Dispersive description of new subprocesses for tensor-meson contribution to muon $g-2$

Eirini Lymperiadou

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PSI

Universität Zürich[UZH]

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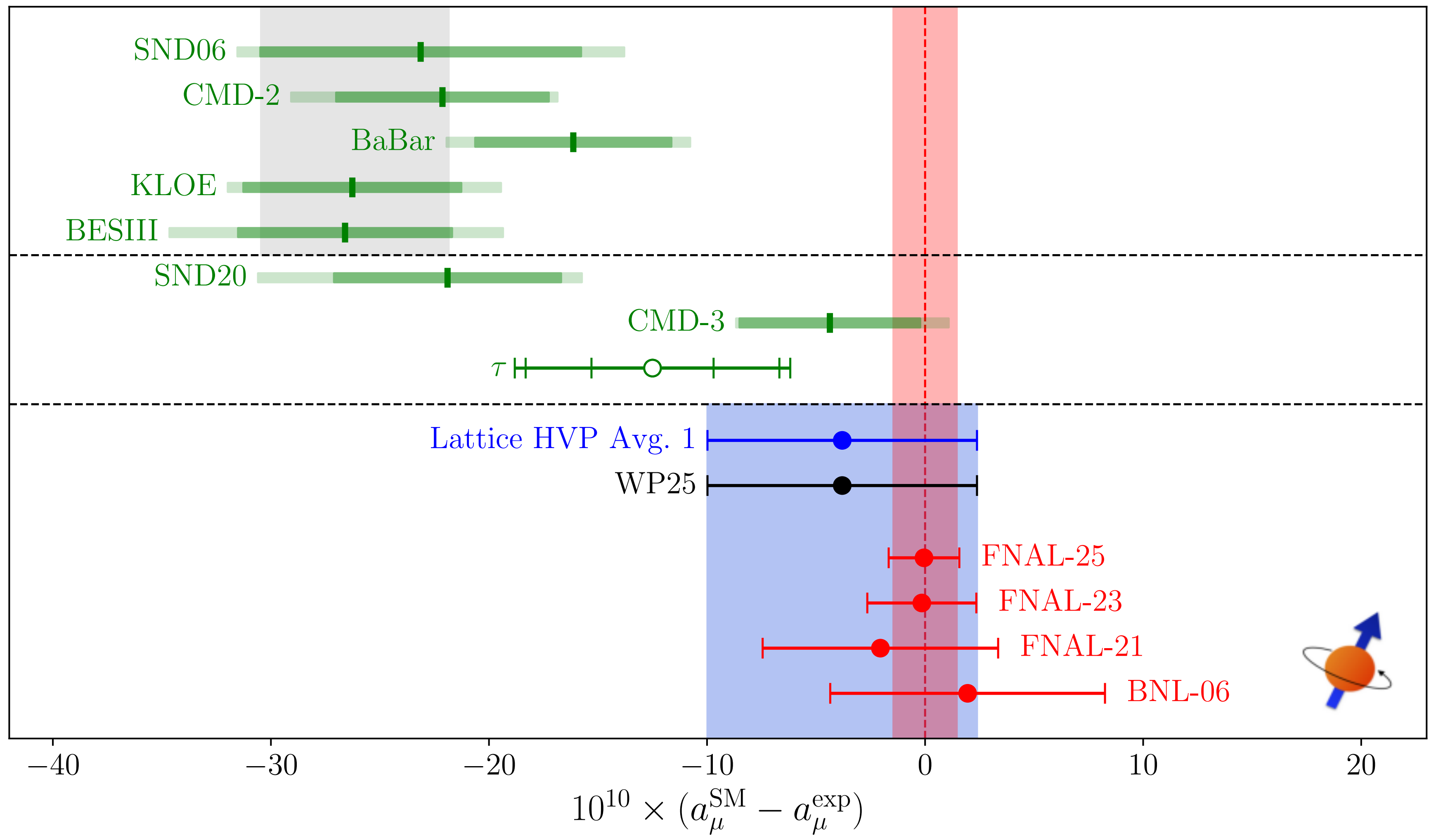
- Triangle kinematics framework
- Current work for contributing subprocesses

Motivation :

Anomalous magnetic moment of the muon (a_μ)

- Deviation between data-drive dispersive prediction & experimental results & lattice prediction
- Discrepancies among values within data-driven approach



[arXiv:2505.21476]

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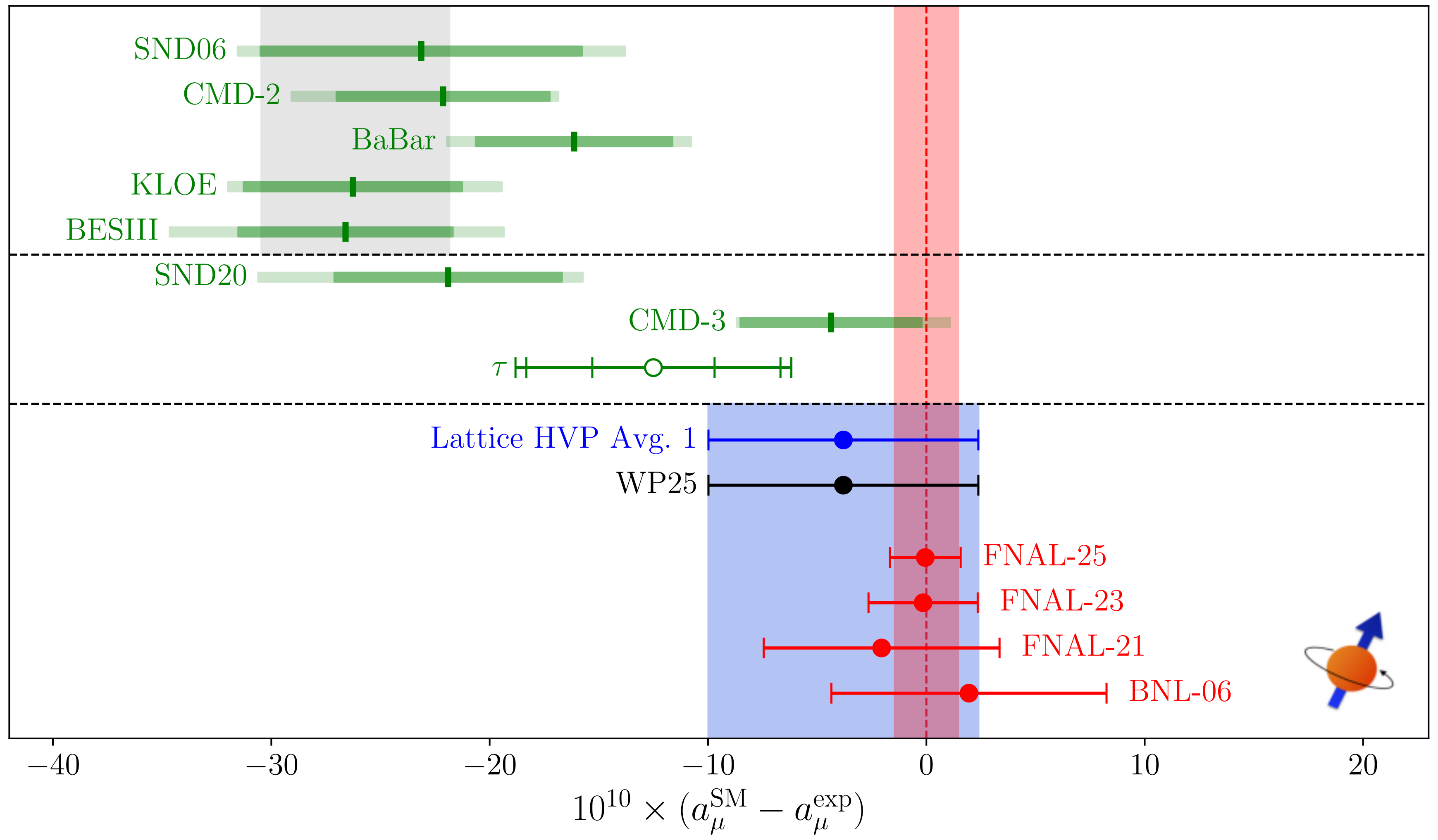
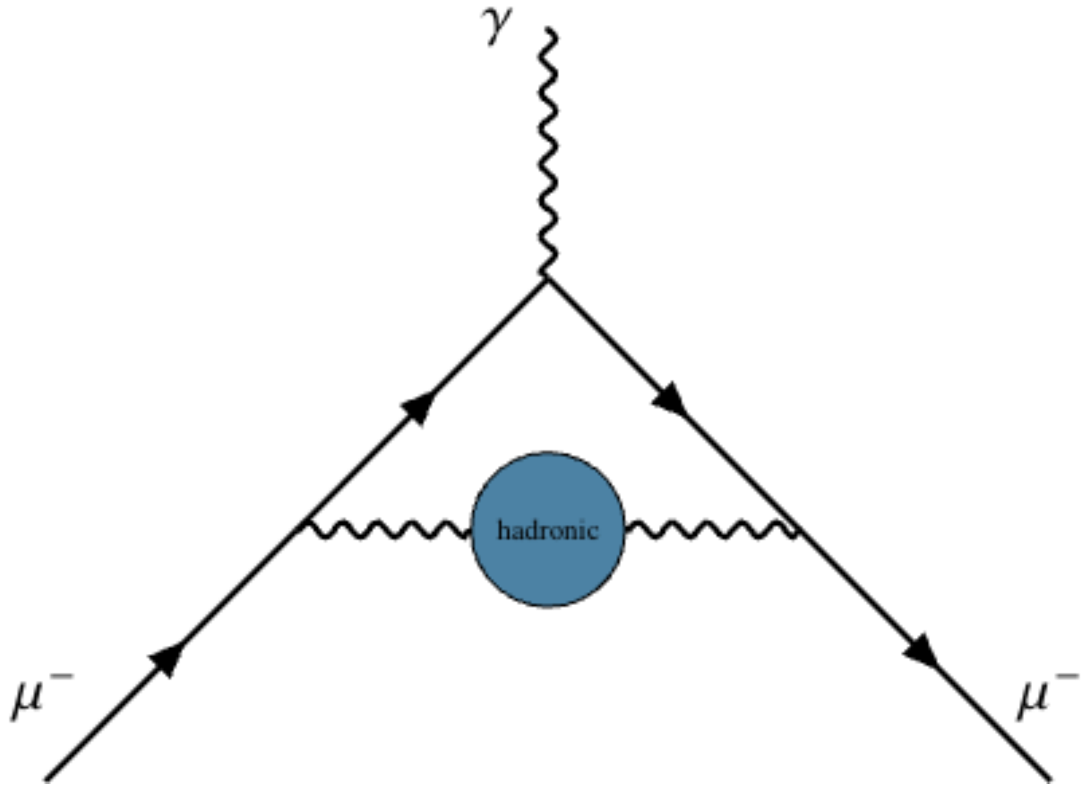


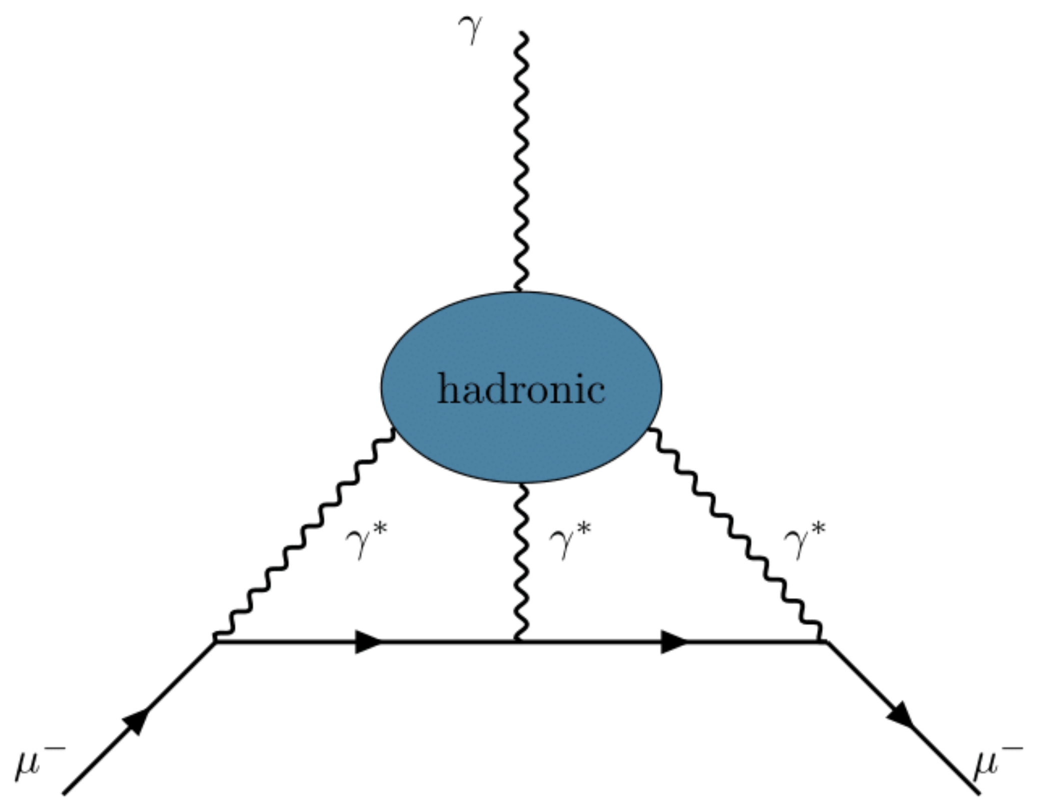
[arXiv:2505.21476]

- Source of error → hadronic part

  ↪ Hadronic vacuum polarization (HVP)

  ↪ Hadronic Light-by-Light (HLbL)

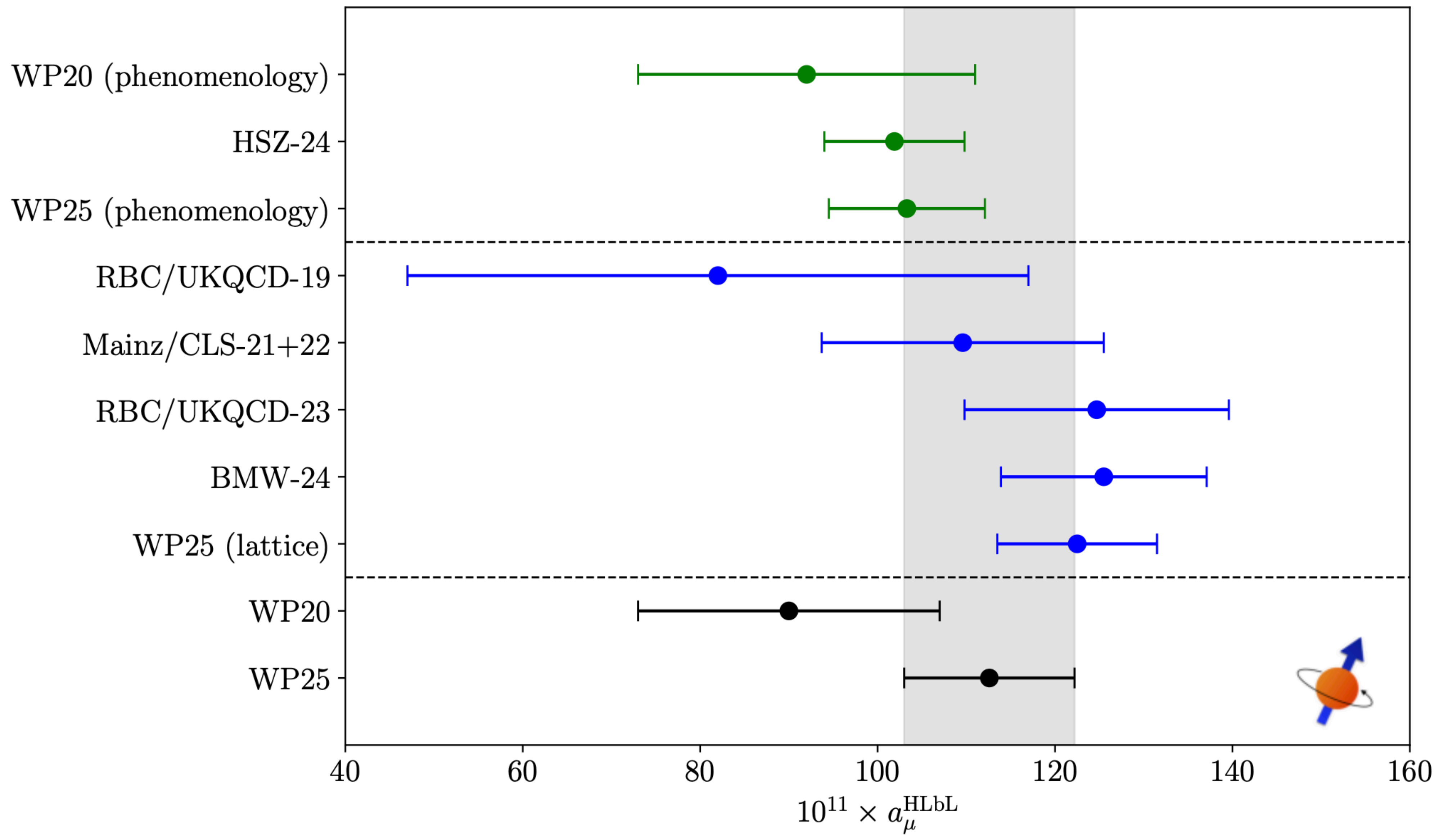




Motivation : tensor meson contribution

Agreement for HLbL

- necessary precision improvement
- still systematic deviations for certain contributions



