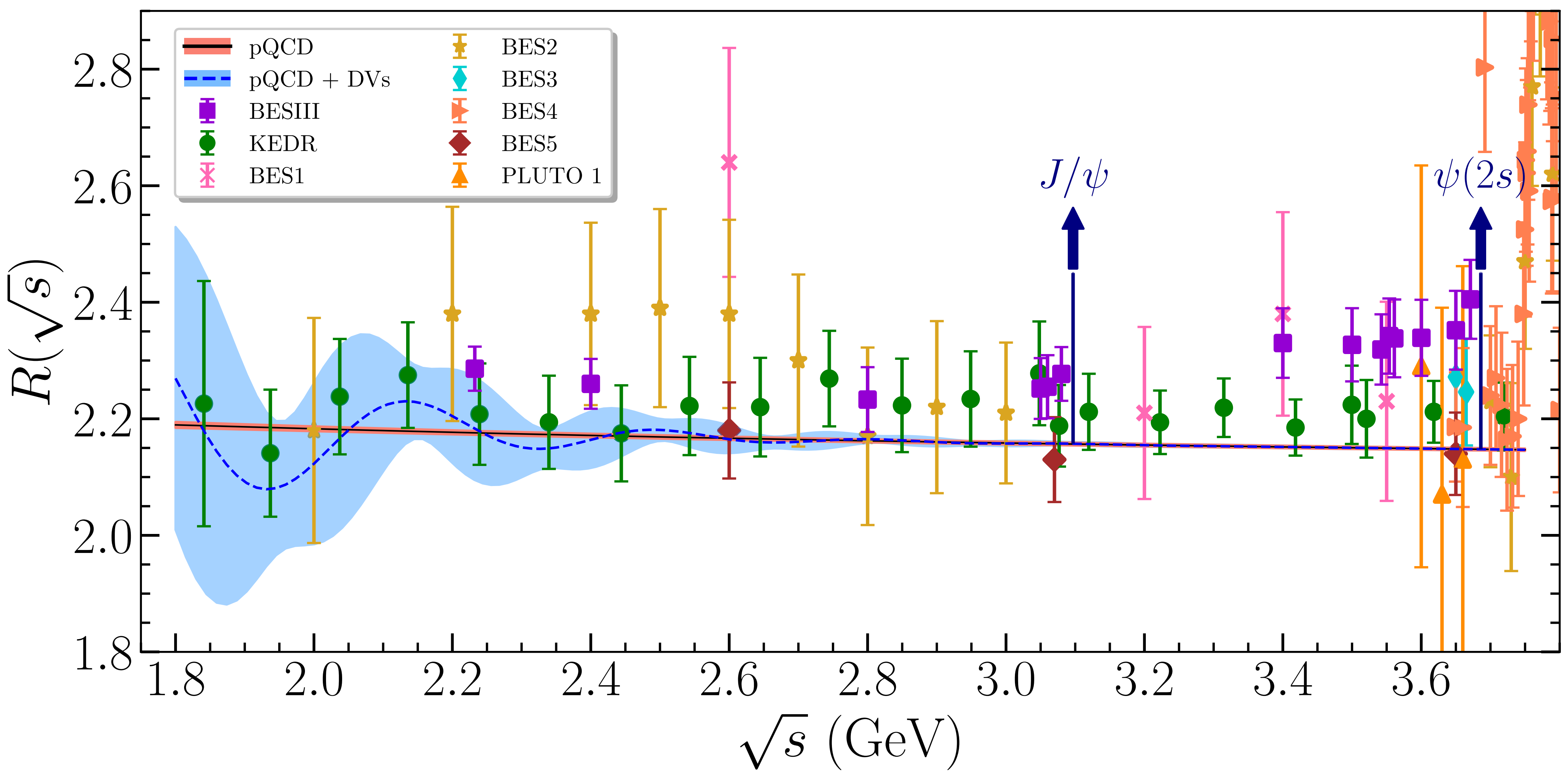
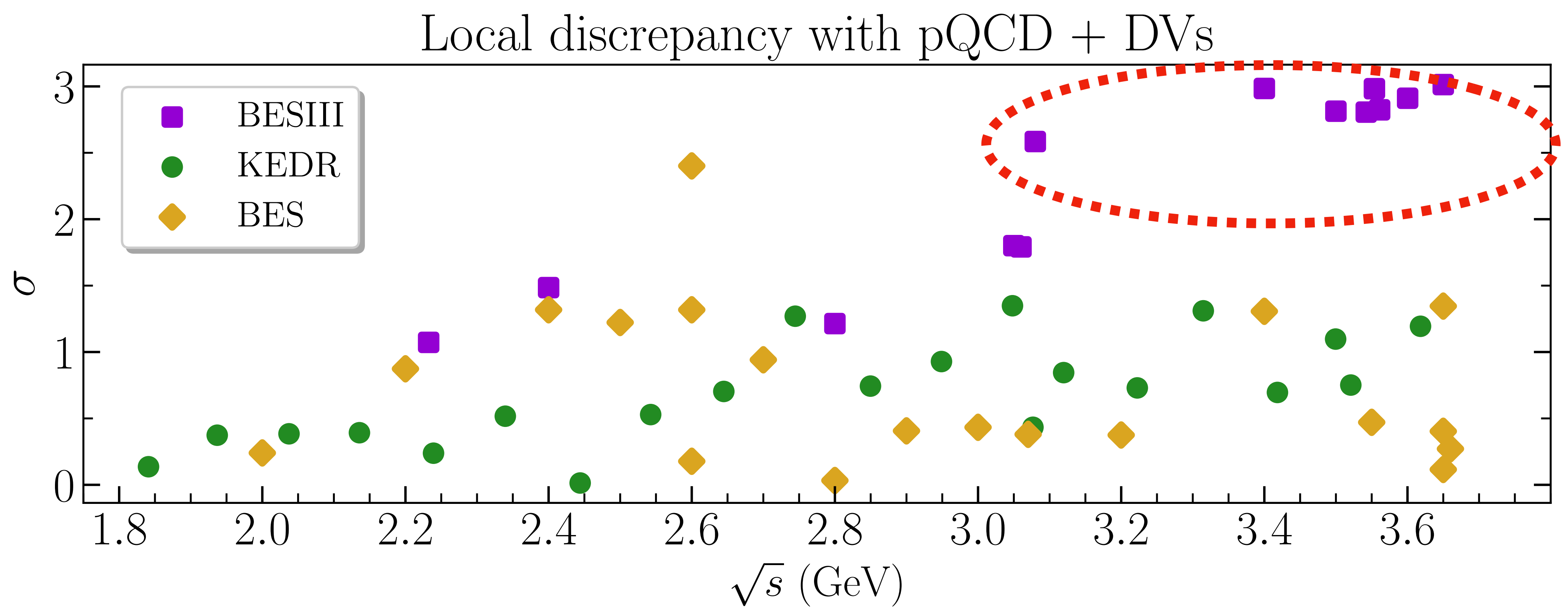




