Coupled-channel Omnès matrix for the D-wave isoscalar $\pi\pi/K\bar{K}$ system

Application: $J/\psi \to \pi^0\pi^0\gamma, J/\psi \to K_S K_S \gamma$

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Overview

- Introduction and Motivation

- Formalism:
  Partial waves and unitarity
  K-matrix approach
  Muskhelishvili-Omnès equation

- Application to $J/\psi \to \pi^0\pi^0\gamma, J/\psi \to K_S K_S \gamma$

- Conclusion and Outlook

Motivation

- **High-precision experiments require precise theory amplitudes**
  unitary, analytic, and consistent with low-energy constraints

- Main goals
  1. **Hadron spectroscopy**: search for exotic states
  2. **Precision SM tests:** look for small deviations that may indicate BSM physics

- Key hadronic **building blocks:**

$$\pi\pi \quad \pi K \quad \pi\eta \quad \cdots$$

Existing coupled-channel dispersive studies mostly focus on the S-wave ($\pi\pi/K\bar{K}$ or $\pi\eta/K\bar{K}$); higher waves are often treated single-channel (e.g. $\rho(770)$, $f_2(1270)$ etc.)

- **This talk:** the missing coupled-channel building block

$$\pi\pi/K\bar{K}, \qquad \mathrm{D-wave}$$

Motivation

- **Why the $\pi\pi/K\bar{K}$ D-wave?**

- According to PDG, below 1.6 GeV the D-wave is dominated by two tensor resonances ⇒ naturally a coupled-channel problem

$$f_2(1270) \overset{85\%}{\to} \pi\pi, \qquad f_2'(1525) \overset{90\%}{\to} K\bar{K}$$

- Only two dispersive applications exist so far:
  $\Upsilon(11020) \to \Upsilon(nS)\pi^+\pi^-$ [TarrusCastella:2021pld]
  Gravitational form factors [Cao:2025dkv]

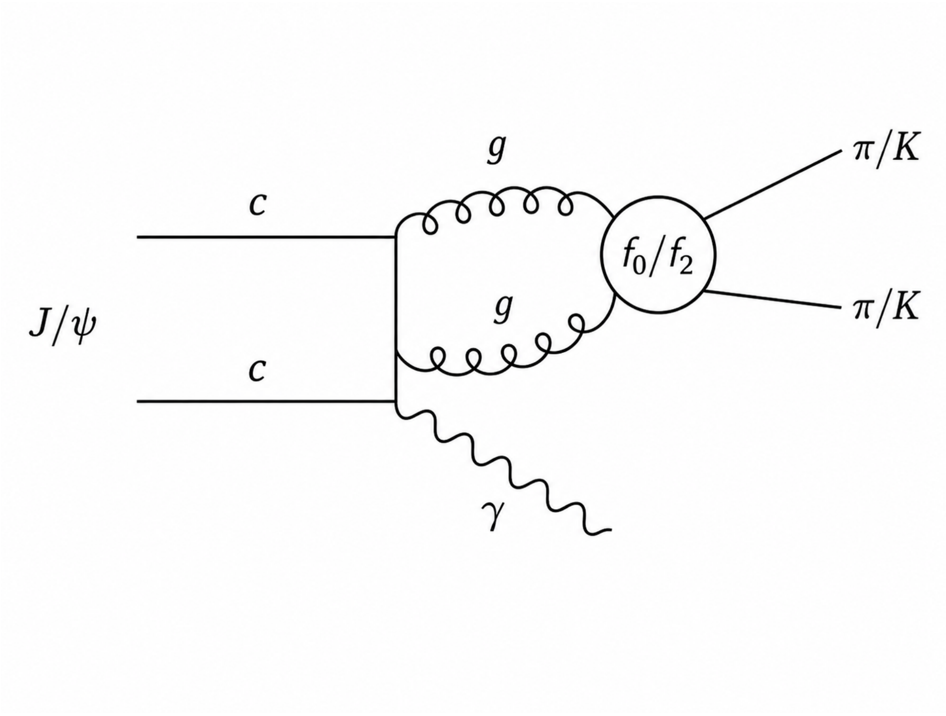
- **Main difficulty**: no direct $K\bar{K} \to K\bar{K}$ scattering data ⇒ input must be constrained indirectly

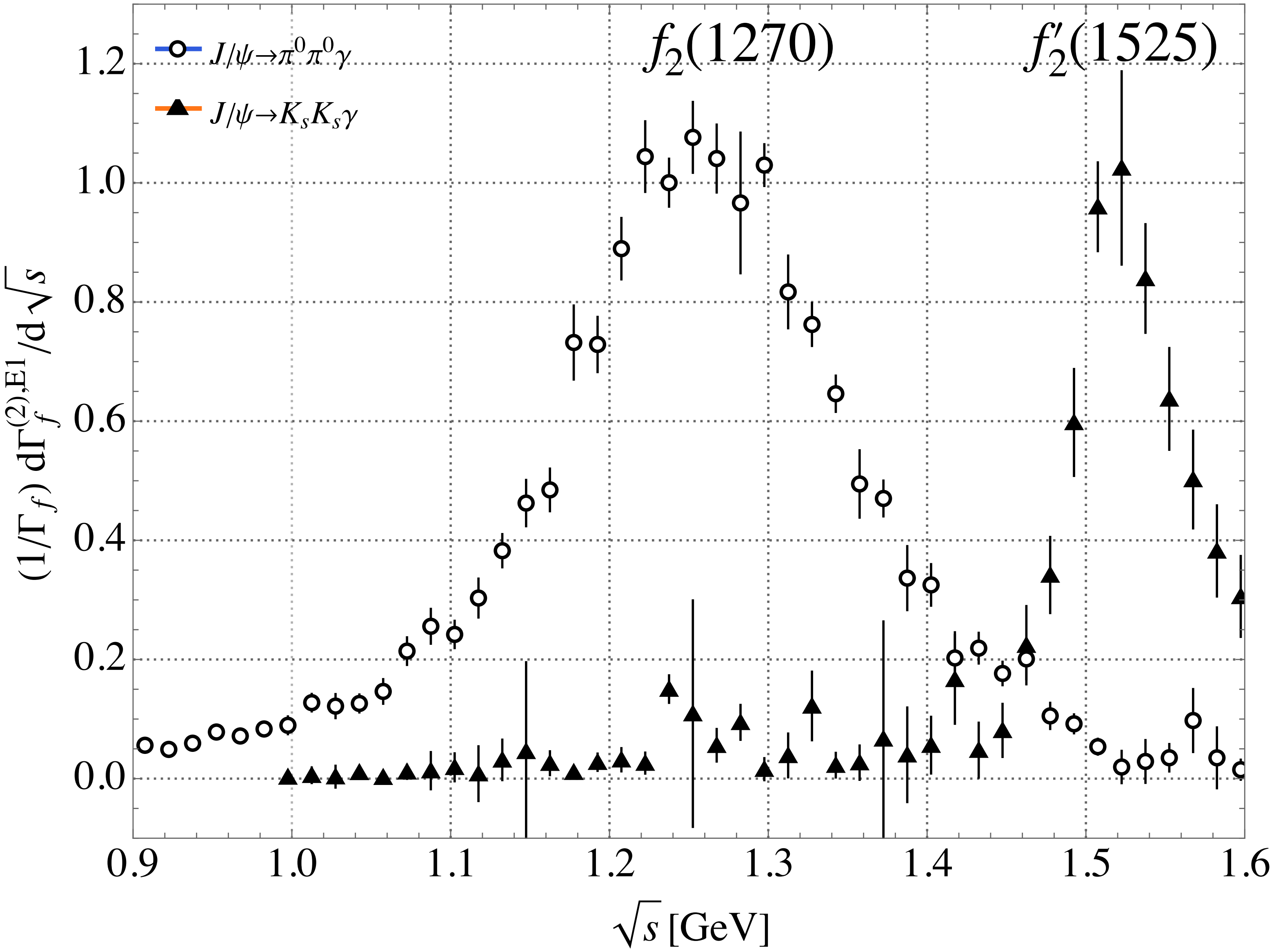
  Previous analyses relied on Breit-Wigner-type fits for $\pi\pi \to K\bar{K}$ [Pelaez:2018qny, Pelaez:2020gnd] ⇒ uncontrolled extraction of $K\bar{K} \to K\bar{K}$ phase shift