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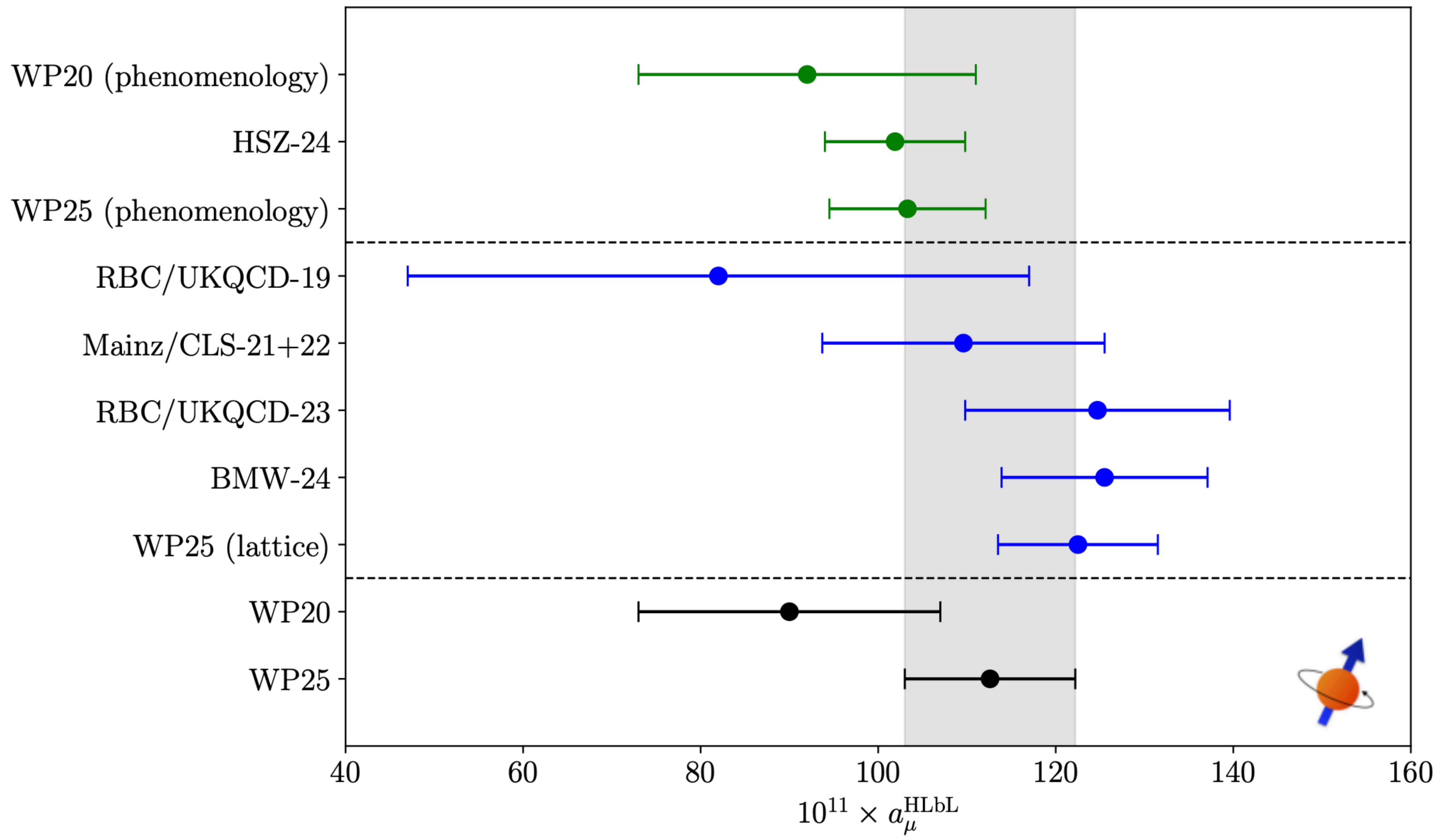
Tensor meson contribution

$$a_\mu^{\text{HLbL,tensor}} = -1(4) \times 10^{-11}$$

- no model-independent description so far



[arXiv:2505.21476]

Region		Dispersive	hQCD	Regge	DSE/BSE
$Q_i > Q_0$		$6.2^{+0.2}_{-0.3}$	6.3(7)	4.8(1)	2.3(1.5)
Mixed	A, S, T	3.8(1.5)			
	OPE	10.9(0.8)			
	Effective pole	1.2			
	Sum	15.9(1.7)	13.5(2.4)	12.8(5)	10.1(3.0)
$Q_i < Q_0$	$A = f_1, f_1', a_1$	12.2(4.3)	13.1(1.5)	10.9(1.0)	8.6(2.6)
	$S = f_0(1370), a_0(1450)$	−0.7(4)			−0.8(3)
	$T = f_2, a_2$	−2.5(8)	2.9(4)		
	Other	2.0	8.0(9)	3.2(6)	2.8(6)
	Sum	11.0(4.4)	24.0(2.8)	14.1(1.2)	10.6(2.7)
Sum		33.2(4.7)	43.8(5.9)	31.7(1.6)	23.0(7.4)

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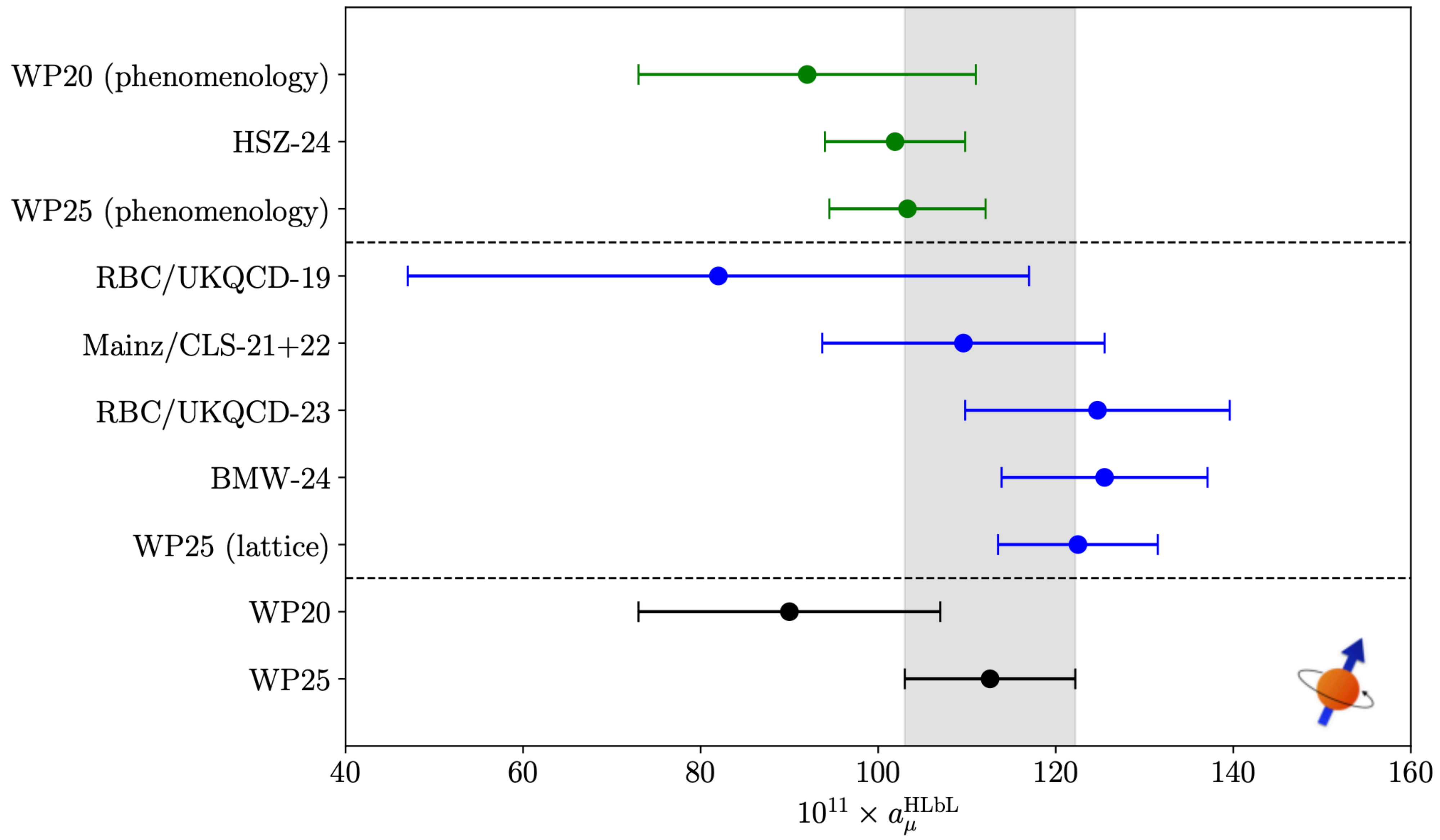
Construction of dispersive description

- minimize error to the $a_\mu^{\text{HLbL, tensor}}$



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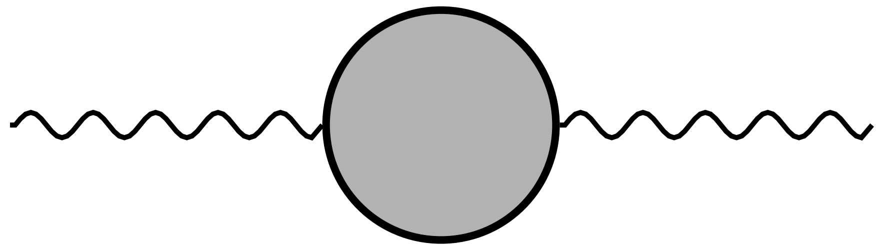
Importance of $\gamma^*\gamma^*\gamma \to \pi\pi$

- Data-driven approach for HVP : main error contribution



↪ cross section data $\sigma(e^+e^- \to \text{hadrons}(+\gamma))$

↪ radiative-return experiments : need of improvement for hadronic modeling for $e^+e^- \to \pi^+\pi^-\gamma$

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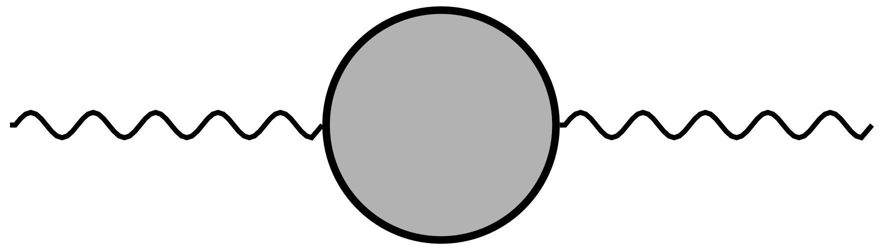

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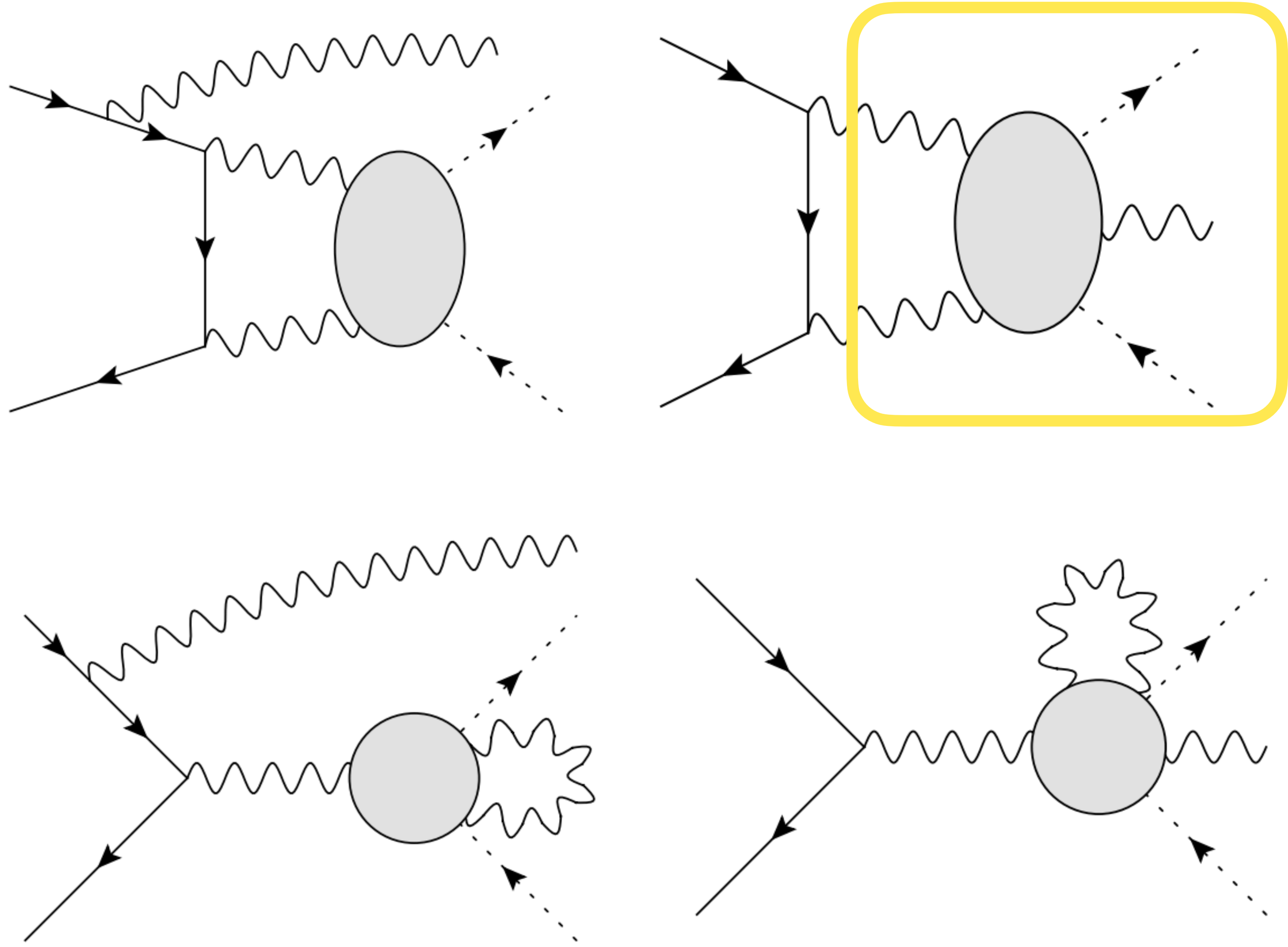
NLO terms : $\gamma^*\gamma^* \to \pi\pi\gamma$ (among others)

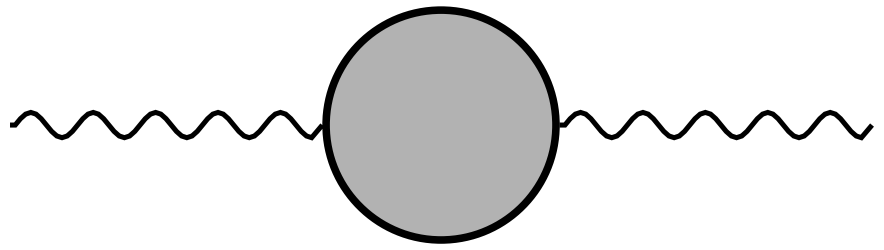
↪ FsQED : not sufficient, need to account for important intermediate states

↪ **no** dispersive description **so far**



[arXiv:2410.22882]