DB, Caram '25

Discrepancies with respect to pQCD + DVs



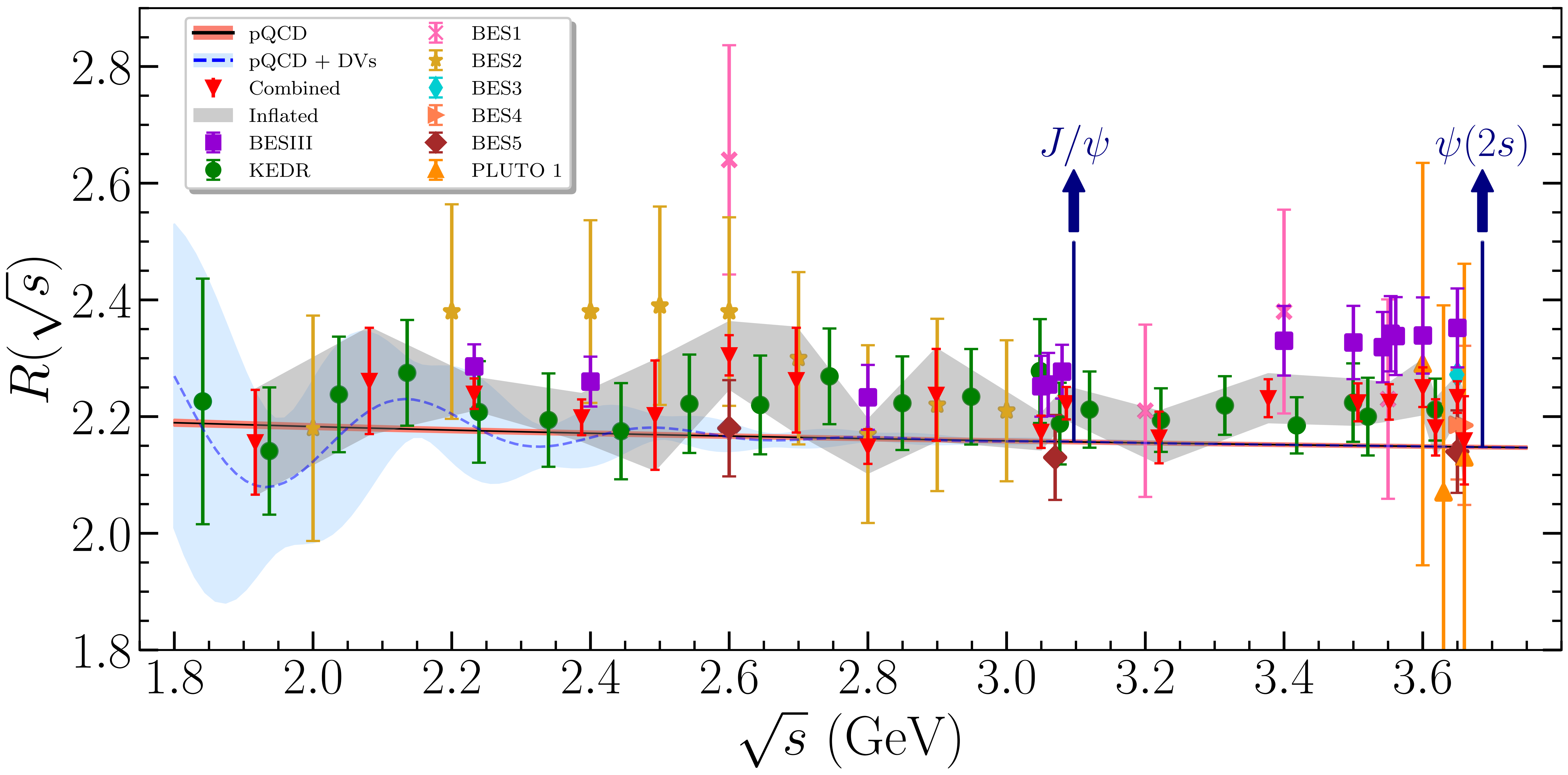


(all results include m_s and EM corrections)

