$V \to 3\pi$ $(V = \omega, \phi, J/\psi)$ with Khuri-Treiman equations

JPAC:

Prog.Part.Nucl.Phys. 127 (2022) 103981;

Eur.Phys.J.C 80 (2020), 1107;

Phys.Rev.D 108, 014035 (2023);

2505.15309 [hep-ph]

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14th International Workshop on e^+e^- collisions from Phi to Psi

Pisa, 8-11 June, 2026

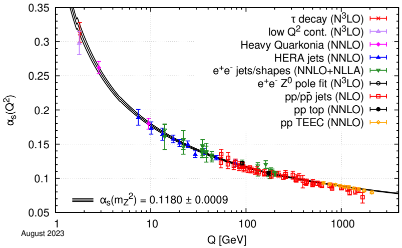
Universitat de Barcelona

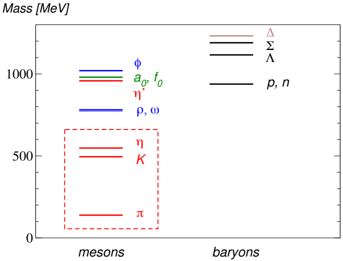
JPAC

QCD: open questions

- **Asymptotic freedom** at high energies (“weak QCD”)
- **Confinement** at low energies (“strong QCD”), no quarks+gluons, only hadrons
- **Hadron spectrum** and **interactions**: why these states? scattering, decays of hadrons?

⟶ (perturbative) QCD does not answer any of these questions!



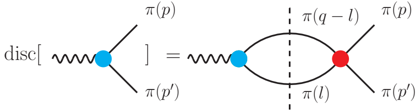


Remedies:

- Effective Field Theories: symmetries, separation of scales
- Dispersion Theory: unitarity, analyticity, crossing symmetry
- Lattice QCD: approach to solving a discretized version of QCD on a computer

Warm-up: the pion vector form factor

- **Unitarity**:



$$\mathrm{disc}F_\pi(s) = 2i\mathrm{Im}F_\pi(s) = 2i\sigma_\pi(s)F_\pi(s)t_1^{1*}(s) = 2iF_\pi(s)\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\,,$$

- Analytic solution, **Omnès** equation [Omnès, Nuovo Cimento 8, 316 (1958)]

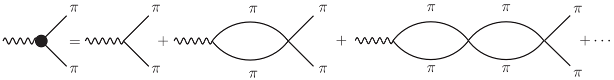
$$F_\pi(s) = \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{disc}F_\pi(s')}{s'-s}ds'\,,$$

$$F_\pi(s) = P(s)\Omega_1^1(s)\,,\quad \Omega_1^1(s) = \frac{f(s)}{f(0)} = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}ds'\frac{\delta_1^1(s')}{s'(s'-s)}\right\}\,,$$