Importance of $\gamma^*\gamma^*\gamma \to \pi\pi$



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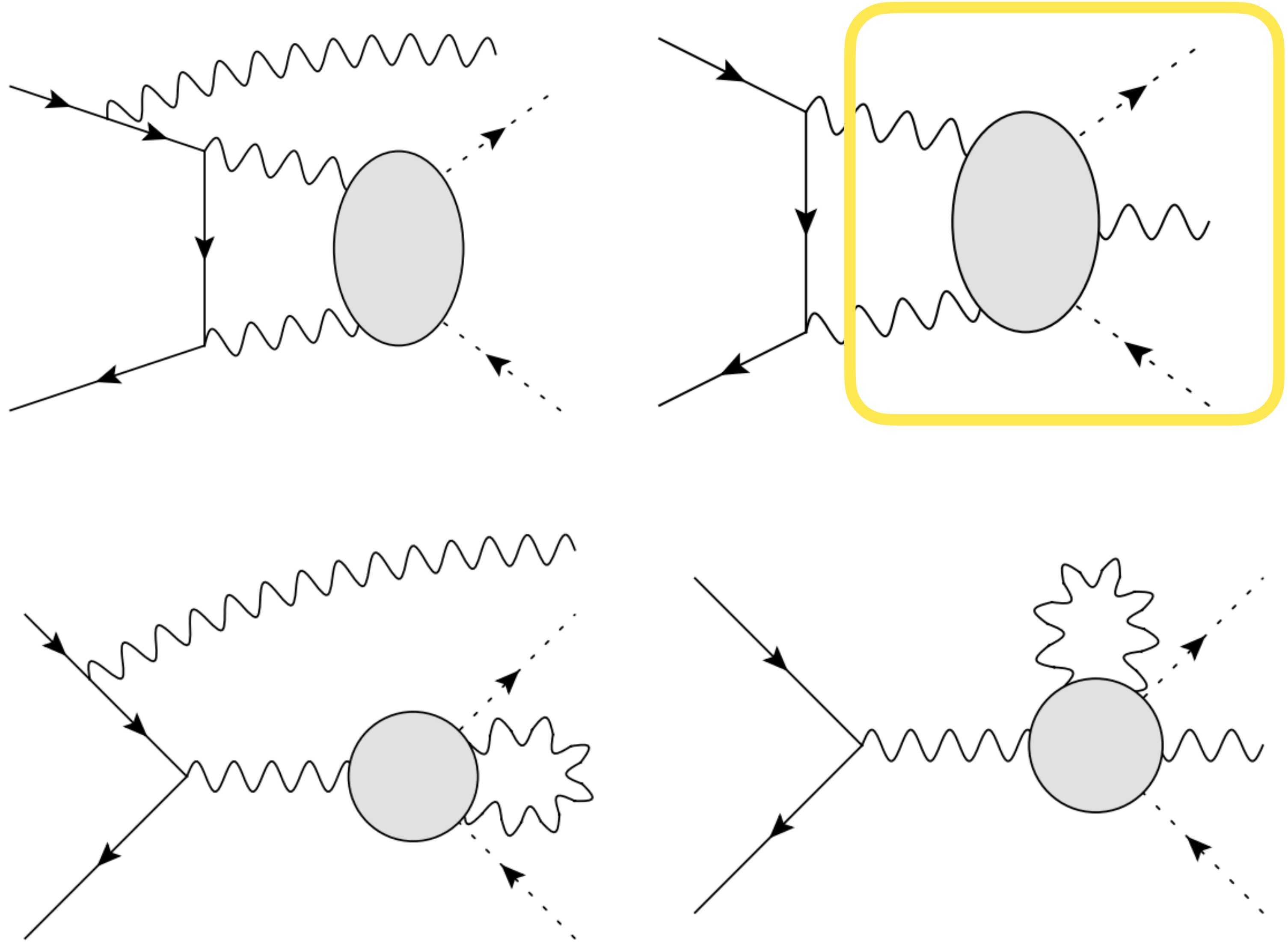

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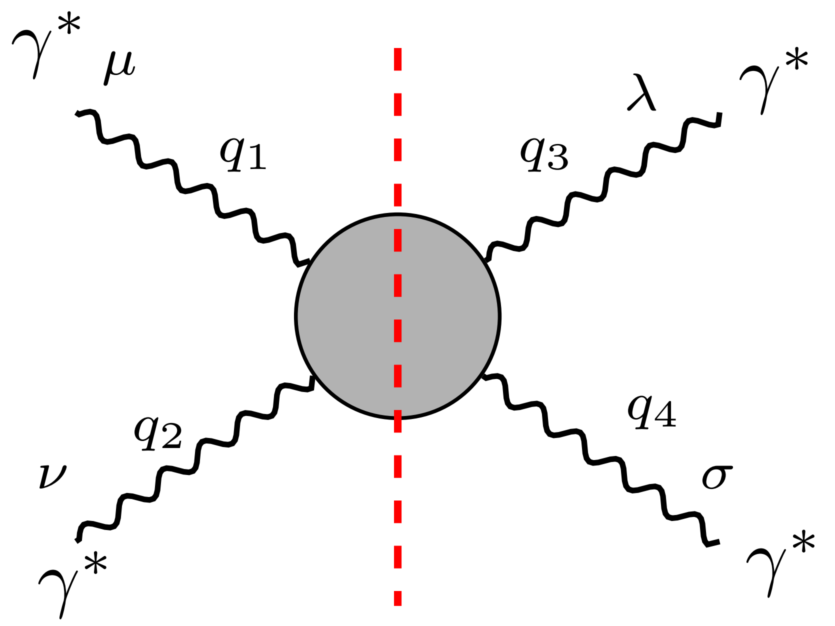
- Start with $\gamma^*\gamma^*\gamma \to \pi\pi$ in HLbL (**soft photon case**)
  - ↪ focus in the tensor-meson effects
  - ↪ **then** consider hard photon case : **useful for HVP**

HLbL : 4-point vs triangle kinematics framework

Usual approach → dispersions relations for HLbL tensor in general **four-point kinematics**

- dispersion relations for singly-on-shell limit for basis of scalar functions



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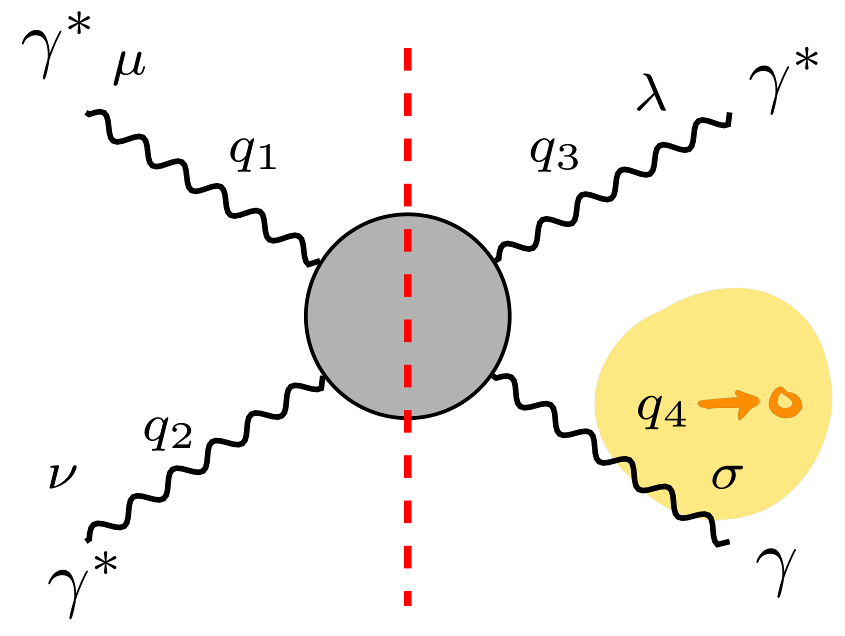

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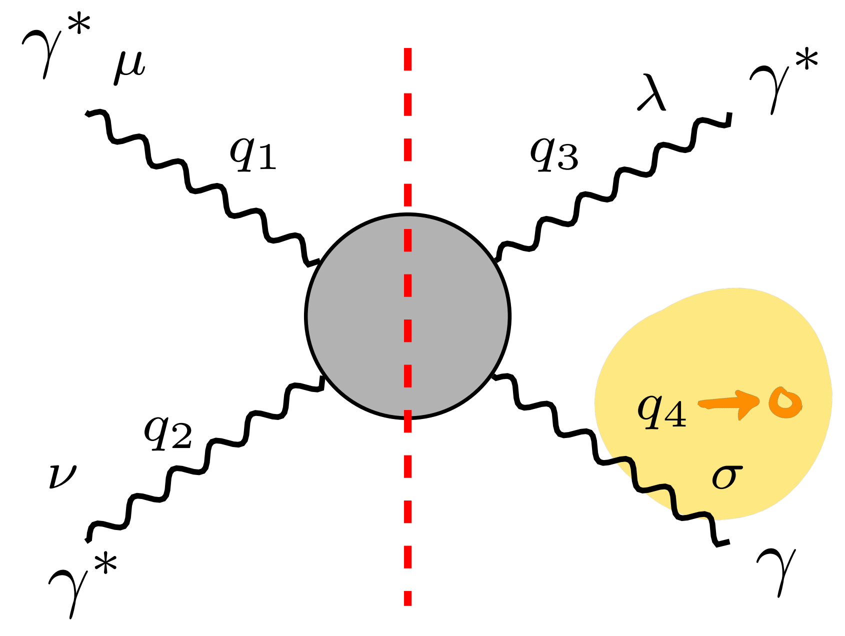

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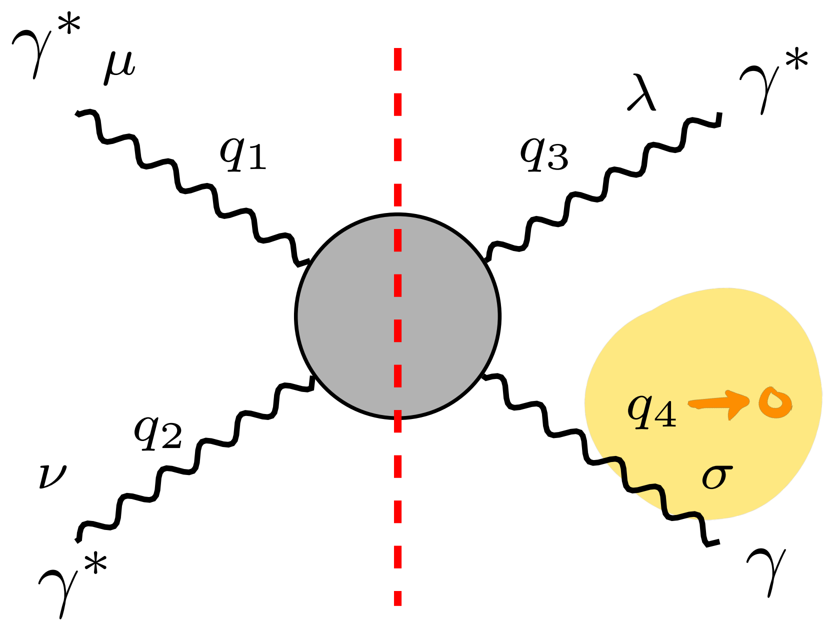

New approach → dispersions relations for HLbL tensor in **three-point kinematics**

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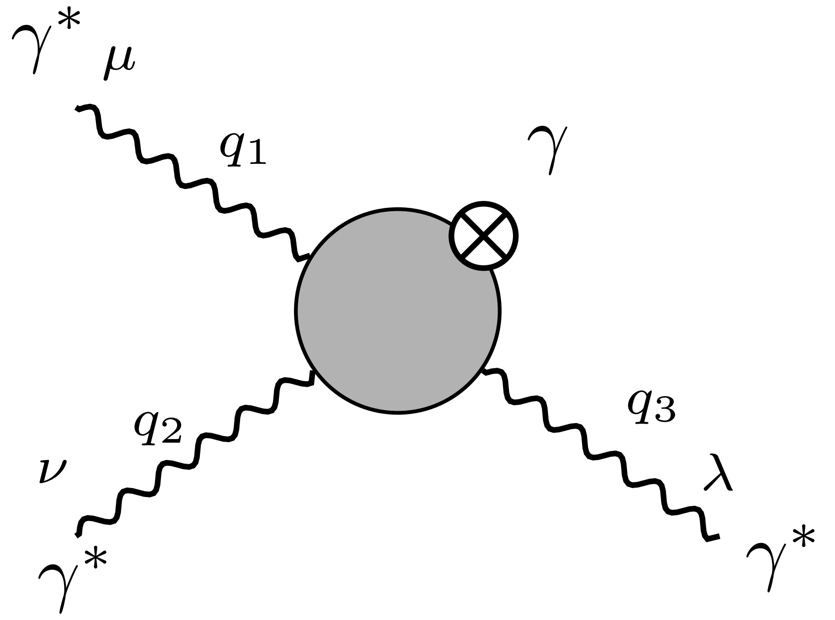

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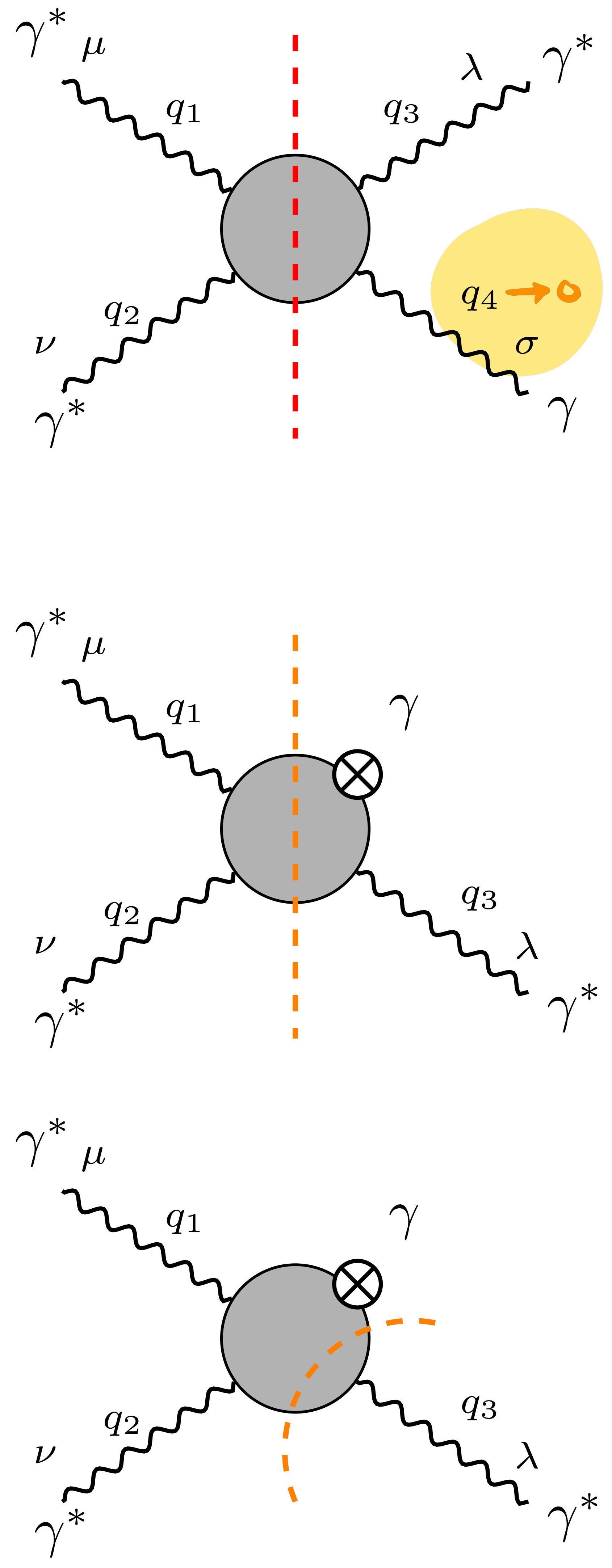
New approach → dispersions relations for HLbL tensor in **three-point kinematics**

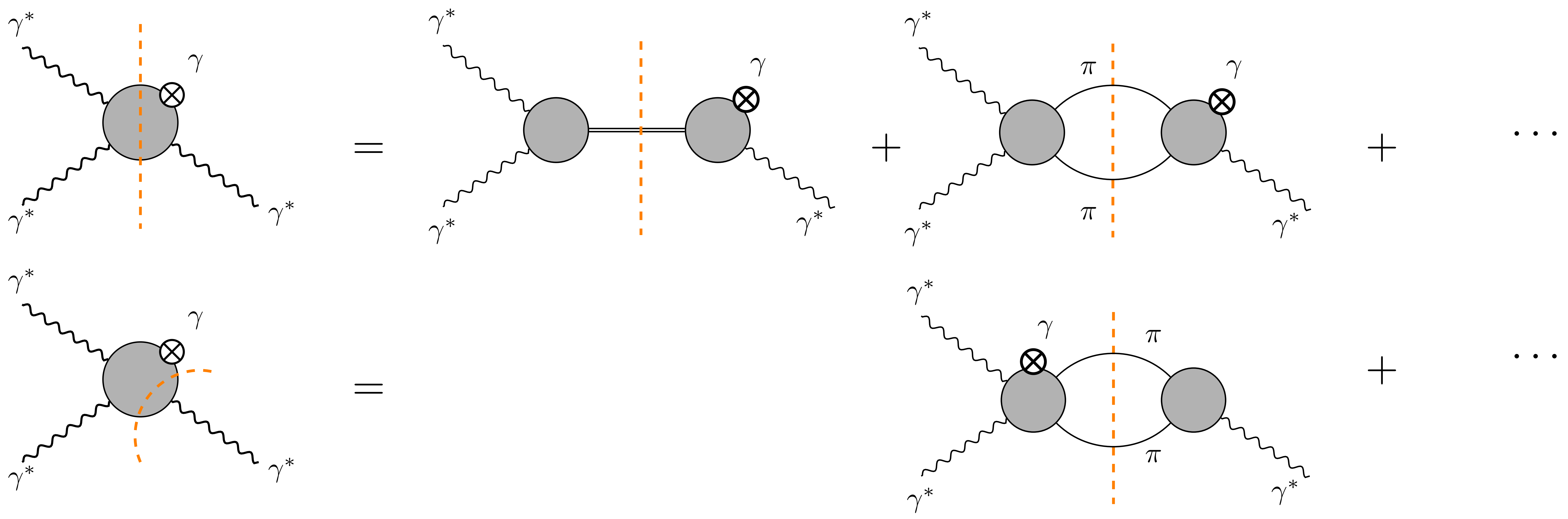
- first take $q_4 \to 0$ limit then write dispersion relations