Combination of the experimental data

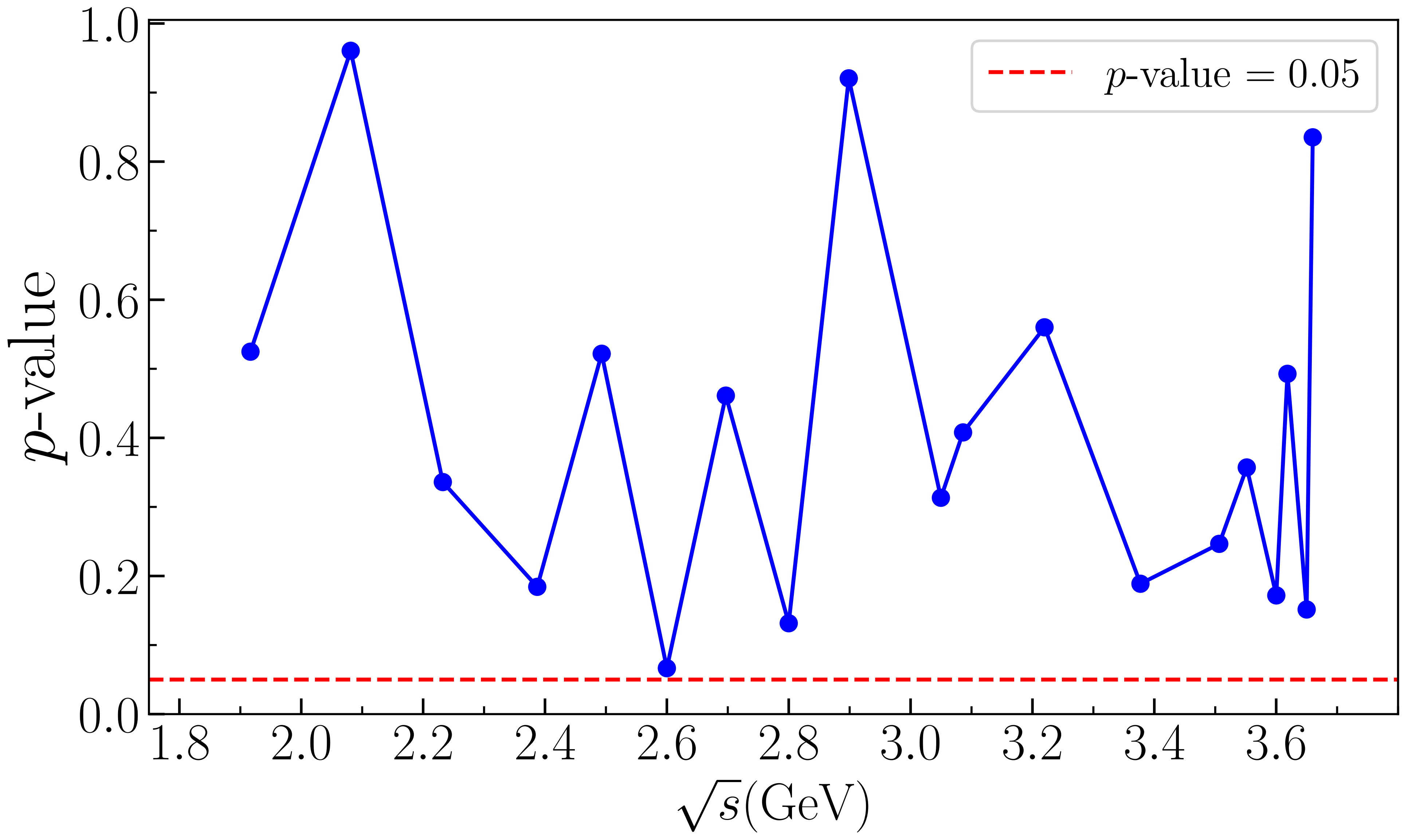
- To investigate local discrepancies in the experimental data and obtain integrated results, we have combined the available data for R_{uds}.
- We used the KNT algorithm (but more modest analysis, restricted to the the interval 1.8 GeV to 3.66 GeV)
  Keshavarzi, Nomura and Teubner '18

$\chi^2/\mathrm{dof} = 41.7/37 = 1.13 \qquad p = 27.5\%$



(BES-III data above 3.4 GeV are highly positively correlated)