- **Our aim**: build a unitary two-channel K-matrix input constrained by existing scattering data and PDG resonance information

Motivation

- Radiative J/ψ decays as a **testing ground** [BESIII:2015rug, BESIII:2018ubj]





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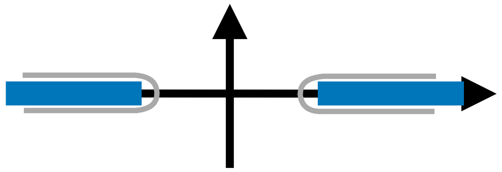
Partial waves and unitarity

- Use partial wave amplitudes **free of kinematic constraints**

$$T_{ab}(s,t) = 16\pi\,\mathcal{N}_{ab}\sum_{J=0}^{\infty}(2J+1)\,(p_a p_b)^J\,t_{ab}^{(J)}(s)\,P_J(\cos\theta)$$

$$a,b = 1{,}2 \qquad 1 \equiv \pi\pi, \qquad 2 \equiv K\bar{K} \qquad J = 2$$

- Unitarity relation for the p.w. amplitudes on the right-hand cut reads



$$\mathrm{Im}\,\boldsymbol{t}(s) = \boldsymbol{t}^{*}(s)\,\boldsymbol{\rho}(s)\,\boldsymbol{t}(s)$$

$$\mathrm{Im}\,\boldsymbol{t}^{-1}(s) = -\boldsymbol{\rho}(s), \quad \boldsymbol{\rho}(s) = \begin{pmatrix} \rho_1(s)\,\theta(s-s_{\text{th}1}) & 0 \\ 0 & \rho_2(s)\,\theta(s-s_{\text{th}2}) \end{pmatrix}$$

$$\rho_a(s) = (p_a(s))^{2J}\,\frac{p_a(s)}{2\sqrt{s}}$$

Two-channel t-matrix parametrization

- In the physical region, a unitary two-channel amplitude can be parametrized by

$$\boldsymbol{t}(s) = \begin{pmatrix} \dfrac{\eta e^{2i\delta_1} - 1}{2i\rho_1} & |t_{12}| e^{i\delta_{12}} \\ |t_{12}| e^{i\delta_{12}} & \dfrac{\eta e^{2i\delta_2} - 1}{2i\rho_2} \end{pmatrix} ,$$

$$s > s_{K\bar{K}}$$

$$\eta = \sqrt{1 - 4\rho_1\rho_2 |t_{12}|^2}$$

$$\delta_{12} = \delta_1 + \delta_2$$

- **Why not just use data directly for $\delta_1, \delta_2, |t_{12}|$?**

- **In practice**
  1). Data may not cover all required quantities
  2). Inputs from different channels/analyses may not be mutually consistent
  3). Analytic continuation should give consistent pole positions in different channels
  4). Dispersive applications also need t_{12} below $K\bar{K}$ threshold, where $|t_{12}|$ is not observable