↪ need to take both cuts → reconstruct additional hadronic sub-processes

↪ no spurious singularities → evaluation of additional contributions



Equivalent unitary cuts in **three-point kinematics**



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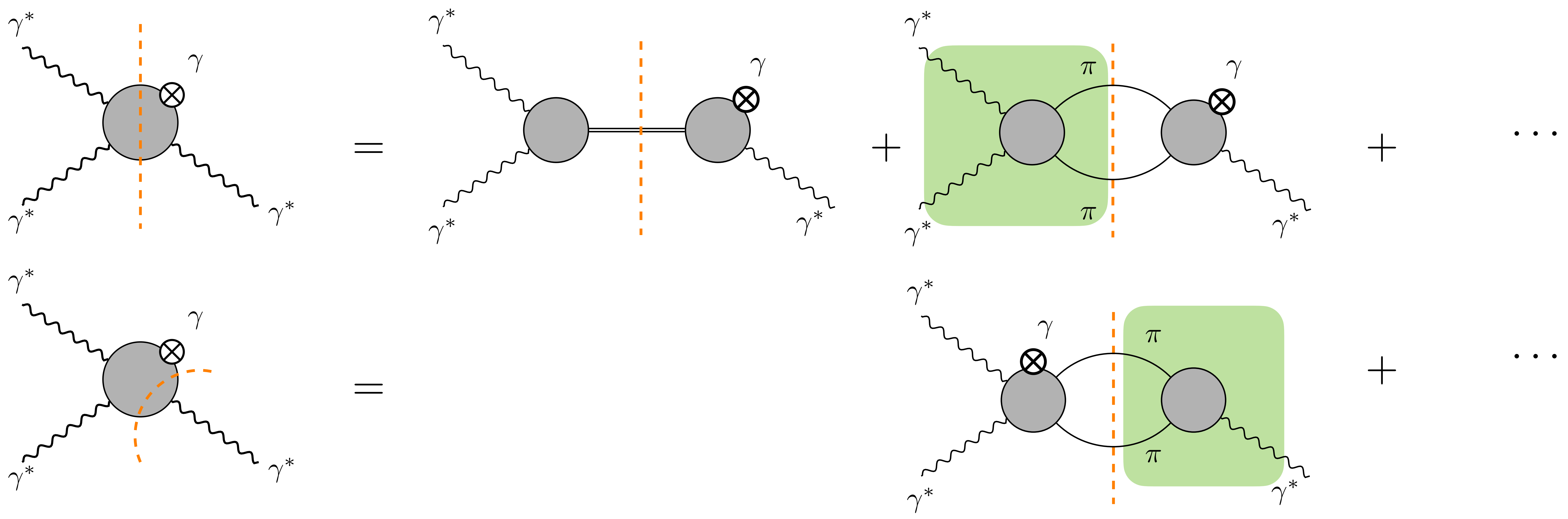

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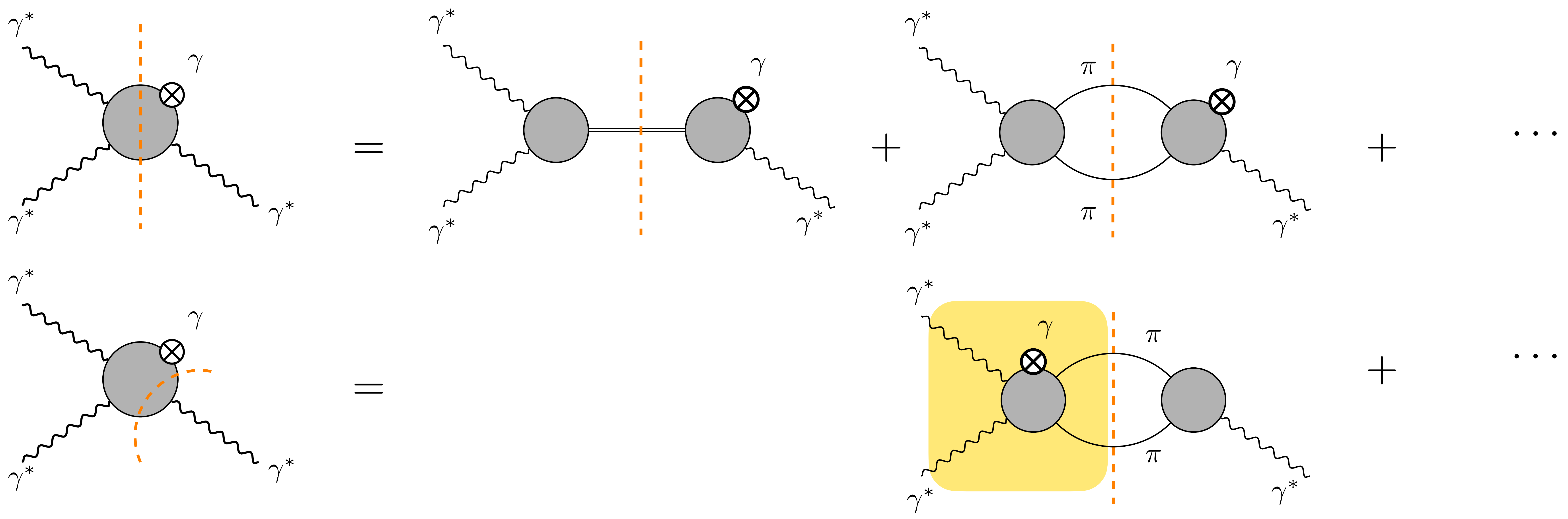

- 5-particle process $\gamma^*(q_1)\gamma^*(q_2)\gamma^*(q_3) \to \pi(p_1)\pi(p_2)$: **no** helicity partial-wave expansion

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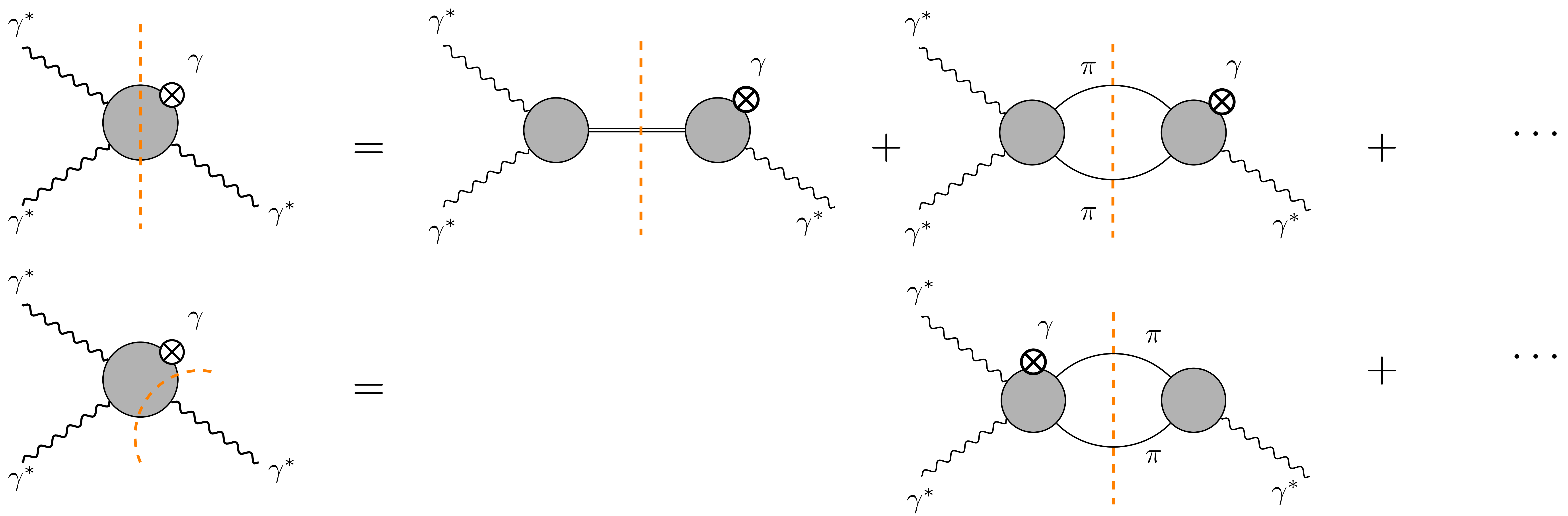

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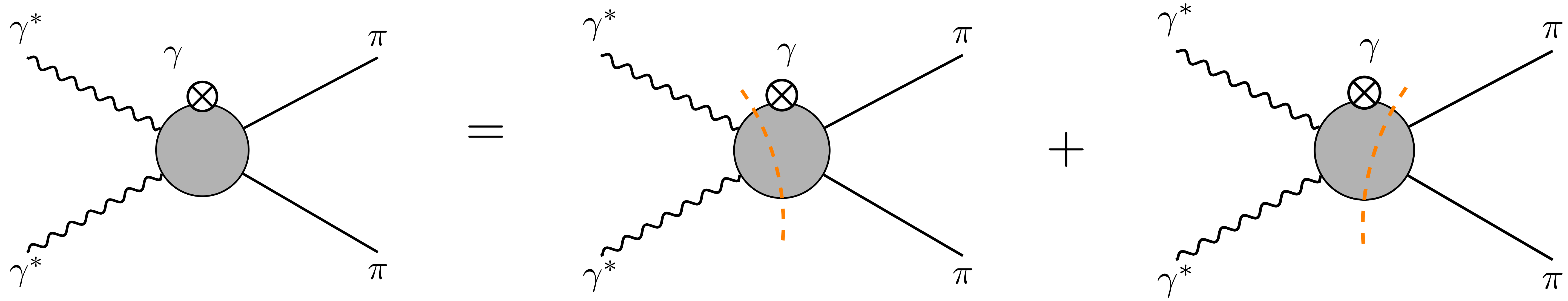
  ↪ reconstruct dispersively via hadronic sub-processes

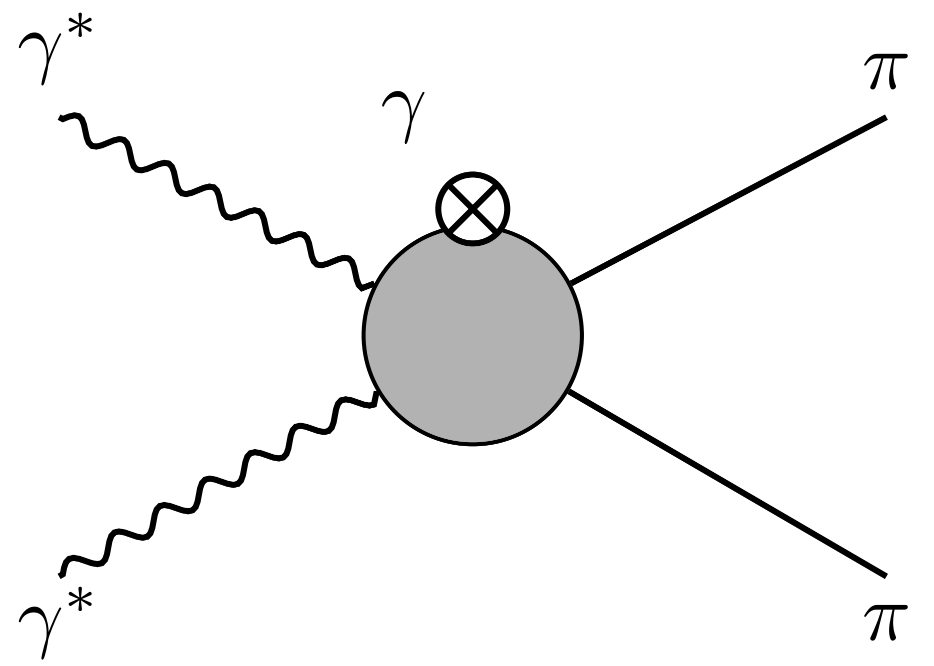


Subprocesses for $\gamma^*\gamma^*\gamma \to \pi\pi$

- Basis for $\gamma^*\gamma^*\gamma \to \pi\pi$ in the soft photon limit : **constructed**

$$\frac{\partial}{\partial q_{3\sigma}} \mathcal{M}^{\mu\nu\lambda}_{\text{reg}}(p_1, p_2, q_1, q_2)\Big|_{q_3=0} = \sum_{i=1}^{27} \mathcal{B}_i^{\mu\nu\lambda;\sigma}(q_1, q_2, q_5)\bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2)$$

- Projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$: **constructed**



[arXiv:2302.12264]

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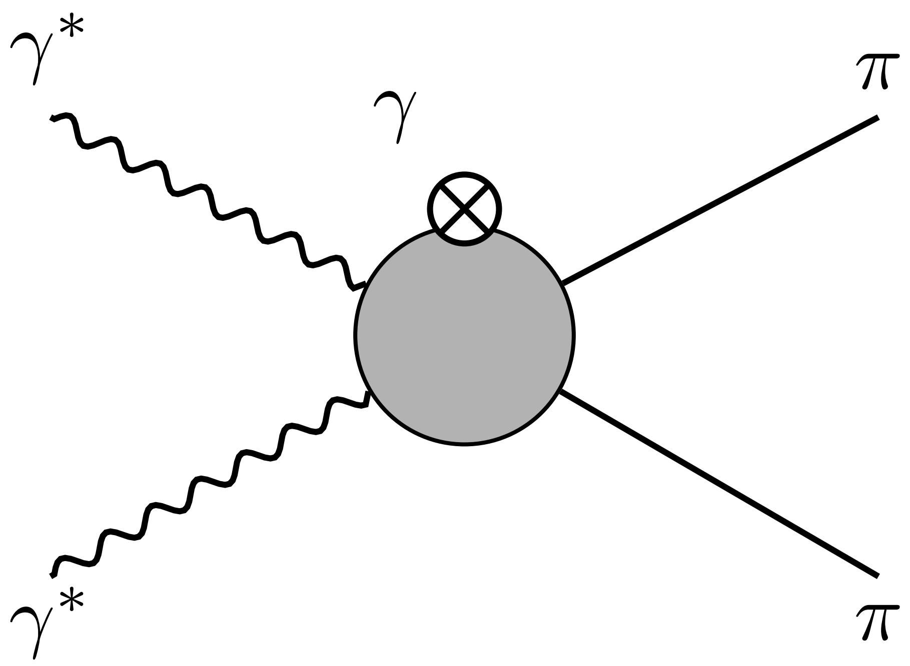

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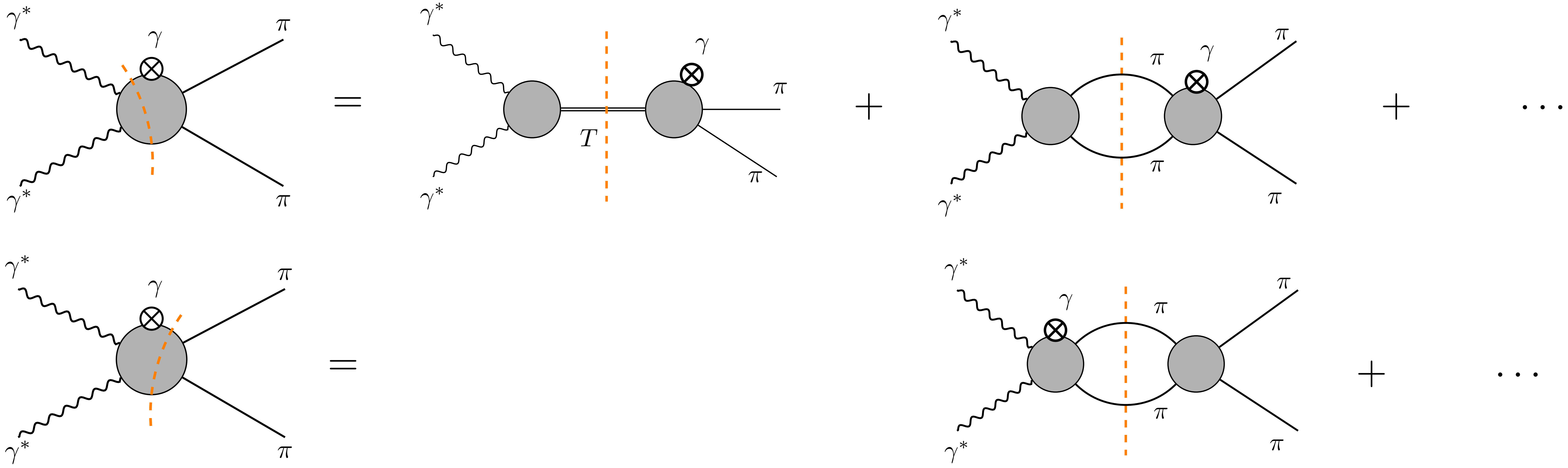
- Contributions :