Local *p*-value per cluster



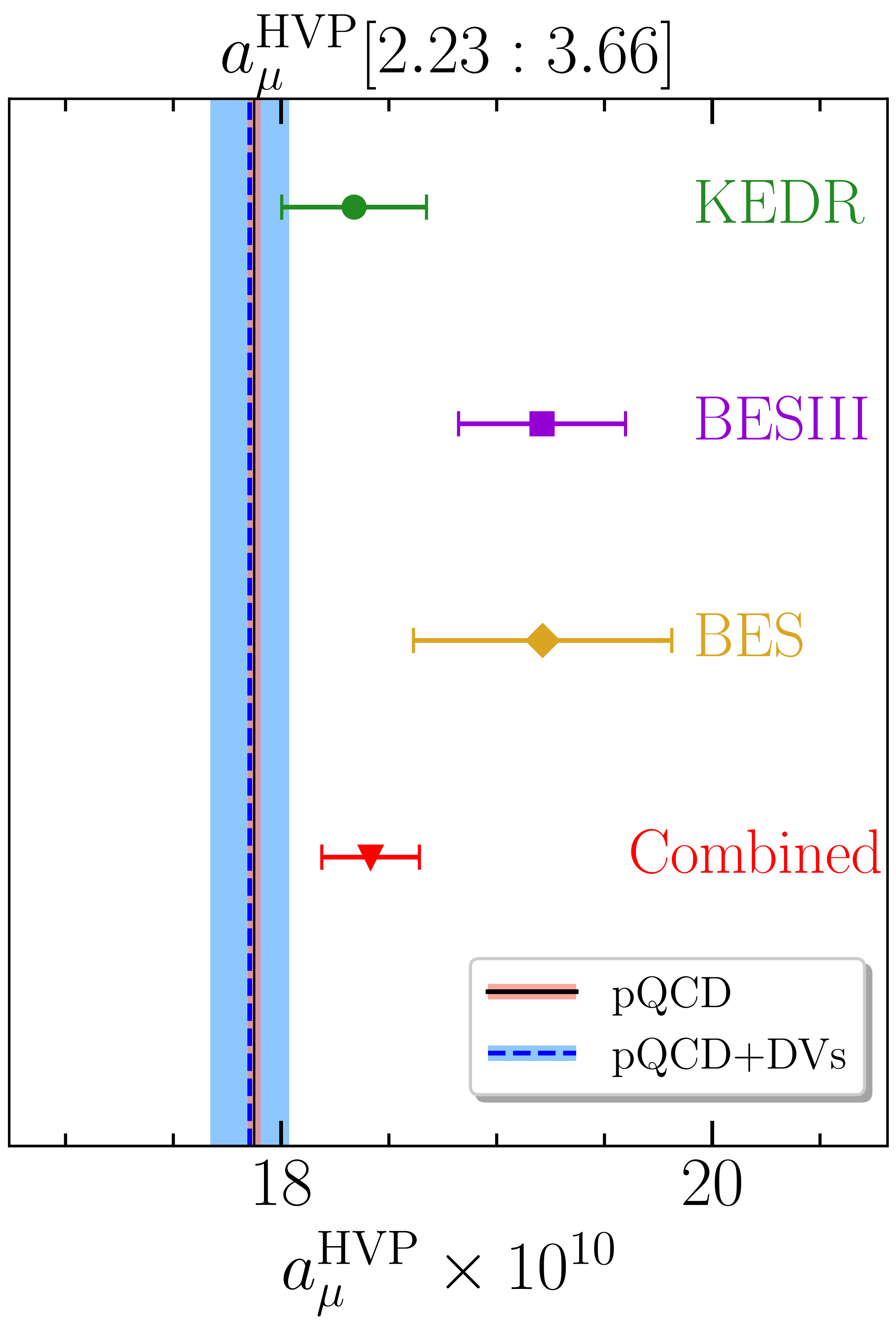
- we do error inflation (à la KNT), we checked linear vs quadratic interpolation, no sign of d'Agostini bias, and we checked different procedures to deal with correlations.

Contributions to $a_\mu^{\rm HVP}$

$a_\mu^{\rm HVP}[s_1 : s_2] \times 10^{10}$

$\sqrt{s}$ [GeV]	pQCD	pQCD+ DVs	BESIII	KEDR	BES	Comb.
[1.8 : 3.66]	33.135(51)	33.0(1.3)	—	33.91(78)	—	33.86(56)
[2.23 : 3.66]	17.875(23)	17.85(18)	19.21(39)	18.34(34)	19.21(60)	18.42(23)

$$a_\mu^{\rm HVP}[s_1; s_2] = \left(\frac{\alpha_{\rm EM} m_\mu}{3\pi}\right)^2 \int_{s_1}^{s_2} ds\, \frac{\hat{K}(s)}{s^2} R(s),$$