$\Rightarrow$ **need a unitary two-channel model input (K-matrix, N/D, etc.)**

Two-channel t-matrix parametrization

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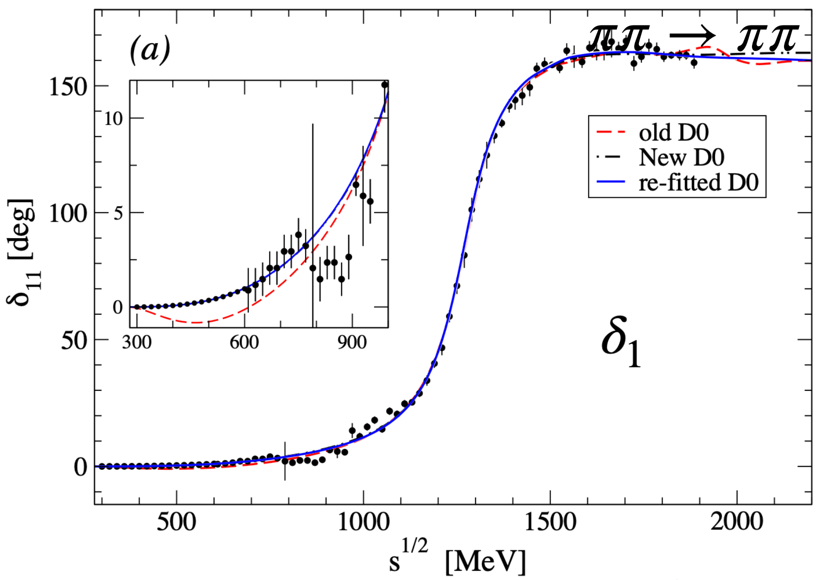
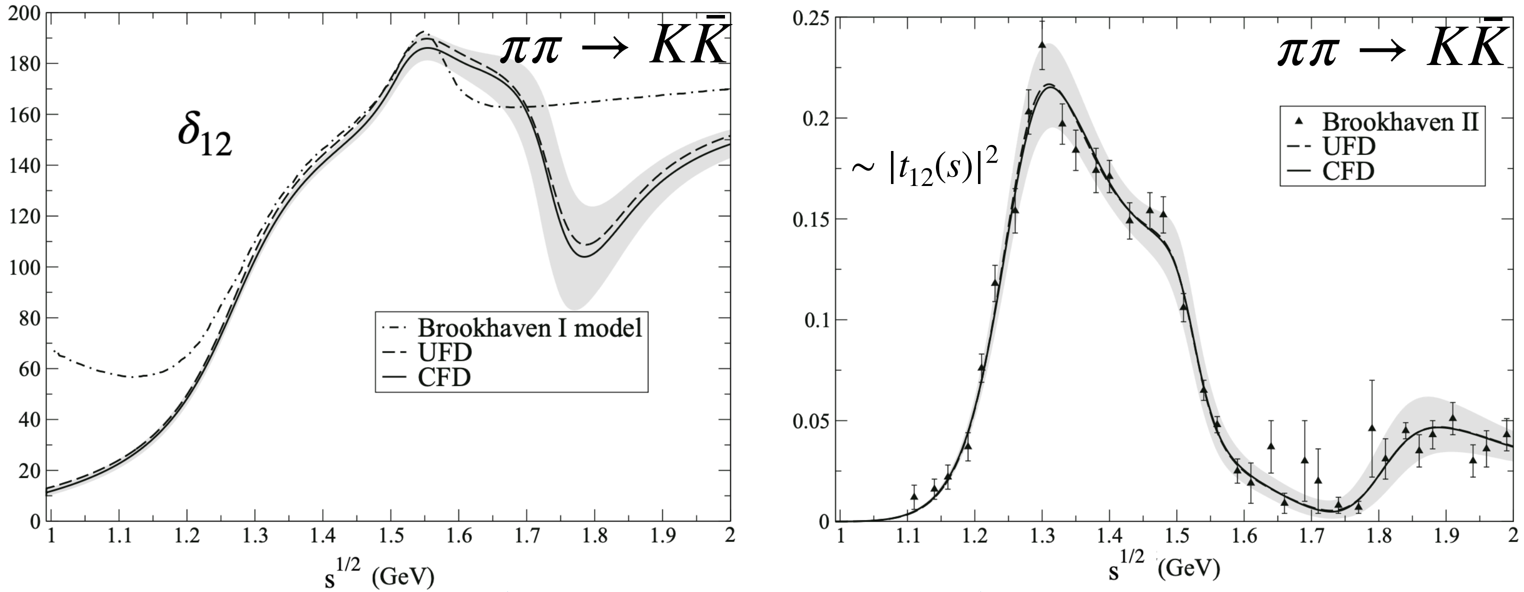
$$\boldsymbol{t}(s)=\begin{pmatrix}\dfrac{\eta e^{2i\delta_1}-1}{2i\rho_1} & |t_{12}|\,e^{i\delta_{12}}\\ |t_{12}|\,e^{i\delta_{12}} & \dfrac{\eta e^{2i\delta_2}-1}{2i\rho_2}\end{pmatrix},\qquad \begin{aligned}&s>s_{K\bar K}\\ &\eta=\sqrt{1-4\rho_1\rho_2\,|t_{12}|^2}\\ &\delta_{12}=\delta_1+\delta_2\end{aligned}$$

- Input currently used in the recent D-wave dispersive analyses of [TarrusCastella:2021pld, Cao:2025dkv]



[Bydzovsky:2016vdx]

Breit-Wigner-type fits for t_{12}



[Pelaez:2018qny, Pelaez:2020gnd]

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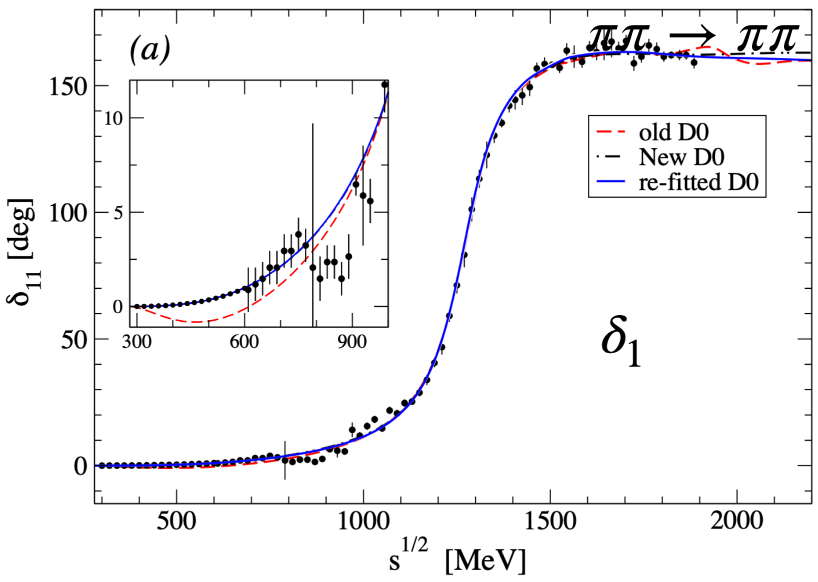
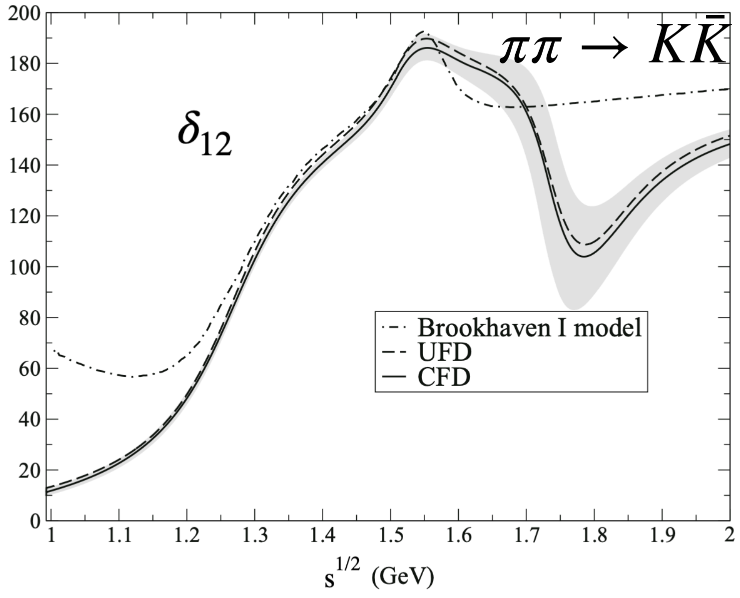
- Input currently used in the recent D-wave dispersive analyses of [TarrusCastella:2021pld, Cao:2025dkv]