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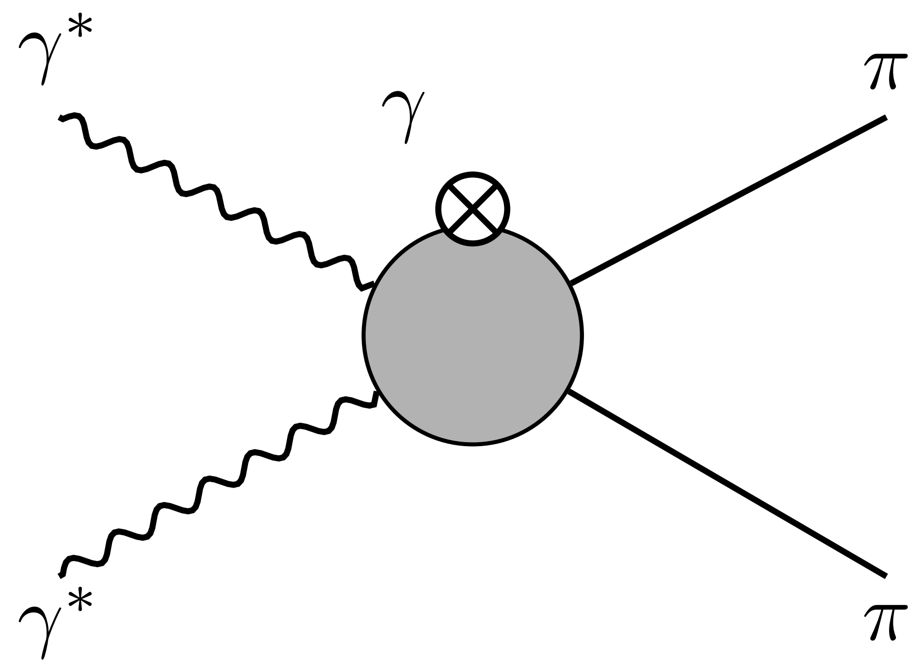

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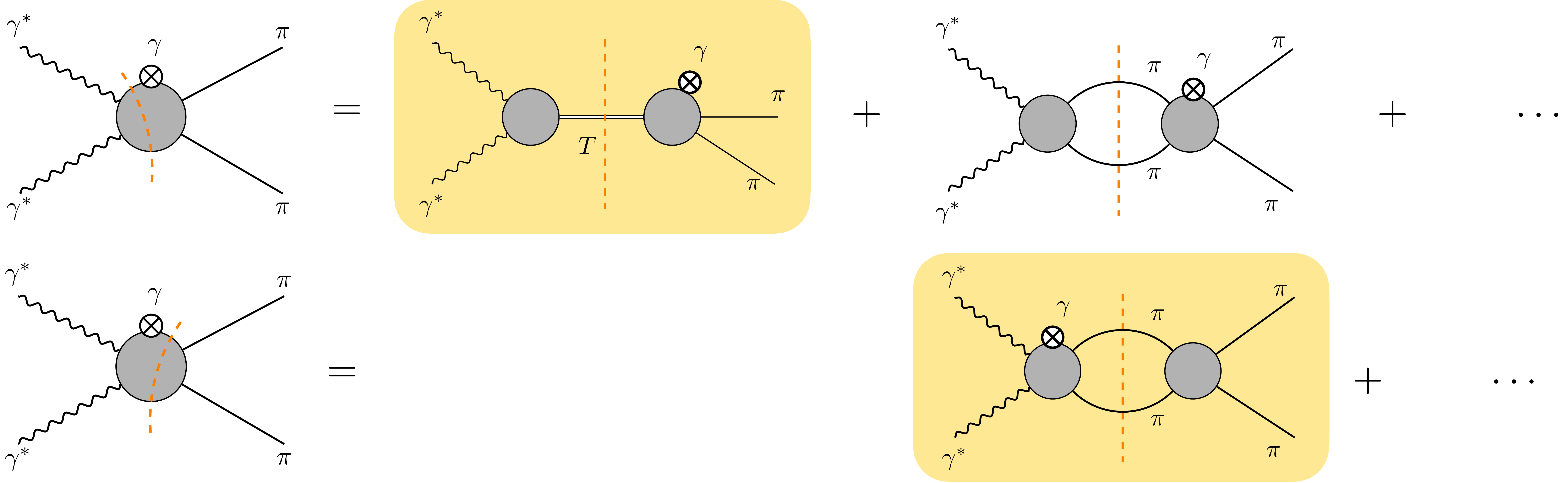
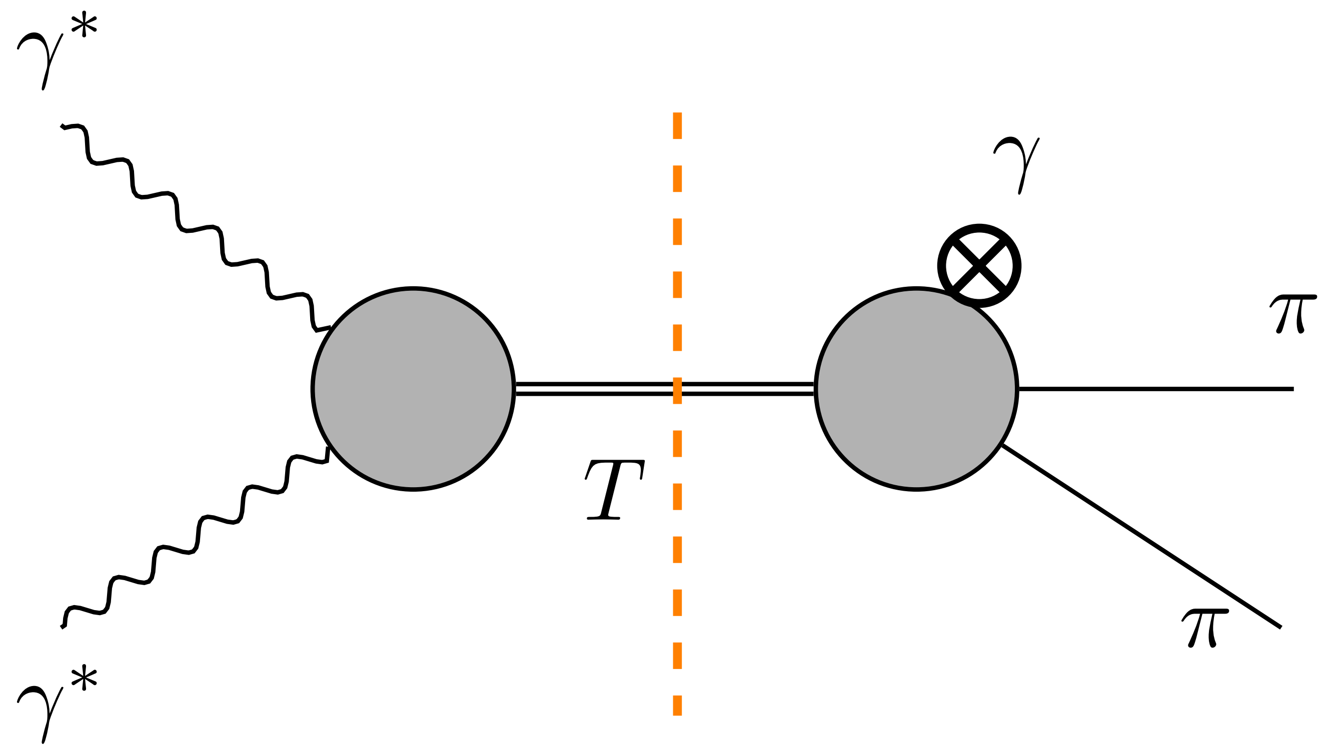
- Contributions : tensor-meson pole & two-pion cut



The tensor-meson cut

Construct bases for sub-processes :



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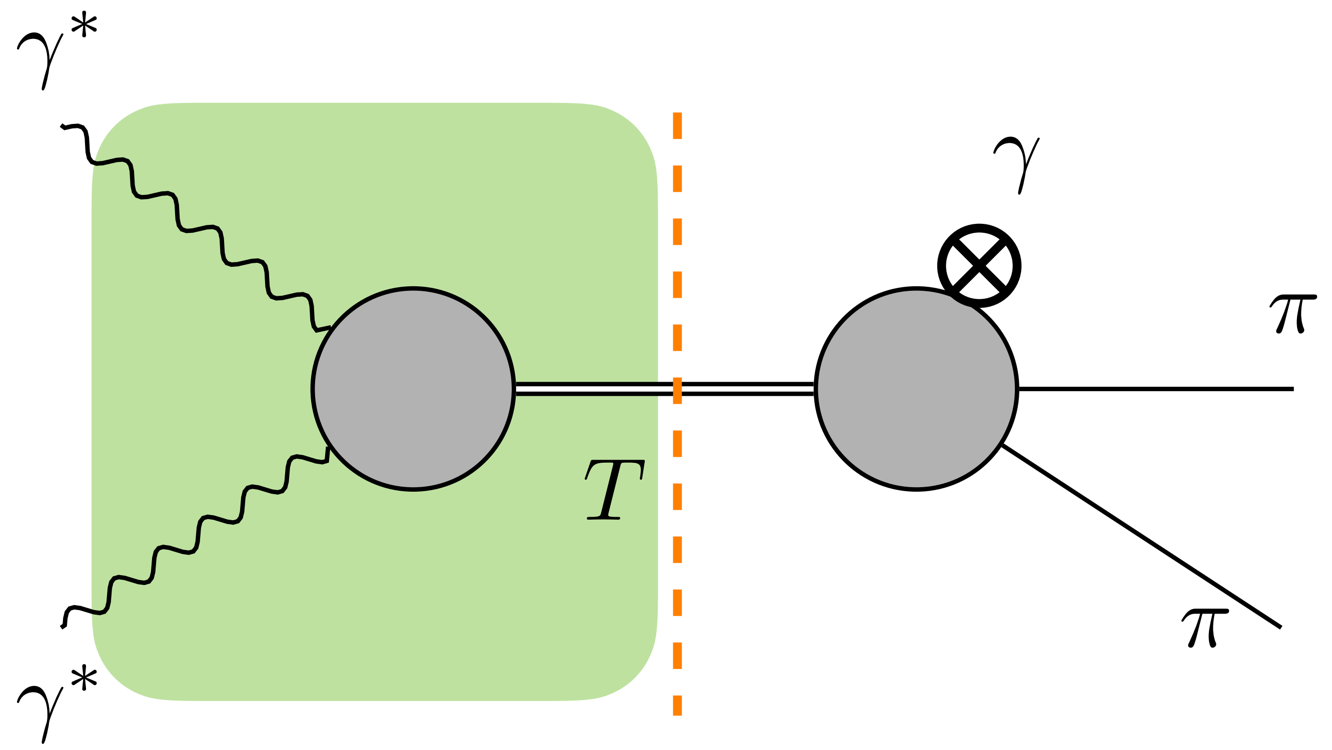
Construct bases for sub-processes :

- tensor meson TFFs [arXiv:2004.06127] : **5 BTT**



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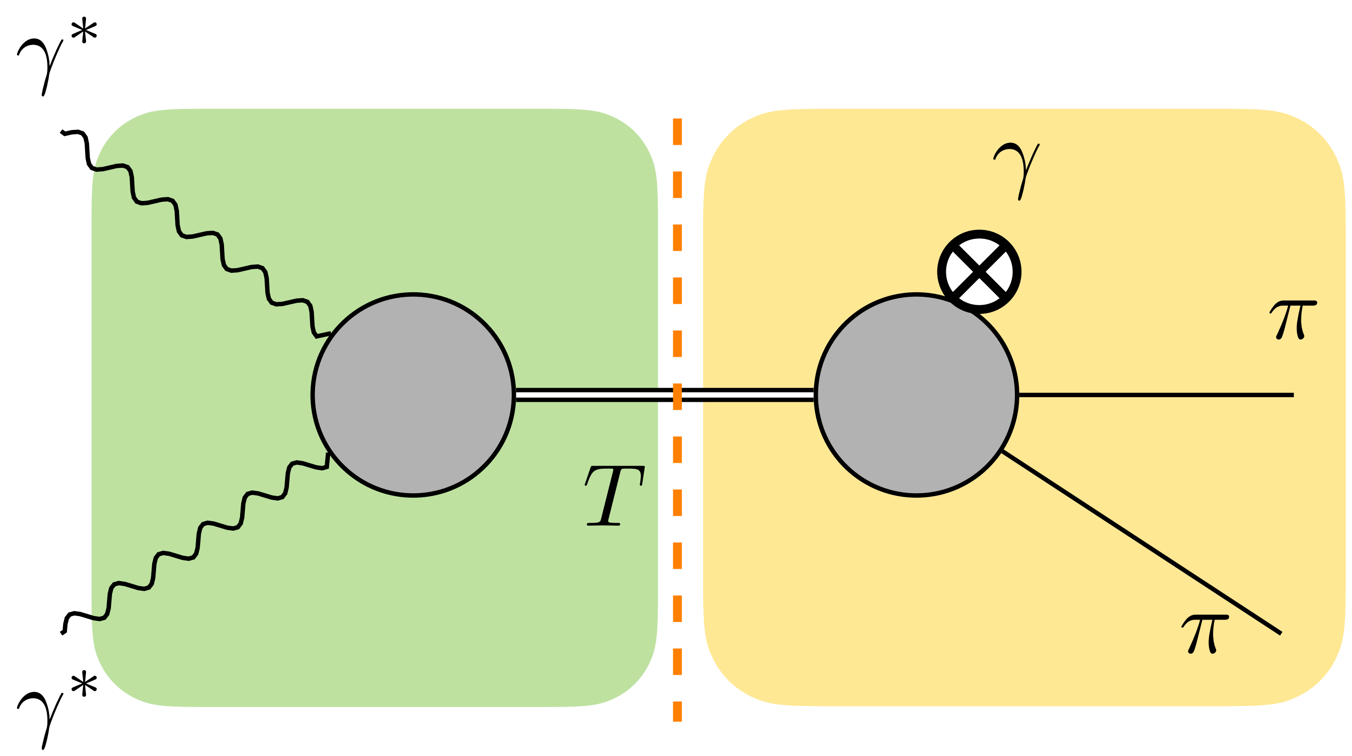

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- description of scattering amplitude for $T\gamma^* \to \pi\pi$

  ↪ **BTT** prescription : gauge invariance, remove poles, eliminate redundancies
  8 structures

  ↪ in the soft-photon limit: **only 2** survive

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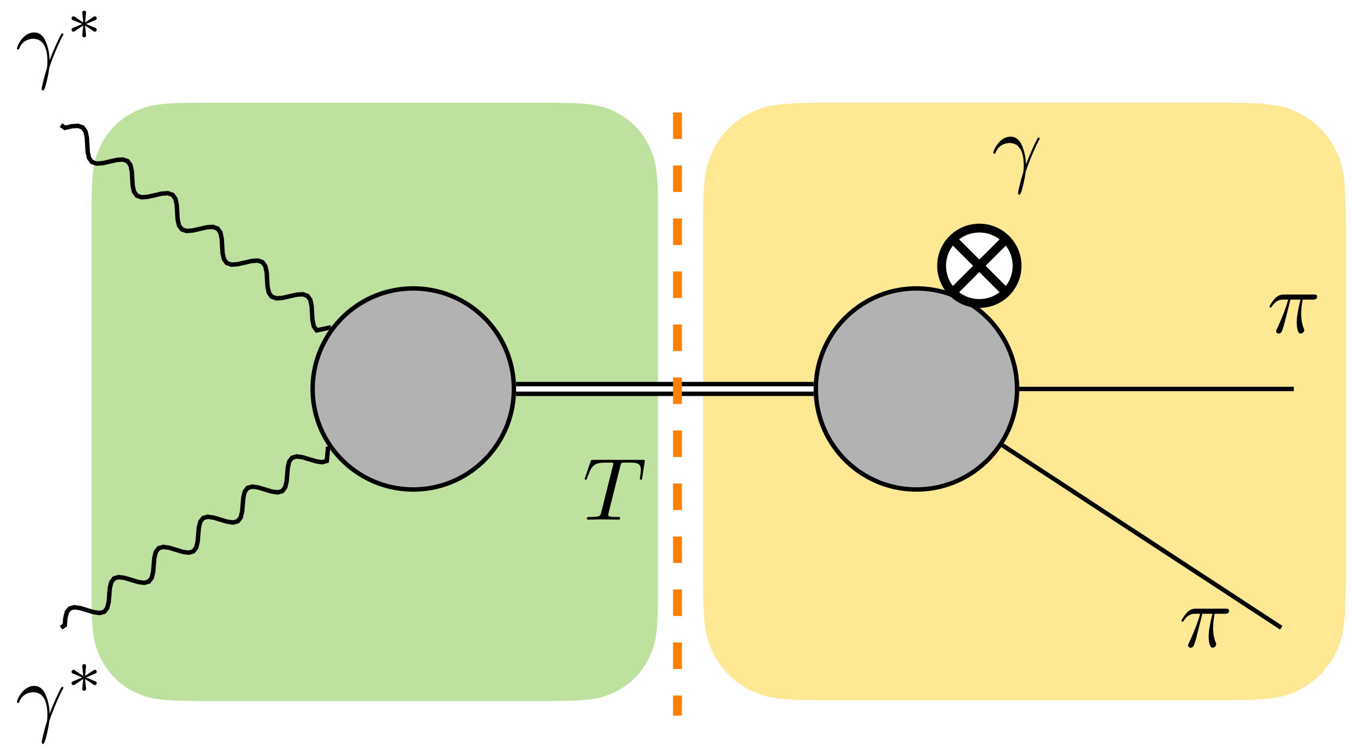