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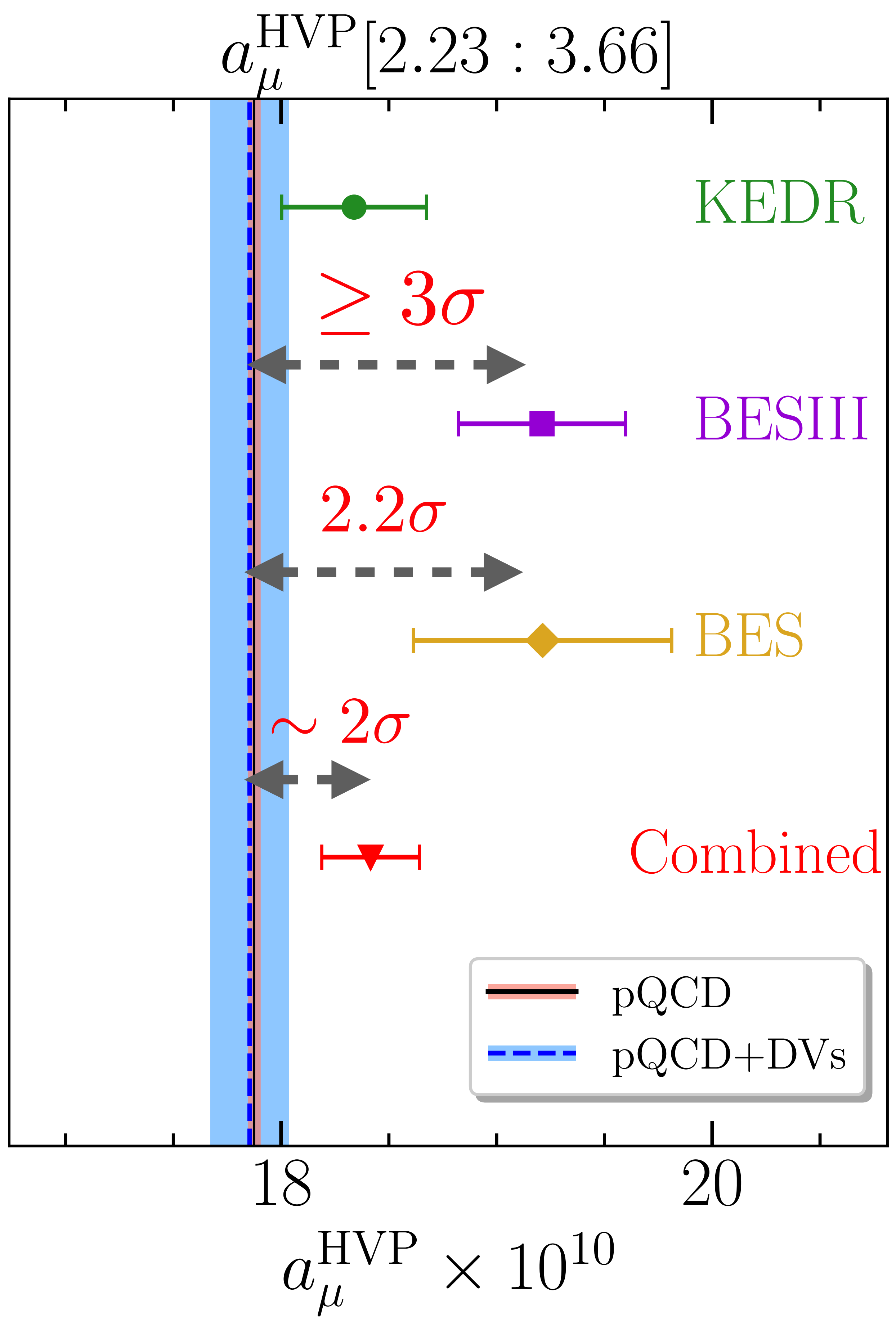

Conclusions

- pQCD **in good shape**: several indications that there is nothing problematic with the series

- Very little room for changes in pQCD. Uncertainty under control.

- DVs significant below 2.5 GeV and improve agreement between theory and data (but large errors)

- Local discrepancies reach 3σ for the BES-III data set (mainly for $\sqrt{s} \geq 3.4\text{GeV}$, but data are correlated)

- Data combination suggests that data sets are locally compatible

- We cannot provide a mechanism to account for the discrepancies between QCD and BES-III data points (mainly above 3.4 GeV)

- More BES-III data and KEDR data will be very welcome

see talks by Thatiana Kharlamova and Weiping Wang

Extra

FOPT and CIPT

$$\delta^{(0)} = \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}W_i(x)\sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} c_{n,k}\log^{k-1}\left(\frac{-s_0 x}{\mu^2}\right)$$

$x = s/s_0$

Fixed renormalization scale, strict Fixed Order expansion (Fixed Order Pt. Theory)

FOPT $\mu^2 = s_0$

$$\delta^{(0)}_{\mathrm{FO},w_i} = \sum_{n=1}^{\infty} a(s_0)^n \sum_{k=1}^{n} k\, c_{n,k}\, J^{\mathrm{FO},w_i}_{k-1}$$

coefficients

expansion in powers of the coupling

$$J_n^{\mathrm{FO},w_i} \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}W_i(x)\log^n(-x)$$

integrals over logs

Running renormalization scale, no longer a strict power expansion in the coupling

CIPT $\mu^2 = -s_0 x$

$$\delta^{(0)}_{\mathrm{CI},w_i} = \sum_{n=1}^{\infty} c_{n,1} J_n^{\mathrm{CI},w_i}(s_0)$$

no obvious expansion parameter

$$J_n^{\mathrm{CI},w_i}(s_0) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}W_i(x)a^n(-s_0 x)$$

integral over the running coupling

Pich & Le Diberder '92
Pivovarov '92

pQCD: renormalon-free gluon-condensate scheme

- FOPT not the only choice for the renormalization scale.

- Another popular choice is Contour Improved Perturbation Theory (CIPT), with a running scale that ressums the running of the strong coupling along the contour integration. But it is well known that FOPT and CIPT **do not agree.** It became clear that standard CIPT is **incompatible** with the OPE. Hoang & Regner '20

- For CIPT to work it is necessary to switch to a scheme that consistently removes the gluon-condensate renormalon.

Hoang & Regner '20, Benitez-Rathgeb, DB, Hoang '22, Golterman, Maltman and Peris '23

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- We switch to the renormalon-free (RF) gluon-condensate scheme and and use a modified version of CIPT: **RF-CIPT:**

Benitez-Rathgeb, DB, Hoang, Jamin '22

$$\langle G^2\rangle^{(n)} = \langle G^2\rangle^{\mathrm{RF}} - R^4 N_g \sum_{n=0} r_n^{(4)} \alpha_s^{n+1}(R^2) + N_g c_0(R^2)$$

with

$$c_0(R^2) = R^4 \left(\frac{2\pi}{\beta_1}\right) \mathrm{PV} \int_0^\infty du\, e^{-u/\bar{\alpha}_s(R^2)} \frac{1}{2-u}$$

This function is introduced for the new GC to be independent of the IR scale R (in the RG sense) and such that the Borel integral of the series is unchanged

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new RF gluon condensate

gluon condensate norm (renormalon residue)