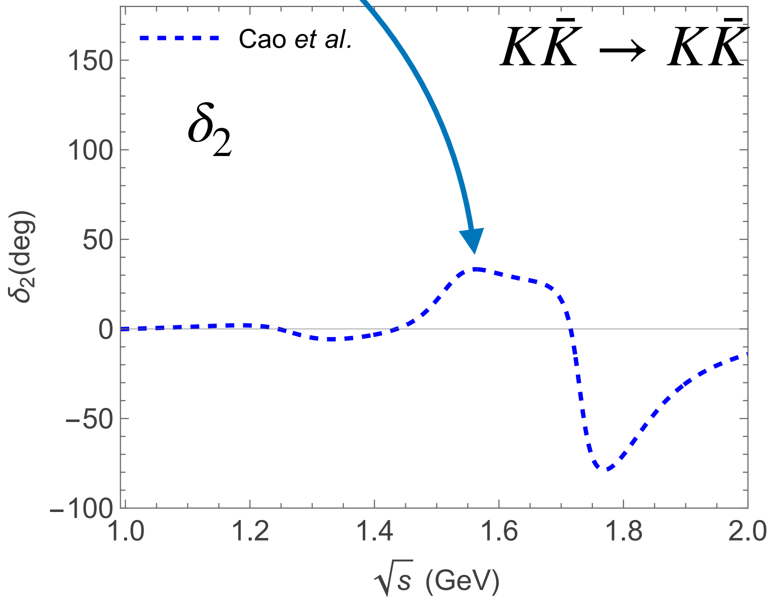
$$f_2(1270) \overset{85\%}{\rightarrow} \pi\pi, \qquad f_2'(1525) \overset{90\%}{\rightarrow} K\bar{K}$$

physically implausible





$\Rightarrow$



K-matrix model

- Construct a unitary two-channel t-matrix:

$$\boldsymbol{t}^{-1}(s) = \boldsymbol{K}^{-1}(s) - i\,\boldsymbol{\rho}(s)$$

- Motivated by Lattice analysis [Briceno:2017qmb] of the same channel (but with $m_\pi \sim 391$ MeV), we use a minimal two-pole form

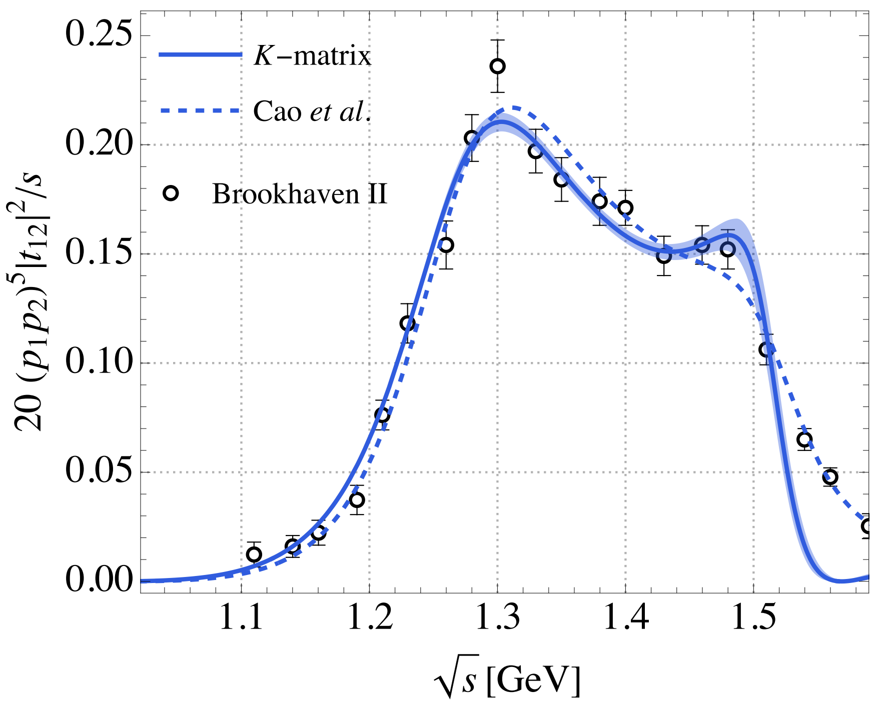
$$K_{ab}(s) = \sum_{R=1}^{2} \frac{\tilde{g}_{R,a}(s)\,\tilde{g}_{R,b}(s)}{m_R^2 - s}\,, \qquad \tilde{g}_{R,a}(s) = g_{R,a}\,\frac{F_2(p_a(s)\,R_{int})}{F_2(p_a(m_R^2)\,R_{int})}$$