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Compute imaginary parts:

$$\mathrm{Im}_s^{\mathrm{T}}\, \bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2) = \pi\delta(s - m_T^2) \sum_{j=1}^{5} \sum_{k=1}^{8} \frac{1}{m_{\mathrm{T}}^3} t^i_{j,k}\, \mathcal{F}_j^T(q_1^2, q_2^2) \hat{\mathcal{F}}_k(q_5^2, 0)$$

- dispersion relations for contribution : possibly **reconstruction theorem**

The two-pion cut

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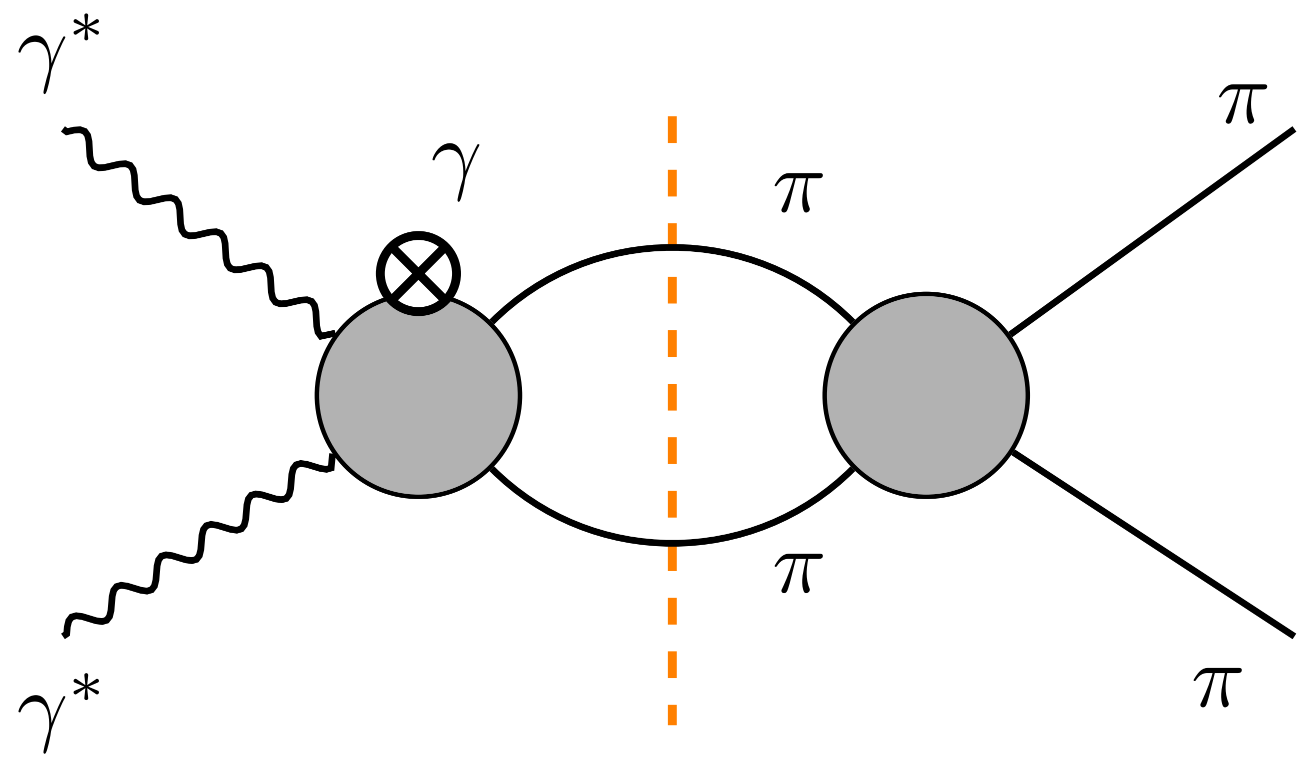
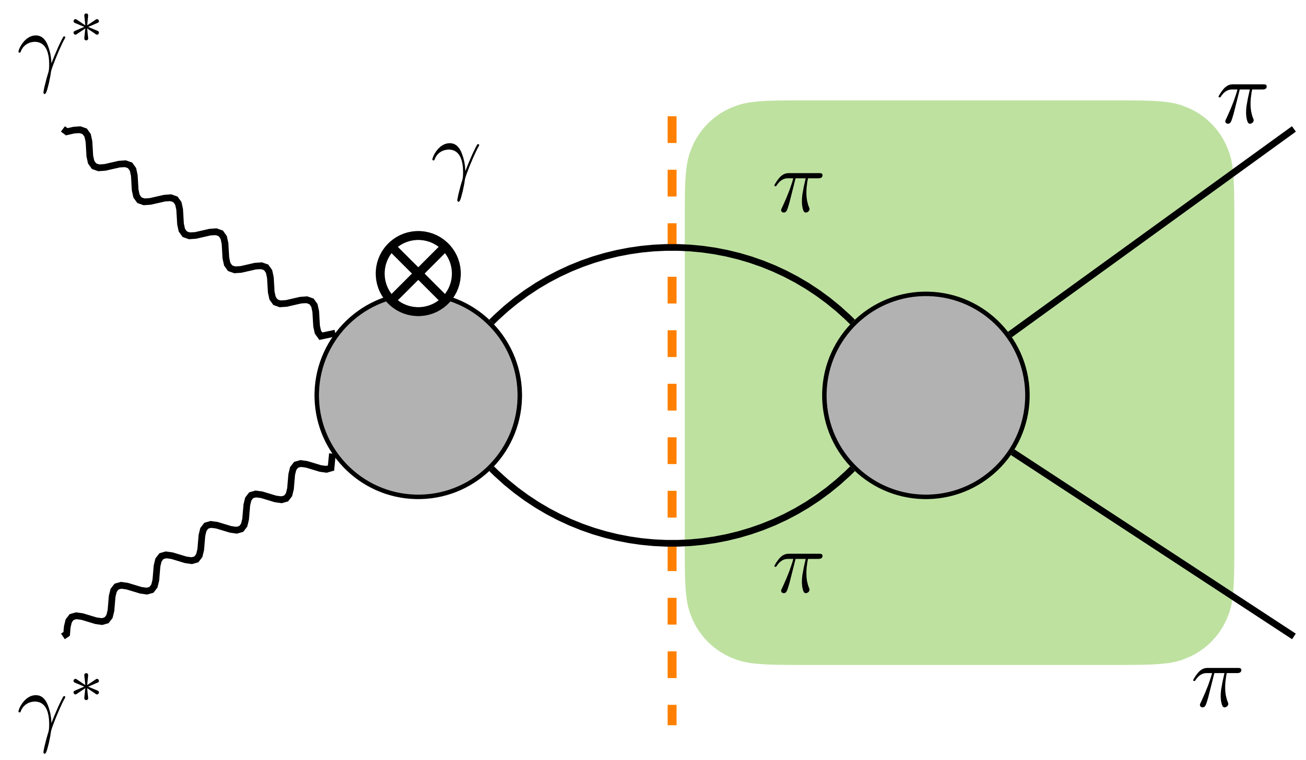


The two-pion cut

Construct bases for sub-processes :

- $\pi\pi-$scattering **:** partial-wave decomposition for fixed isospin I amplitude

  ↪ for $\gamma^*\gamma^*\gamma \to \pi\pi$: only **odd** partial waves



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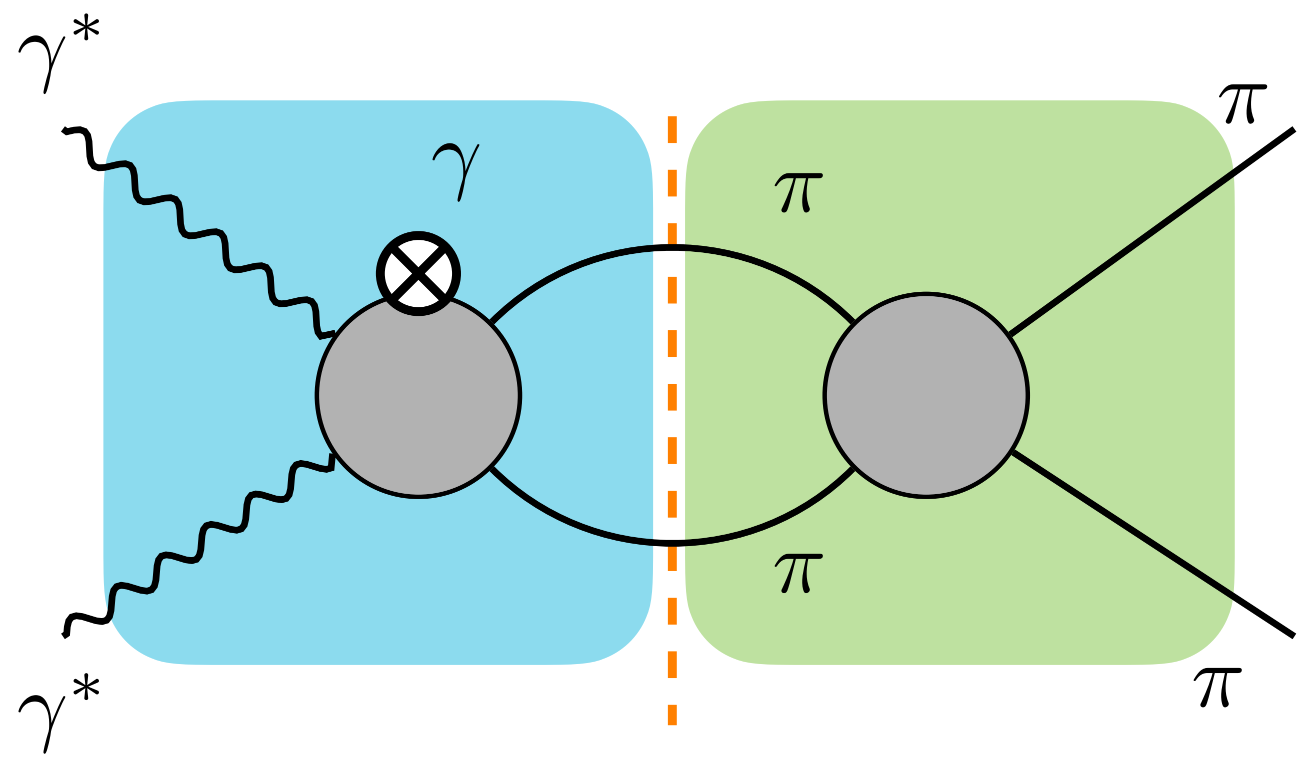
- sub-process $\gamma^*\gamma^*\gamma \to \pi\pi$: **Omnès problem**

  ↪ **no** helicity partial-wave expansion : general expansion over $P_j(z)$

$$\bar{A}_i = \sum_j a_j^i(s) \cdot P_j(z)$$



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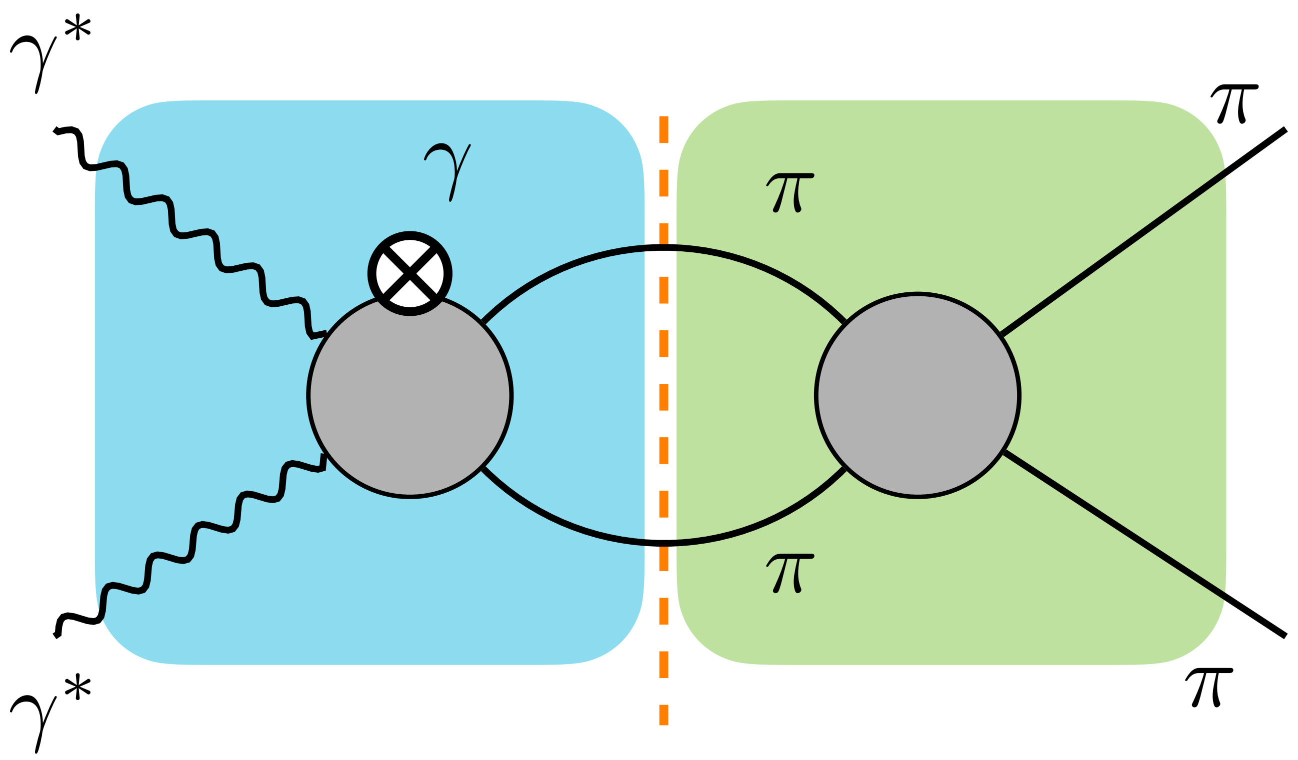
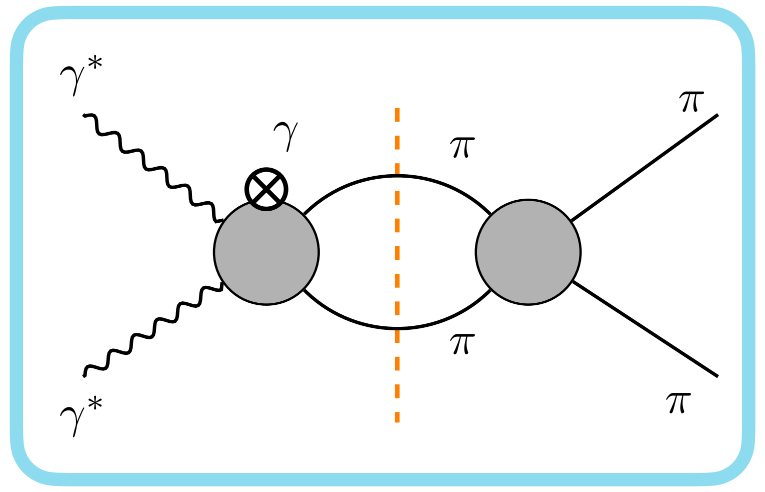
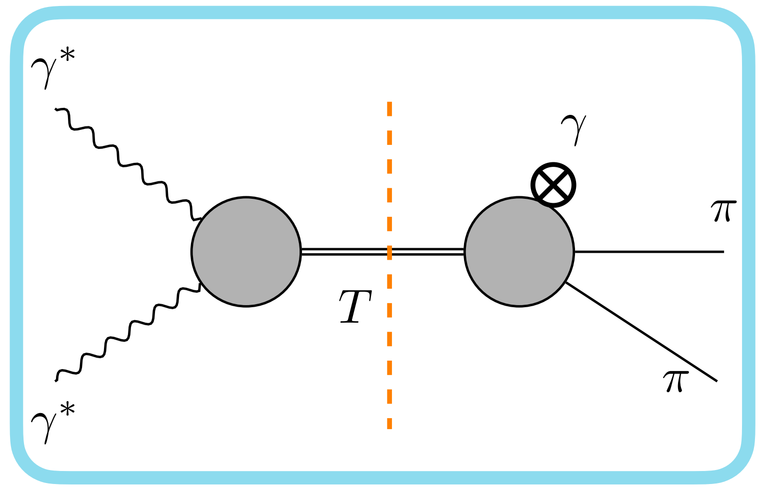
↪ inhomogeneous contribution from tensor-pole



$$\bar{A}^{\text{full}}_{i'}(s,z') = \bar{A}^{\text{resc.}}_{i'}(s,z') + \bar{A}^{\text{tensor}}_{i}(s,z')$$