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IR subtraction scale

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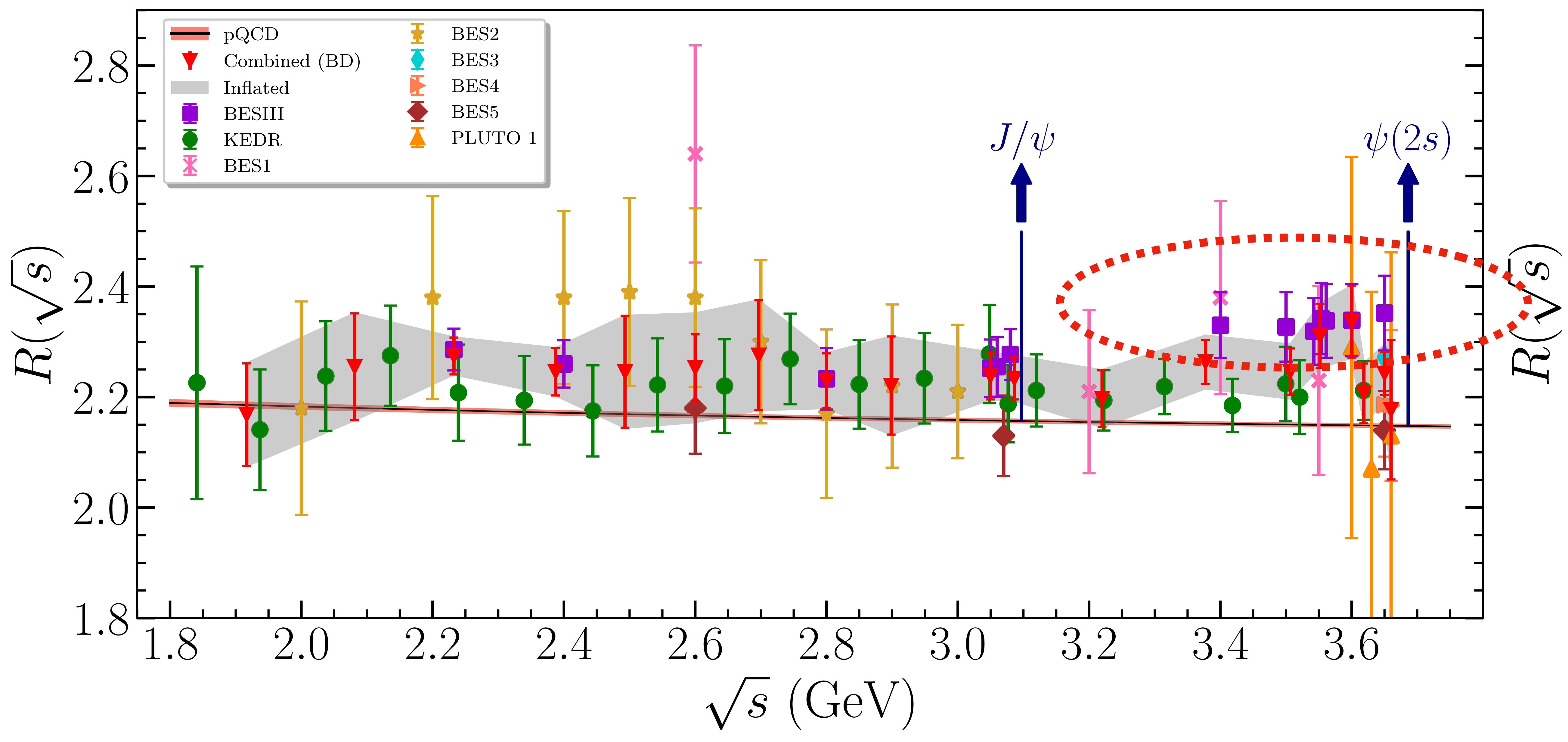
This defines another scheme for the same perturbative series