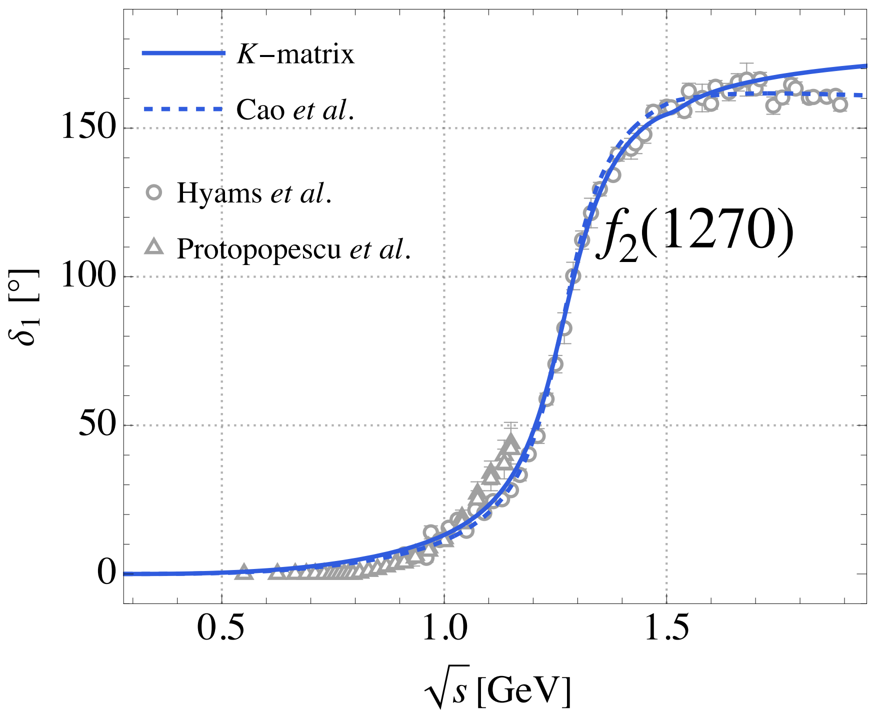
$$R = f_2(1270), \quad f_2'(1525)$$

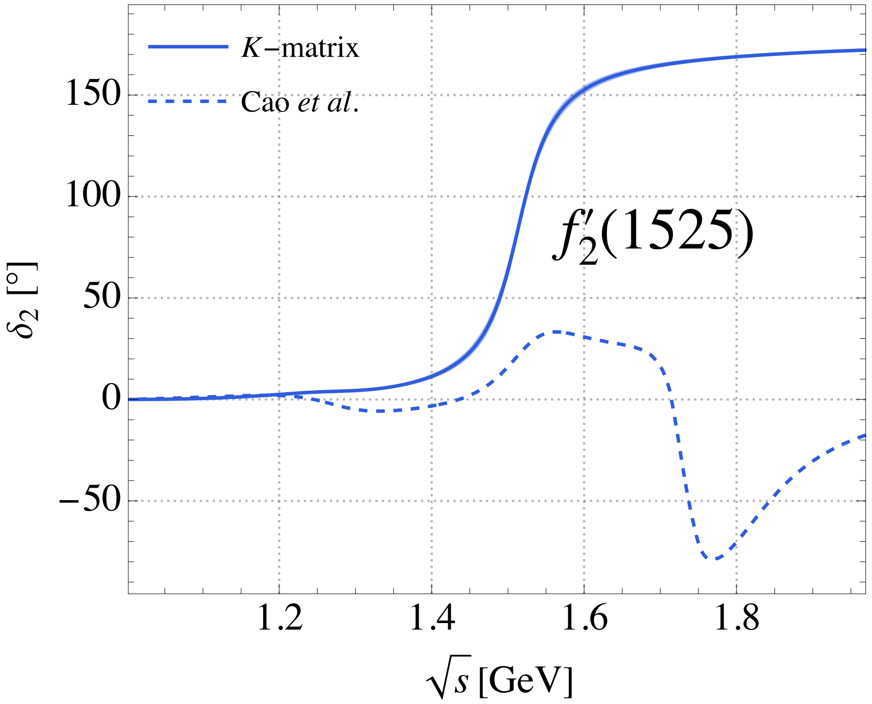
$$\Gamma_R = \sum_a \Gamma_a, \quad \Gamma_a = \frac{g_{R,a}^2}{m_R}\,\rho_a(m_R^2)$$

- Input m_R, Γ_R from PDG; couplings $g_{R,a}$ (including the relative sign of $g_{R,1}\,g_{R,2}$ between two resonances) fitted to data on $|t_{12}(s)|$

K-matrix model







$\chi^2/N_{\mathrm{dof}} = 1.3$

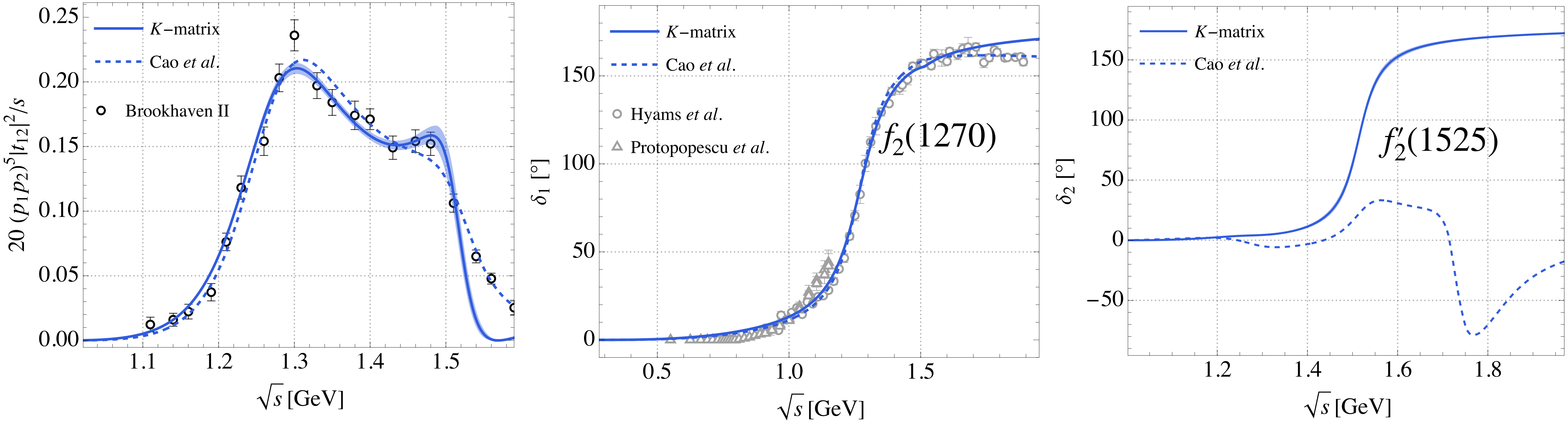
	K-matrix parameters		
	m_R [MeV]	Γ_R [MeV]	$g_{R,\pi\pi}$, $g_{R,K\bar{K}}$ [GeV]
$f_2(1270)$	1275.4(8)	185.8(2.8)	1.25(1), 0.77(1)
$f_2'(1525)$	1517.3(2.4)	72(7)	0.09(1), −1.14(5)

$$t_{ab}(s) \sim \frac{g_{p,a}\, g_{p,b}}{s_p - s}, \qquad \sqrt{s_p} \equiv M_p - \frac{i}{2}\Gamma_p, \qquad \mathcal{B}_a = \frac{1}{\Gamma_p}\left(\frac{|g_{p,a}|^2}{M_p}\,\rho_a(M_p^2)\right)$$

Pole position and couplings			From pole
M_p [MeV]	Γ_p [MeV]	$\lvert g_{p,\pi\pi}\rvert$, $\lvert g_{p,K\bar{K}}\rvert$ [GeV]	$\mathcal{B}_{\pi\pi}[\%]$, $\mathcal{B}_{K\bar{K}}[\%]$
1267.1(0.8)	185.7(2.5)	1.26(1), 0.81(1)	96.3(1), 4.2(1)
1514.6(2.5)	71.5(6.4)	0.11(1), 1.14(5)	3.1(2), 96.8(2)

two-channel approximation

K-matrix model



- The main difference with the input used in previous analyses [TarrusCastella:2021pld, Cao:2025dkv] is the $K\bar{K}$ phase shift, which now shows clear $f_2'(1525)$ phase motion

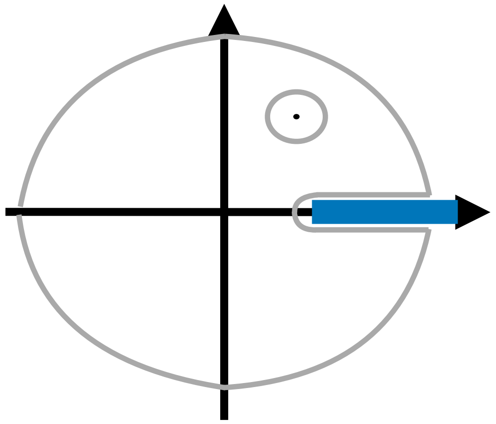
Mushkhelishvili-Omnès equation

Omnès matrix applications

$$t_{\pi\pi/K\bar{K}}(s) \quad \Rightarrow \quad \boxed{\Omega(s)} \quad \Rightarrow \quad J/\psi \to \pi^0\pi^0\gamma,\ K_S K_S \gamma$$

- The Omnès matrix $\mathbf{\Omega}(s)$ is defined as the solution of a homogeneous 2×2 Muskhelishvili-Omnès (MO) problem that encapsulates the **right-hand cut contributions** of the scattering t-matrix

$$\mathbf{\Omega}(s) = \int_{s_{\text{th}1}}^{\infty} \frac{ds'}{\pi} \frac{\operatorname{Im} \mathbf{\Omega}(s')}{s' - s}, \qquad \operatorname{Im} \mathbf{\Omega}(s) = \boldsymbol{t}^*(s)\, \boldsymbol{\rho}(s)\, \mathbf{\Omega}(s)$$