Compute imaginary parts: for only P-wave contributing to $\pi\pi$—scattering

$$\mathrm{Im}_s^{\pi\pi}\bar{\mathcal{A}}_i = \mathrm{Im}_s^{\pi\pi} a_0^{i\,\mathrm{resc.}}(s) = \left[a_0^{i\,\mathrm{resc.}}(s) + \hat{F}_i^{\mathrm{inh.}}(s)\right] \cdot \sin\delta_1^1(s) e^{-i\delta_1^1(s)}$$

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- scalar function contributions : inhomogeneous Omnès solution

$$\bar{\mathcal{A}}_i^{\mathrm{resc.}} = \Omega_1(s)\left[P(s) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_1^1(s')}{|\Omega_1(s')|(s'-s)}\hat{F}_i^{\mathrm{inh.}}(s')\right]$$

, with $\hat{F}_i^{\mathrm{inh.}}(s) = \bar{\mathcal{A}}_i^{\mathrm{tensor,P}}$

↪ account for asymptotic behavior

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$$\mathrm{Im}_s^{\mathrm{T}}\,\bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2) = \pi\delta(s - m_T^2)\sum_{j=1}^{5}\sum_{k=1}^{8}\frac{1}{m_{\mathrm{T}}^3} t_{j,k}^i\,\mathcal{F}_j^T(q_1^2, q_2^2)\hat{\mathcal{F}}_k(q_5^2, 0)$$

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↪ only s-dependence (for $J = 1$)

$$\hat{\mathcal{F}}_i^{\mathrm{inh.}}(s) = \bar{\mathcal{A}}_i^{\mathrm{tensor,P}}(s) = \frac{\pi}{m_T^3(s-m_T^2)}\sum_{j=1}^{5} t_{j,1}^i \mathcal{F}_j^T(q_1^2, q_2^2)\hat{\mathcal{F}}_1(m_T^2, 0)$$

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↪ need TFFs and $\hat{\mathcal{F}}_k$ → Emilis' talk

Conclusions & Outlook

- Dispersive construction of scalar functions for $\gamma^*\gamma^*\gamma \to \pi\pi$: tensor-meson pole & two-pion cut

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✦ Fully study and understand the results of scalar functions $\bar{\mathcal{A}}_i$

  ↪ compute contribution to a_μ^{HLbL} (including tensor-meson effects)

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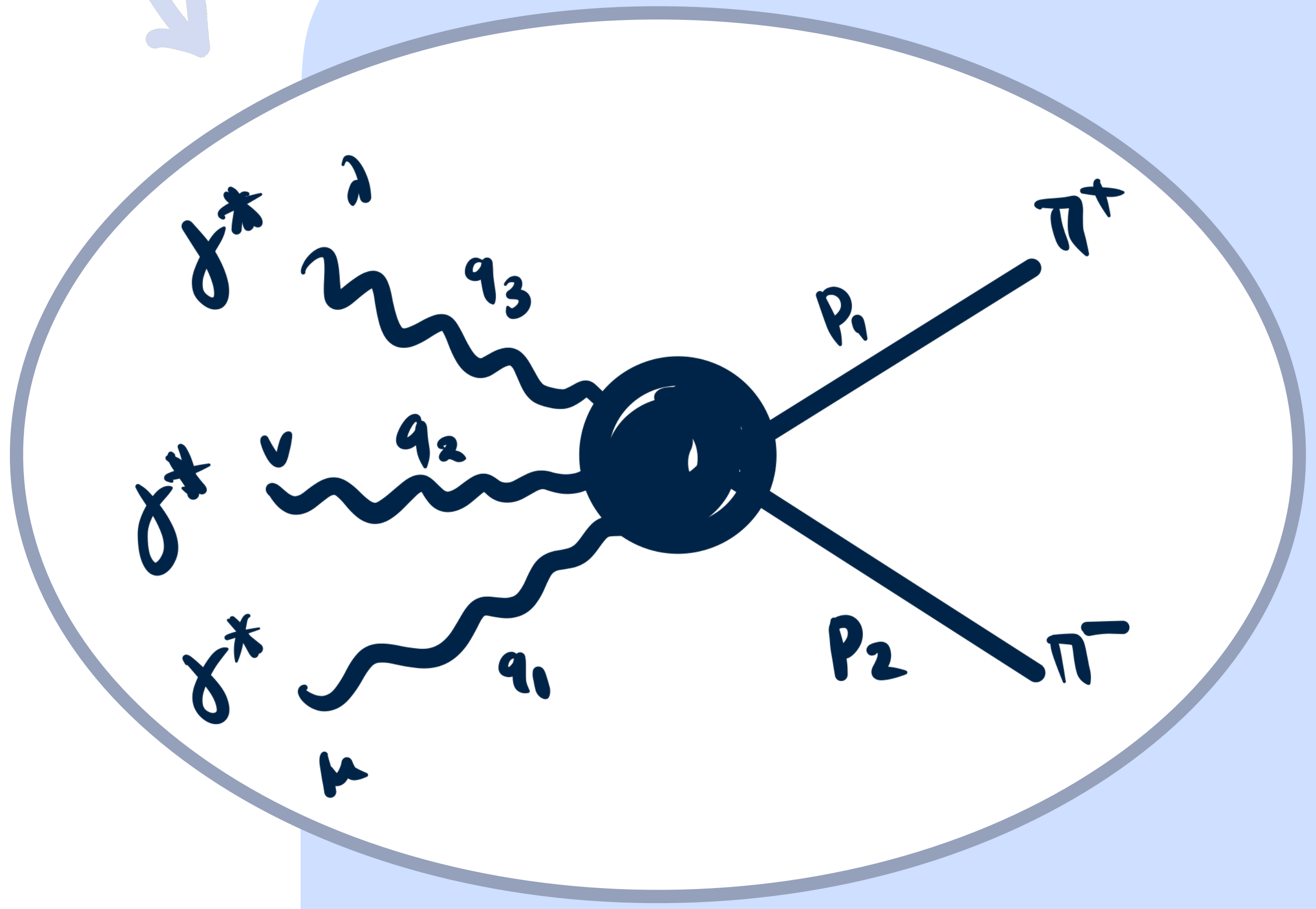
✦ Steps towards the hard-photon case : 5-particle process

  ↪ useful hadronic modeling for radiative corrections

Thank you for your attention!

B1. Bardeen-Tung-Tarrach

- Bardeen-Tung-Tarrach (BTT) prescription for the scattering amplitude



Lorentz decomposition of the amplitude $M^{\mu\nu\lambda} = \sum_i T_i^{\mu\nu\lambda} A_i$

1. consider all possible combinations of Lorentz structures

   ↪ building blocks: external momenta, metric, Levi-Civita

2. ensure gauge invariance (via projectors): $I^{\mu\nu} = g^{\mu\nu} - \frac{v^\mu q_i^\nu}{v \cdot q_i}$

3. remove poles (via linear combinations)

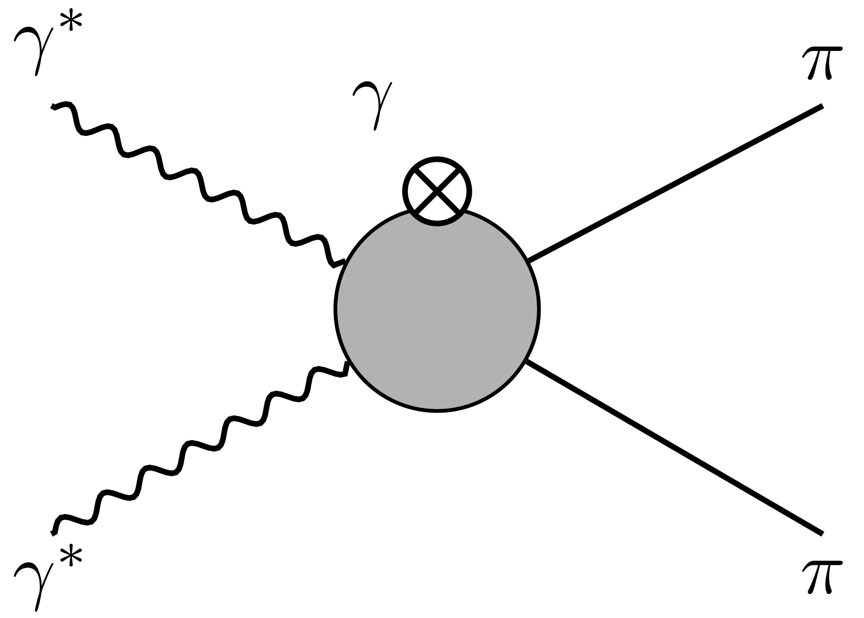
4. consider physical properties of the process

   ↪ photon crossing symmetry, Bose symmetry

5. identify and deal with Tarrach redundancies in the basis

B.2 Construction of $\gamma^*\gamma^*\gamma \to \pi\pi$ basis

Focusing on the process $\gamma^*\gamma^*\gamma \to \pi\pi$ in the soft photon limit



[arXiv:2302.12264]

$$\frac{\partial}{\partial q_{3\sigma}} \mathcal{M}^{\mu\nu\lambda}_{\text{reg}}(p_1,p_2,q_1,q_2)\Big|_{q_3=0} = \sum_{i=1}^{28} \tilde{T}_i^{\mu\nu\lambda;\sigma}(q_1,q_2,q_5)\tilde{\mathcal{A}}_i(s,t-u,q_1^2,q_2^2)$$

↪ with one redundancy raining:

$$0 = (t-u)^2\tilde{T}_{22}^{\mu\nu\lambda;\sigma} - (q_1^2-q_2^2-s)(q_1^2-q_2^2+s)\tilde{T}_{27}^{\mu\nu\lambda;\sigma} + (q_1^2+q_2^2-s)\tilde{T}_{28}^{\mu\nu\lambda;\sigma}$$

- Change of basis

$$\frac{\partial}{\partial q_{3\sigma}} \mathcal{M}^{\mu\nu\lambda}_{\text{reg}}(p_1,p_2,q_1,q_2)\Big|_{q_3=0} = \sum_{i=1}^{27} \mathcal{B}_i^{\mu\nu\lambda;\sigma}(q_1,q_2,q_5)\bar{\mathcal{A}}_i(s,t-u,q_1^2,q_2^2)$$

↪ with certain kinematic constraints for scalar functions

$$\bar{\mathcal{A}}_{22} = (q_1^2+q_2^2-s)\tilde{\mathcal{A}}_{22} - (t-u)^2\tilde{\mathcal{A}}_{28}$$
$$\bar{\mathcal{A}}_{27} = (q_1^2+q_2^2-s)\tilde{\mathcal{A}}_{27} - (q_1^2-q_2^2-s)(q_1^2-q_2^2+s)\tilde{\mathcal{A}}_{28}$$
$$\bar{\mathcal{A}}_i = \tilde{\mathcal{A}}_i \quad \text{for } i \in \{1,\cdots,21,23,\cdots,26\}$$

B.3 Projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$

Building projectors for $\gamma^*\gamma^*\gamma \to \pi\pi$

$$\frac{\partial}{\partial q_{3\sigma}} \mathcal{M}^{\mu\nu\lambda}_{\text{reg}}(p_1, p_2, q_1, q_2)\Big|_{q_3=0} = \sum_{i=1}^{27} \mathcal{B}_i^{\mu\nu\lambda;\sigma}(q_1, q_2, q_5)\bar{A}_i(s, t-u, q_1^2, q_2^2)$$

- isolate each scalar function: $\mathcal{P}_{j\,\mu\nu\lambda\sigma}\mathcal{B}_i^{\mu\nu\lambda;\sigma} = \delta_{ij}$

Definition of projectors : $\mathcal{P}_j^{\mu\nu\lambda\sigma} = C_{jk} T_k^{\mu\nu\lambda\sigma}$

↪ contracting & inverting matrix of coefficients : $C_{jk} = I_{27}\,\mathrm{CTM}^{-1}$

↪ preservation of matrix rank: $\mathrm{CTM}_{(27\times 27)} := T_{k\,\mu\nu\lambda\sigma}\mathcal{B}_i^{\mu\nu\lambda;\sigma}$

✅ inversion of matrix via C++ program [arXiv:1904.00009v2]

B.4 Expansion with Legendre polynomials

$$\bar{\mathcal{A}}_i = \sum_j a^i_j(s) \cdot P_j(z) \rightarrow \quad a^i_j(s) = \frac{2j+1}{2} \int_{-1}^{+1} dz \; P_j(z)\, \bar{\mathcal{A}}_i(z)$$

- Taking Imaginary part:

$$\mathrm{Im}^{\pi\pi}_s a^i_j(s) = \frac{2j+1}{2} \int_{-1}^{+1} d\,z\, P_j(z)\, \mathrm{Im}^{\pi\pi}_s \bar{\mathcal{A}}_i(z)$$

$$= \frac{2j+1}{2} \int_{-1}^{+1} d\,z\, P_j(z)\, \mathrm{Im}^{\pi\pi}_s \left(\mathcal{P}_{i,\,\mu\nu\lambda;\sigma} \cdot \sum_{k=1}^{27} \mathcal{B}^{\mu\nu\lambda;\sigma}_k \bar{\mathcal{A}}_k \right) \Rightarrow$$

$$\mathrm{Im}^{\pi\pi}_s a^i_j(s) = \frac{2j+1}{2} \int_{-1}^{+1} d\,z\, P_j(z)\, \mathcal{P}_{i,\,\mu\nu\lambda;\sigma} \cdot \sum_{k=1}^{27} \mathcal{B}^{\mu\nu\lambda;\sigma}_k \cdot \mathrm{Im}^{\pi\pi}_s \bar{\mathcal{A}}_k(s,z)$$

replace the unitarity relation in here

B.5 Tensor-meson pole cut (extra results)

$$\mathrm{Im}_s^{\mathrm{T}}\, \bar{\mathcal{A}}_i(s, t-u, q_1^2, q_2^2) = \pi\delta(s - m_T^2) \sum_{j=1}^{5}\sum_{k=1}^{8} \frac{1}{m_{\mathrm{T}}^3} t^i_{j,k}\, \mathcal{F}_j^T(q_1^2, q_2^2)\hat{\mathcal{F}}_k(q_5^2, 0)$$

with

$$\boxed{\mathrm{Im}_s^{\mathrm{T}} \bar{\mathcal{A}}_i = 0 \quad \text{for } i \in \{11, 12, 21, 22, 24, 27\}}$$ *