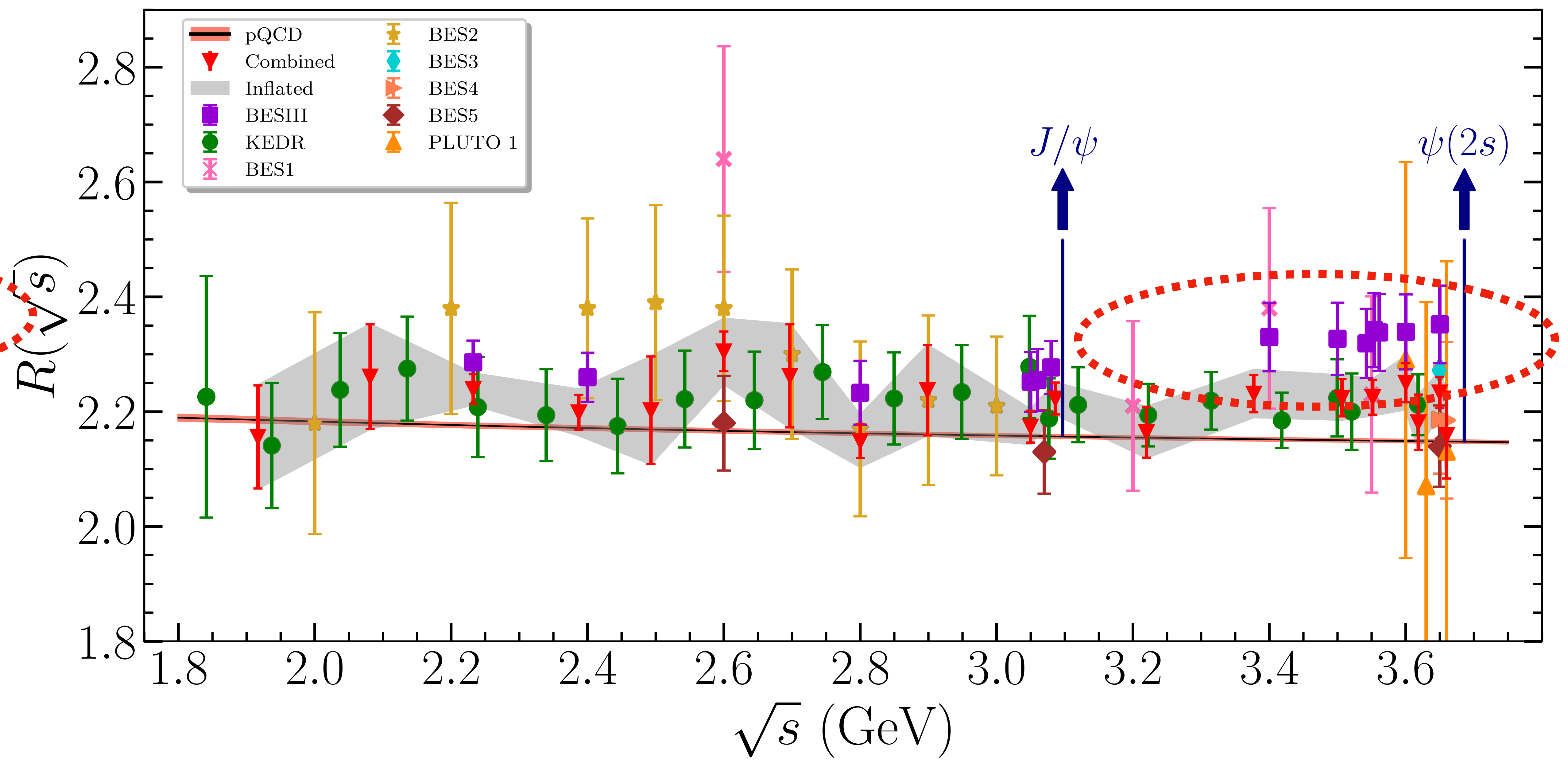
Combining the data: correlations

- Data combination considering only the correlations within a given cluster (*block-diagonal* combination)

Block-diagonal



Standard fit



- Block-diagonal data combination closer to BES-III
- Long-distance correlations not included in the fit (only included in error propagations), but statistical analysis possible using Bruno & Sommer '22

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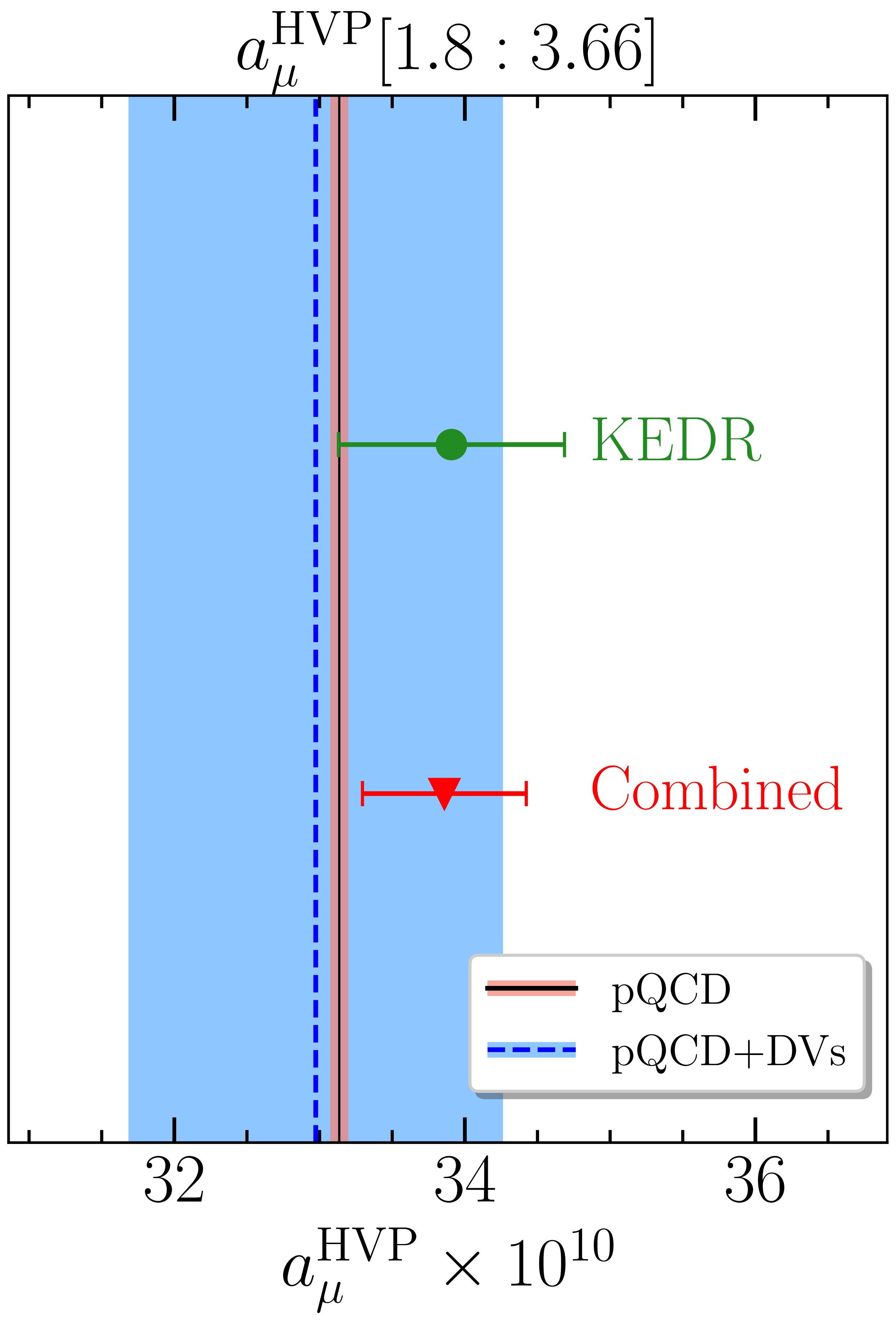