- In the coupled-channel case no closed analytic expression exists, except for the determinant

$$\det \mathbf{\Omega}(s) = \exp \left[\frac{s}{\pi} \int_{s_{\text{th}1}}^{\infty} \frac{ds'}{s'} \frac{\delta_{12}(s')}{s' - s} \right]$$

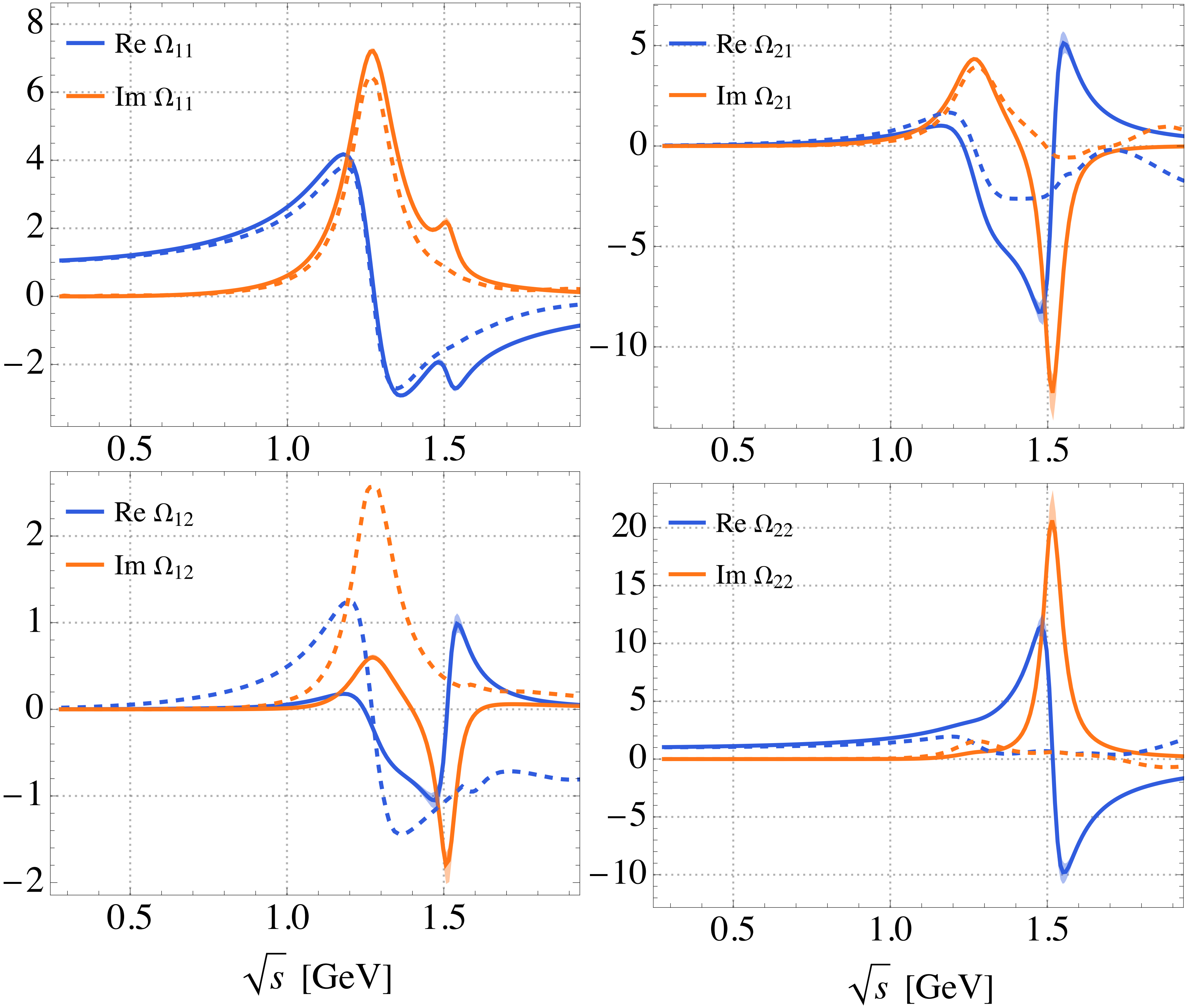
- Boundary / asymptotics: $\Omega(0) = 1, \quad \Omega(s) \sim \frac{1}{s} \Leftrightarrow \delta_1(\infty) + \delta_2(\infty) = 2\pi$

naturally satisfied in our
K-matrix construction

Omnès matrix

Omnès matrix applications

$t_{\pi\pi/K\bar{K}}(s)$ ⇒ $\Omega(s)$ ⇒ $J/\psi \to \pi^0\pi^0\gamma,\ K_S K_S \gamma$



Solid: this work
Dashed: [Cao:2025dkv]

Changed $K\bar{K}$ phase strongly affects $\Omega_{12}, \Omega_{21}, \Omega_{22}$

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$J/\psi \to \pi^0\pi^0\gamma, J/\psi \to K_S K_S \gamma$

- BESIII performed mass-independent amplitude analyses [BESIII:2015rug, BESIII:2018ubj] for both final states ⇒ ideal testing ground

$$\frac{d\Gamma_f}{d\sqrt{s}} = \frac{d\Gamma_f^{(0),\mathrm{E1}}}{d\sqrt{s}} + \sum_{X=\mathrm{E1,M2,E3}} \frac{d\Gamma_f^{(2),X}}{d\sqrt{s}}$$

- In our work we focused on the dominant $J = 2$, E1 term, which can be expressed in terms of $I = 0$ amplitudes

$$\frac{d\Gamma^{(2),\mathrm{E1}}_{J/\psi\to\pi^0\pi^0\gamma}}{d\sqrt{s}} \sim \left|\frac{h_0^{(2),\mathrm{E1}}(s)}{\sqrt{3}}\right|^2, \qquad \frac{d\Gamma^{(2),\mathrm{E1}}_{J/\psi\to K_S K_S\gamma}}{d\sqrt{s}} \sim \left|\frac{k_0^{(2),\mathrm{E1}}(s)}{2}\right|^2$$

- For dispersive application one needs to factor out kinematic constraints and implement unitarity relation

$$\begin{pmatrix} h_0^{(2),\mathrm{E1}}(s) \\ k_0^{(2),\mathrm{E1}}(s) \end{pmatrix} \equiv (s - q^2) \begin{pmatrix} p_1^2(s) & 0 \\ 0 & p_2^2(s) \end{pmatrix} \begin{pmatrix} \hat{h}(s) \\ \hat{k}(s) \end{pmatrix}, \quad \mathrm{Im} \begin{pmatrix} \hat{h}(s) \\ \hat{k}(s) \end{pmatrix} = \boldsymbol{t}^*(s)\,\boldsymbol{\rho}(s) \begin{pmatrix} \hat{h}(s) \\ \hat{k}(s) \end{pmatrix}$$