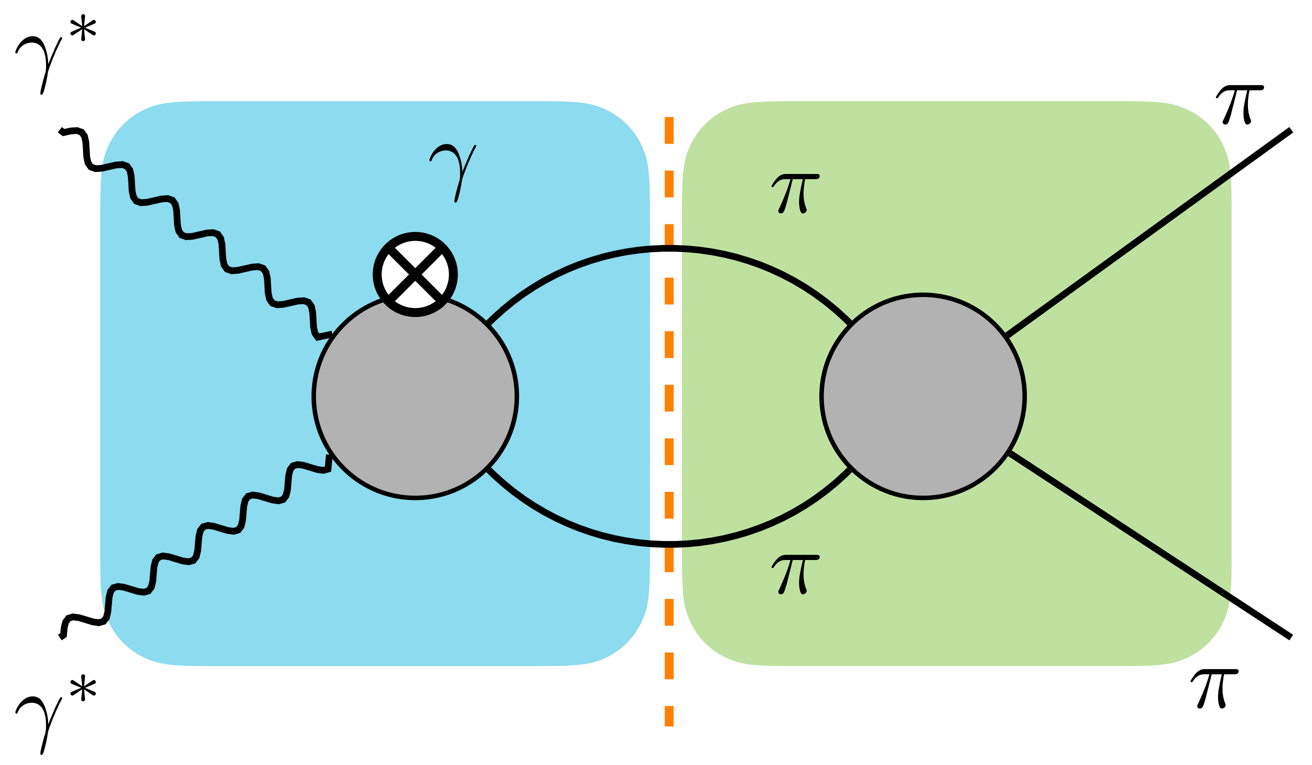
* structures 22 and 27 do not have contributing terms :)

- s,t,u dependance in the coefficients $t^i_{j,k}$

↪ 21 non-zero 5×8 matrices of coefficients, e.g.

$$t^1_{3,1} = 2m_T^2\,, \quad t^1_{4,1} = q_1^2 - q_2^2 - m_T^2\,, \quad t^3_{3,5} = (t-u)^2\,, \quad t^3_{4,5} = \frac{(t-u)^2(m_T^2 - q_1^2 + q_2^2)}{2(q_1^2 - q_2^2)}$$

B.6 Two-pion cut (extra calculation)



Computation of imaginary parts:

$$\mathrm{Im}_s^{\pi\pi} a_j^{i\,\text{resc.}}(s) = \frac{2j+1}{2}\int_{-1}^{+1} d\,z\,P_j(z)\,\mathcal{P}_{i,\,\mu\nu\lambda;\sigma}(s,t_1-u_1)\cdot\int d\Pi_2\left(\mathcal{T}^{(I)}(p_1p_2\to k_1k_2)\right)^*\sum_{i'=1}^{27}\mathcal{B}_{i'}{}^{\mu\nu\lambda}(q_1,q_2,k_5)\cdot\bar{A}_{i'}(s,t_3-u_3) \Rightarrow$$

$$\bar{A}_{i'}^{\text{full}}(s,z') = \bar{A}_{i'}^{\text{resc.}}(s,z') + \bar{A}_{i}^{\text{tensor}}(s,z')$$

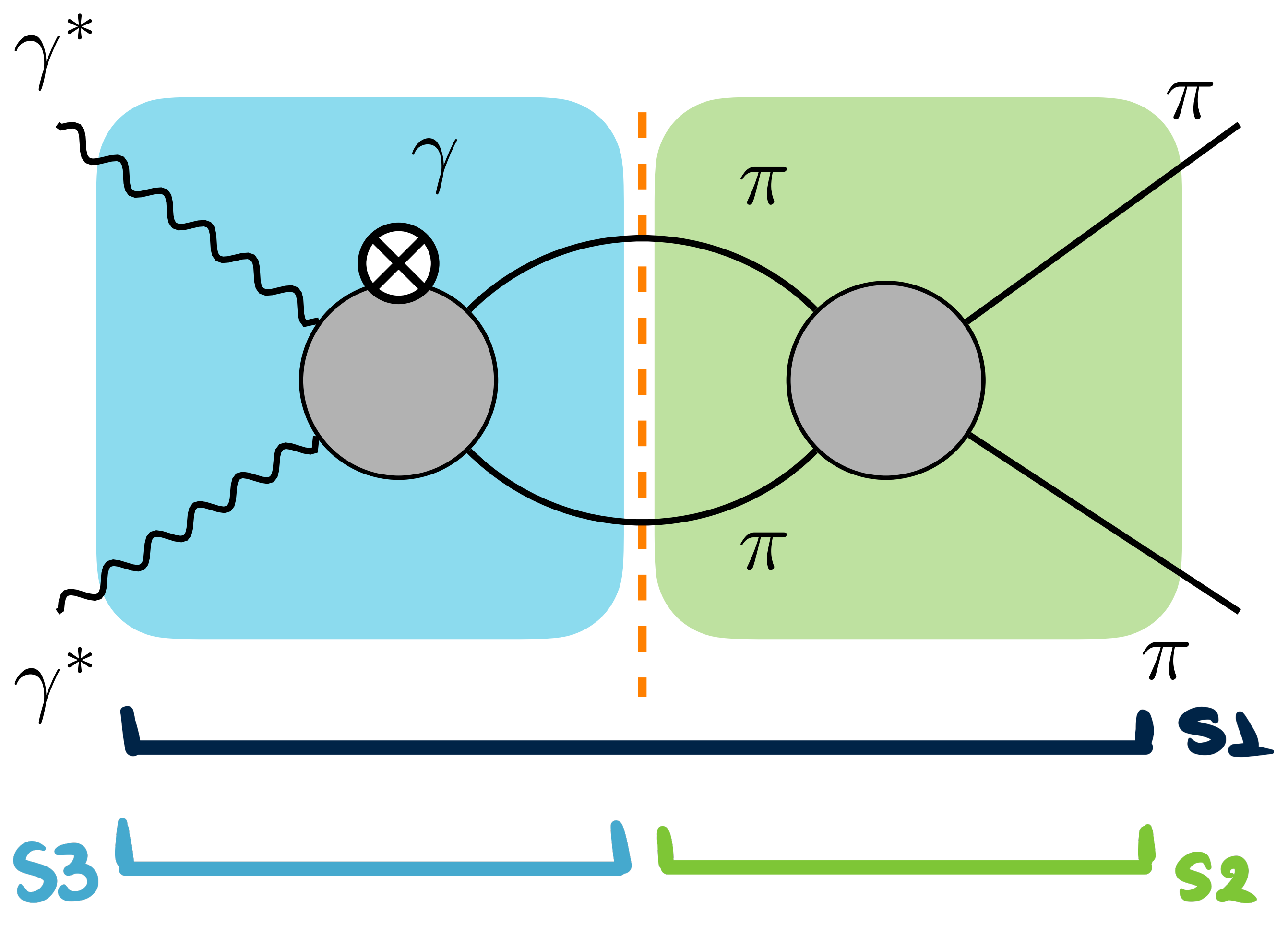
$$\mathrm{Im}_s^{\pi\pi} a_j^{i\,\text{resc.}}(s) = \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\int_{-1}^{+1}dz\int_{-1}^{+1}dz'\int_0^{2\pi}d\phi'\,P_j(z)P_J(z_0)\sum_{i'=1}^{27}C_{ii'}(s,z,z',z_0)\cdot A_{i'}^{\text{full}}(s,z')$$

$$\bar{A}_i = \sum_j a_j^i(s)\cdot P_j(z)$$

$$P_J(z_0) = P_J(z')P_J(z) + 2\sum_{m=1}^{J}\frac{(J-m)!}{(J+m)!}P_J^m(z')P_J^m(z)\cos m(\phi-\phi')$$

Kinematics for subprocesses:

$$\gamma^*(q1)\gamma^*(q_2)\gamma(q_3) \to \pi(k_1)\pi(k_2) \to \pi(p_1)\pi(p_2)$$



$$s_1 = (q_1+q_2)^2 = (p_1+p_2)^2 \qquad s_2 = (k_1+k_2)^2 = (p_1+p_2)^2 \qquad s_3 = (q_1+q_2)^2 = (k_1+k_2)^2$$

$$t_1 = (q_1-p_1)^2 = (q_2-p_2)^2 \qquad t_2 = (k_1-p_1)^2 = (k_2-p_2)^2 \qquad t_3 = (q_1-k_1)^2 = (q_2-k_2)^2$$

$$u_1 = (q_1-p_2)^2 = (q_2-p_1)^2 \qquad u_2 = (k_1-p_2)^2 = (k_2-p_1)^2 \qquad u_3 = (q_1-k_2)^2 = (q_2-k_1)^2$$

$$z = \cos\theta = \frac{t_1-u_1}{\sigma_\pi(s)\lambda^{1/2}(s,q_1^2,q_2^2)}$$

$$z_0 = \cos\theta'\cos\theta + \cos(\phi-\phi')\sin\theta\sin\theta' = \frac{t_2-u_2}{(s-4M_\pi^2)}$$

$$z' = \cos\theta' = \frac{t_3-u_3}{\sigma_\pi(s)\lambda^{1/2}(s,q_1^2,q_2^2)}$$

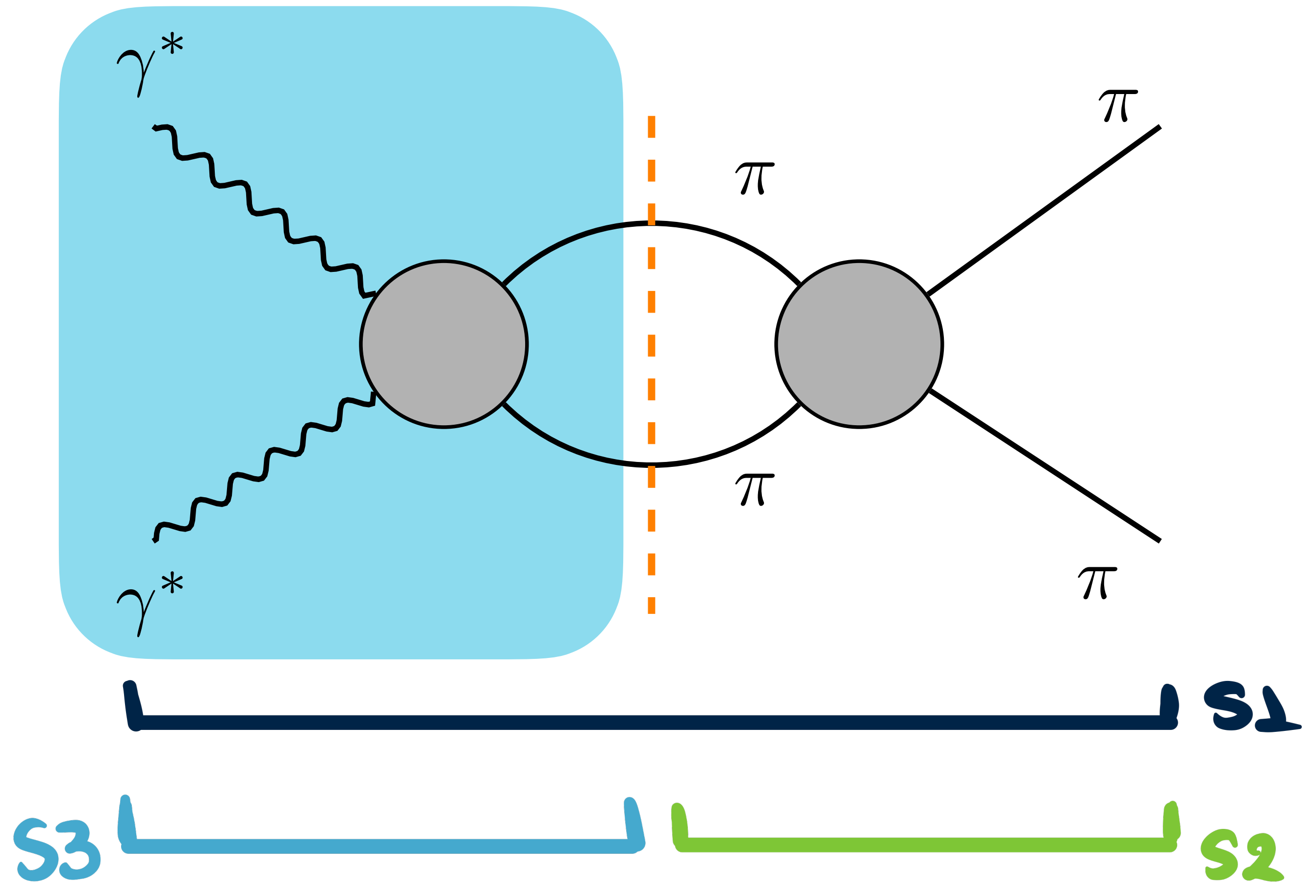
✅ cross-check for $\gamma^*\gamma^* \to \pi\pi$

↪ for $J = 0$:

$$\mathrm{Im}_s^{\pi\pi} a_0^{1\,\mathrm{resc.}}(s) = \left[a_0^{1\,\mathrm{resc.}}(s) + \hat{F}_1^{\mathrm{inh.}}(s)\right] \cdot \sin\delta_0^0(s) e^{-i\delta_0^0(s)}$$

$$\mathrm{Im}_s^{\pi\pi} a_0^{2\,\mathrm{resc.}}(s) = \left[a_0^{2\,\mathrm{resc.}}(s) + \hat{F}_2^{\mathrm{inh.}}(s)\right] \cdot \sin\delta_0^0(s) e^{-i\delta_0^0(s)}$$

$$\mathrm{Im}_s^{\pi\pi} a_0^{3\,\mathrm{resc.}}(s) = \mathrm{Im}_s^{\pi\pi} a_0^{4\,\mathrm{resc.}}(s) = \mathrm{Im}_s^{\pi\pi} a_0^{5\,\mathrm{resc.}}(s) = 0$$



$$A_{i'}^{\mathrm{full}}(s,z') = A_{i'}^{\mathrm{resc.}}(s,z') + F_\pi^V(q_1^2) F_\pi^V(q_2^2)\, A_i^{\mathrm{Born}}(s,z')$$

↪ solution for inhomogeneous Omnès problem :

$$A_1^{\mathrm{resc.}} = a_0^{1\,\mathrm{resc.}}(s) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0^0(s')}{|\Omega_0(s')|} \frac{\hat{F}_1^{\mathrm{inh.}}(s')}{(s'-s)}$$

$$A_2^{\mathrm{resc.}} = a_0^{2\,\mathrm{resc.}}(s) = \frac{\Omega_0(s)}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0^0(s')}{|\Omega_0(s')|} \frac{\hat{F}_2^{\mathrm{inh.}}(s')}{(s'-s)}$$

$$A_3^{\mathrm{resc.}} = A_4^{\mathrm{resc.}} = A_5^{\mathrm{resc.}} = 0$$

same to the results from the helicity partial-wave approach (S-wave contribution) [arXiv:1905.13198]

Two-pion cut (extra calculation) : $\gamma^*\gamma^* \to \pi\pi$

$$\mathrm{Im}_s^{\pi\pi} a_j^{i\,\text{resc.}}(s) = \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\int_{-1}^{+1}dz\int_{-1}^{+1}dz'\int_0^{2\pi}d\phi'\,P_j(z)P_J(z_0)\sum_{i'=1}^{27}C_{ii'}(s,z,z',z_0)\cdot A_{i'}^{\text{full}}(s,z')$$

$$A_{i'}^{\text{full}}(s,z') = A_{i'}^{\text{resc.}}(s,z') + F_\pi^V(q_1^2)F_\pi^V(q_2^2)\,A_i^{\text{Born}}(s,z')$$

$$z_0 = \cos\theta'\cos\theta + \cos(\phi-\phi')\sin\theta\sin\theta' = z'\cdot z + \cos(\phi-\phi')\sin\theta\sin\theta'$$

$$\begin{aligned}
I_{\text{Born}}^i &= \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\int_{-1}^{+1}dz\int d\Omega'\,P_j(z)P_J(z_0)\sum_{i'=1}^{5}C_{ii'}(s,z,z',z_0)\cdot A_{i'}^{\text{Born}}(s,z') \\
&= \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\int_{-1}^{+1}dz\int_{-1}^{+1}dz'\int_0^{2\pi}d\phi' P_j(z) \\
&\quad\times\left(P_J(z)P_J(z') + 2\sum_{m=1}^{J}\frac{(J-m)!}{(J+m)!}P_J^m(z)P_J^m(z')\cos(m(\phi-\phi'))\right)\sum_{i'=1}^{5}C_{ii'}(s,z,z',\phi')\cdot A_{i'}^{\text{Born}}(s,z') \\
&= \frac{(2j+1)}{8\pi}\sum_J (2J+1)\sin\delta_J^I(s)e^{-i\delta_J^I(s)}\left[I_{\text{Born}}^{(\text{P1})} + I_{\text{Born}}^{(\text{P2})}\right]
\end{aligned}$$

B.7 Tensor TFFs