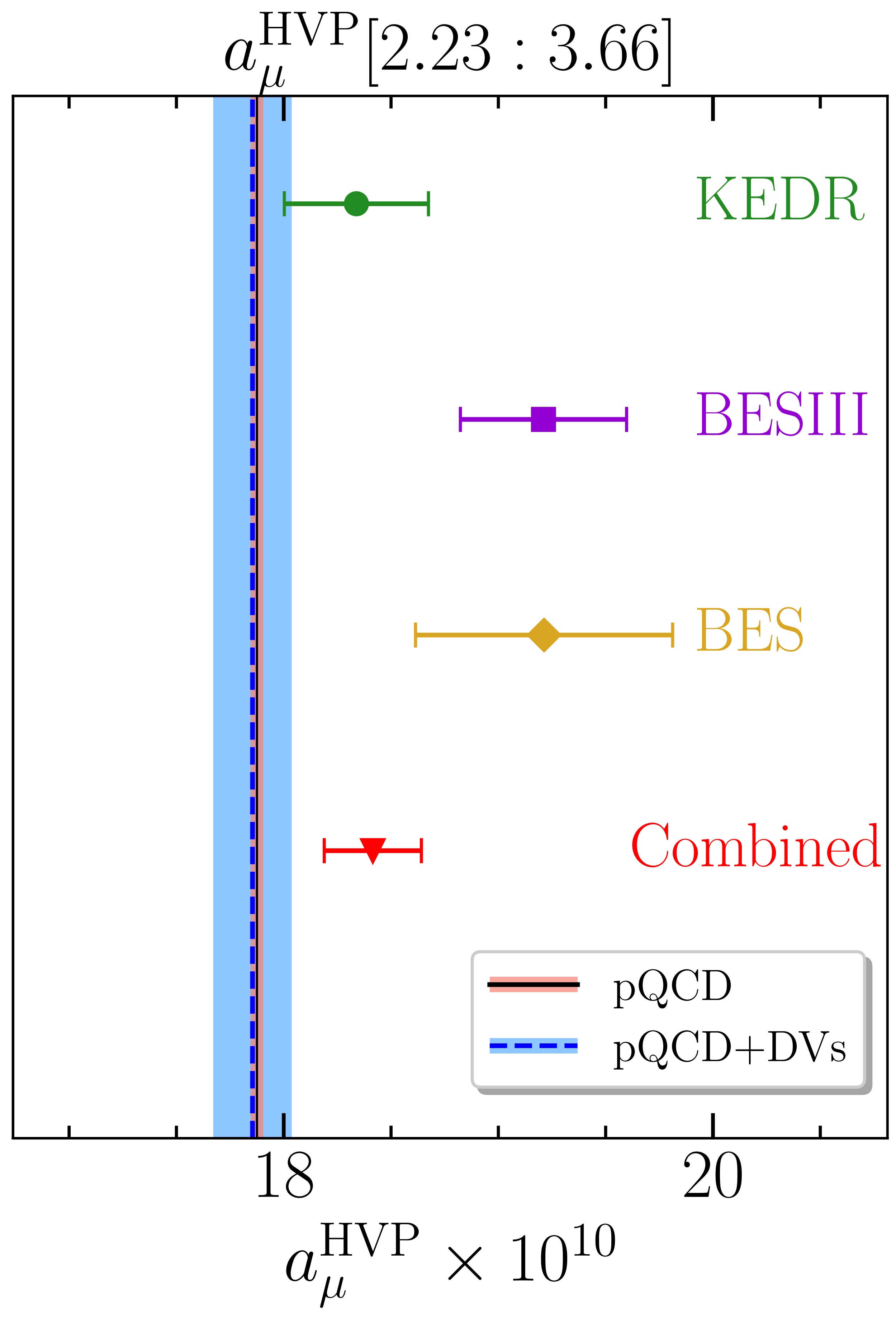

An attempt to fit of the strong coupling

- An attempt to fit the strong coupling to R_{uds}(s) KEDR data with pure pQCD leads to large uncertainties (large exp. errors combined with a not-so-strong sensitivity to alpha_s)

$$\alpha_s^{n_f=5}(m_Z) = 0.1307 \pm (0.0064)_{\rm stat}$$

- Fit with pQCD only, up to $\mathcal{O}(\alpha_s^5)$ use the estimate $c_{5,1} = 280 \pm 140$ for the six-loop coefficient

- Errors of about 5% at the *Z*-boson mass

- This is **just an exploration**, not a complete analysis (no theory error, for example)

- Very good χ^2 if only KEDR is used (bad χ^2 if BES-III data are included: p-value < 5%).

- **Fits to only BES-III data** don’t work: p-value = 0.000002.