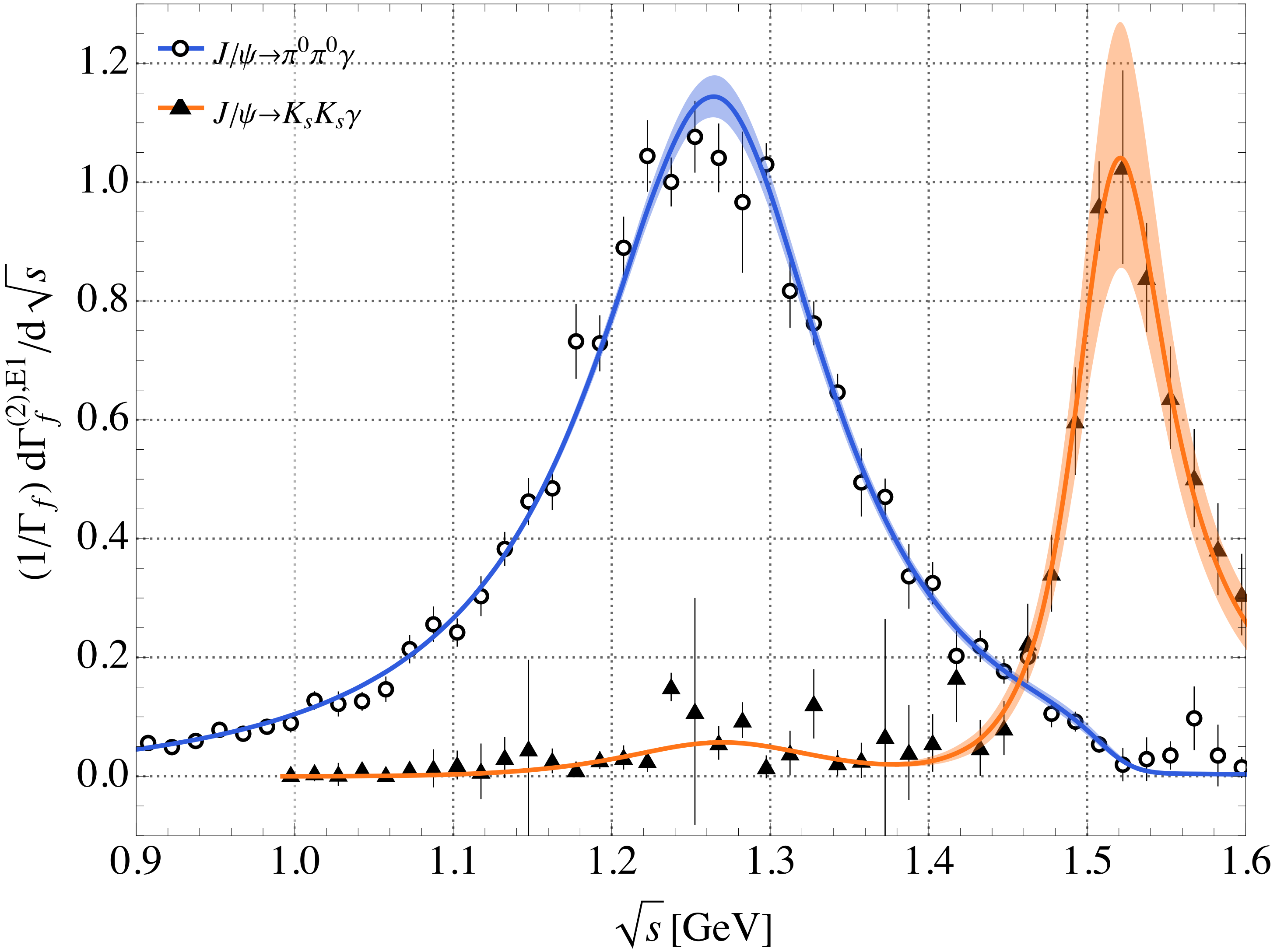
$J/\psi \to \pi^0\pi^0\gamma, J/\psi \to K_S K_S \gamma$

- MO representation [Muskhelishvili1953, Omnès1958]

$$\begin{pmatrix} \hat{h}(s) \\ \hat{k}(s) \end{pmatrix} = \mathbf{\Omega}(s) \left[\int_L \frac{ds'}{\pi} \frac{\mathbf{\Omega}^{-1}(s')}{s'-s} \operatorname{Im} \begin{pmatrix} \hat{h}^L(s') \\ \hat{k}^L(s') \end{pmatrix} \right] \approx \mathbf{\Omega}(s) \begin{pmatrix} a + b\,s \\ c + d\,s \end{pmatrix}$$

where real parameters (a, b, c, d) describe short-distance physics and left-hand-cut effects and fitted to the data.

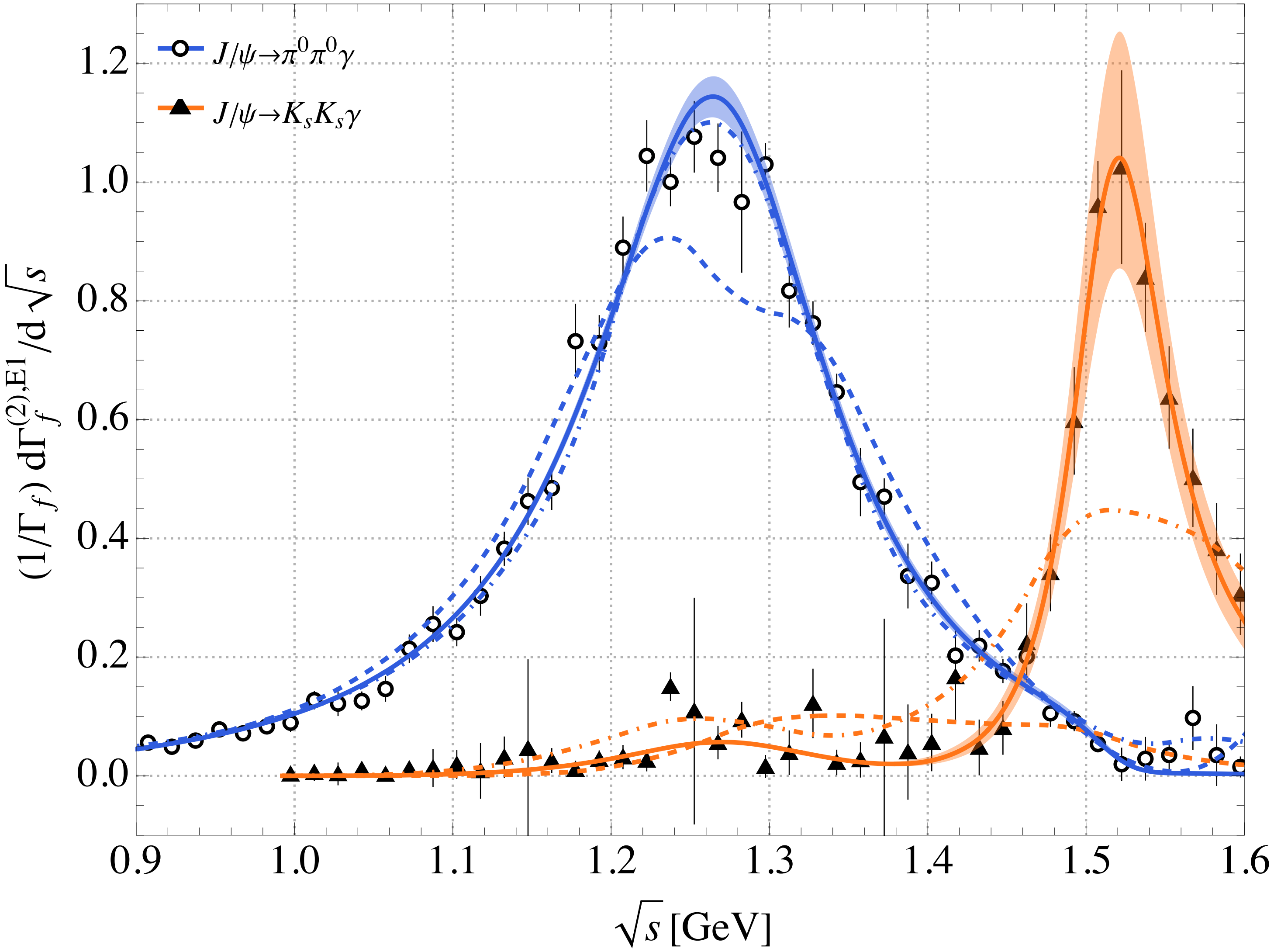
$\chi^2/N_{\text{dof}} = 1.1$



$J/\psi \to \pi^0\pi^0\gamma, J/\psi \to K_S K_S \gamma$

- Same data, but different Omnès matrices: [TarrusCastella:2021pld, Cao:2025dkv]
  The new matrix is not just a better fit; it also has better cross-channel predictive power.

Fit type	χ^2/N_{data}	Ref. [7]	Ref. [8]	This work
$\pi\pi$+$K\bar{K}$	$\pi\pi$	2.7	1.3	1.1
	$K\bar{K}$	12.4	4.5	1.0
	total (fit)	5.8	2.3	1.1



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Conclusion and Outlook

- Constructed a new unitary coupled-channel input for the

$$\pi\pi/K\bar{K}, \qquad I = 0, \qquad J = 2$$

- Used a minimal two-pole K-matrix parametrization constrained by $\pi\pi \to \pi\pi$, $\pi\pi \to K\bar{K}$, and PDG resonance information. Found significant differences with previous analyses [TarrusCastella:2021pld, Cao:2025dkv], especially in the $K\bar{K} \to K\bar{K}$ channel.

- Built the corresponding Muskhelishvili-Omnès matrix, implementing analyticity and two-channel unitarity. Applied the MO formalism to radiative charmonium decays $J/\psi \to \gamma\pi^0\pi^0, \gamma K_S^0 K_S^0$. Achieved an accurate simultaneous description of the BESIII $J = 2$, E1 spectra.

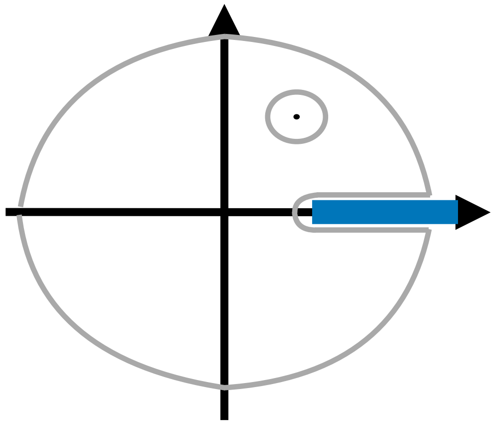
- **Outlook**
  Explore extensions beyond two channels and use this $\Omega(s)$ as a building block for form factors and tensor-meson phenomenology.

Back-up

Numerics

- The Omnès matrix $\boldsymbol{\Omega}(s)$ is defined as the solution of a homogeneous 2×2 MO problem

$$\boldsymbol{\Omega}(s) = \int_{s_{\rm th1}}^{\infty} \frac{ds'}{\pi} \frac{\operatorname{Im} \boldsymbol{\Omega}(s')}{s'-s}\,, \qquad \operatorname{Im} \boldsymbol{\Omega}(s) = \boldsymbol{t}^*(s)\, \boldsymbol{\rho}(s)\, \boldsymbol{\Omega}(s)$$



- Equivalent real-part formulation used numerically

$$\operatorname{Re} \boldsymbol{\Omega}(s) = \mathrm{pv}\int_{s_{\rm th1}}^{\infty} \frac{ds'}{\pi} \frac{\boldsymbol{X}(s') \operatorname{Re} \boldsymbol{\Omega}(s')}{s'-s}, \qquad \boldsymbol{X}(s) = i\left[\mathbf{1} - \left(\mathbf{1} - i\, \boldsymbol{t}^*(s)\, \boldsymbol{\rho}(s)\right)^{-1}\right]$$

$$\boldsymbol{t}^*(s)\, \boldsymbol{\rho}(s) \overset{s>s_{\rm th2}}{=} \begin{pmatrix} \dfrac{1-\eta\, e^{-2i\delta_1}}{2i} & \dfrac{\sqrt{1-\eta^2}}{2}\sqrt{\dfrac{\rho_2}{\rho_1}}\, e^{-i\delta_{12}} \\ \dfrac{\sqrt{1-\eta^2}}{2}\sqrt{\dfrac{\rho_1}{\rho_2}}\, e^{-i\delta_{12}} & \dfrac{1-\eta\, e^{-2i\delta_2}}{2i} \end{pmatrix}$$

- **Key points:**
  1). It is essential to work with partial waves that are free of kinematical constraints
  2). The real part equation is solved with the help of a singular-value decomposition [Moussallam:1999aq]

Numerics

Fit type	χ^2/N_{data}	Ref. [7]	Ref. [8]	This work
$\pi\pi$+$K\bar{K}$	$\pi\pi$	2.7	1.3	1.1
	$K\bar{K}$	12.4	4.5	1.0
	total (fit)	5.8	2.3	1.1
$\pi\pi$-only	$\pi\pi$ (fit)	2.7	1.1	1.1
	$K\bar{K}$ (post)	12.8	136	4.2
	total	5.9	44	2.1
$K\bar{K}$-only	$K\bar{K}$ (fit)	2.6	2.9	0.9
	$\pi\pi$ (post)	$\sim 10^6$	44	2.5
	total	$\sim 10^6$	31	2.0