- Polynomial $P(s)$, not fixed by unitarity: matched to EFT or by data

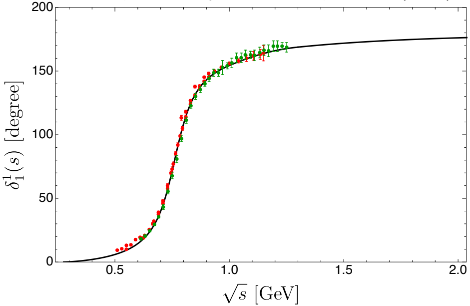
Omnès equation

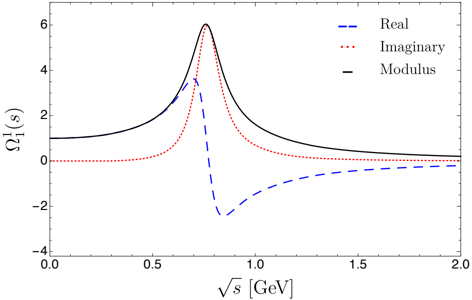
- Diagrammatic interpretation



- Solution: depends solely on the P-wave phase shift of $\pi\pi$

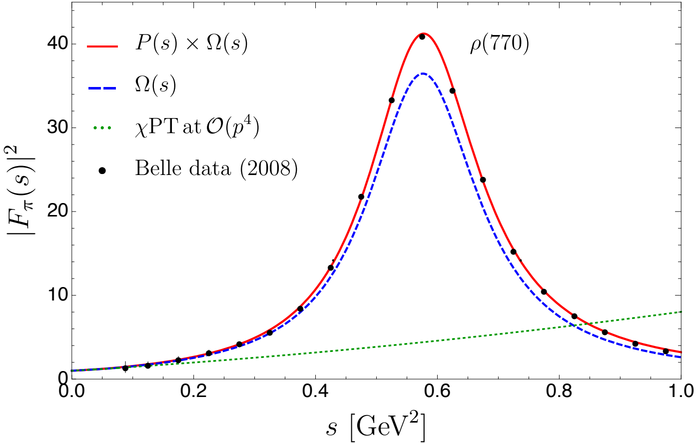
Garcia-Martin, *et.al.* Phys. Rev. D83, 074004 (2011)





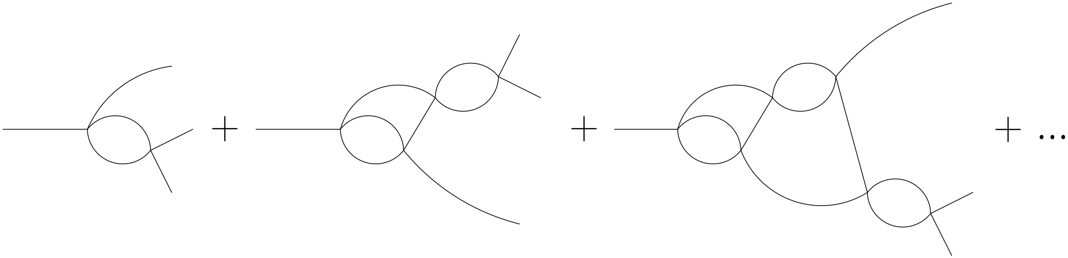
The pion vector form factor

- $P(s) = 1 + \alpha s$, with $\alpha = 0.11\ \text{GeV}^{-2}$ due to inelasticities



Three-particle decay

- In many decay processes one wants to take into account unitarity/FSI in the three possible channels
- Khuri-Treiman equations:
  - Include full rescattering effects



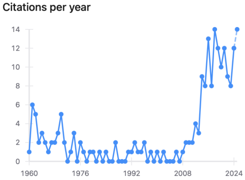
Khuri-Treiman formalism: citation summary

- Developed for $K \to 3\pi$

Pion-Pion Scattering and K + /- --> 3pi Decay

N.N. Khuri (Princeton, Inst. Advanced Study), S.B. Treiman (Princeton U.)
Aug 1, 1960

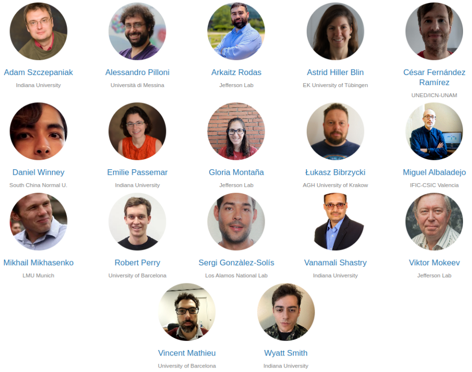
6 pages
Published in: *Phys.Rev.* 119 (1960) 1115-1121
DOI: 10.1103/PhysRev.119.1115



- Most prominent applications:
  - $\eta \to 3\pi$ [Kambor *et al.* 1995, Anisovich *et.al.* 1996, Descotes-Genon *et al.* 2014, JPAC 2015&17, Colangelo 2016, Albaladejo *et al.* 2017]
  - $\eta' \to \eta\pi\pi$ [Isken *et al.* 2017]
  - $\omega/\phi \to 3\pi, \pi\gamma^*$[Kubis *et al.* 2012, JPAC'14'20'25, Dax *et al.* 2018] →this talk
  - $J/\psi \to 3\pi, \pi\gamma^*$ [JPAC 2023] → this talk
  - $e^+e^- \to 3\pi/\pi\gamma$ (a_μ) [Hoferichter *et.al.* 2018&19, Hoid 2020]
  - $D \to K\pi\pi$ [Niecknig *et.al.* 2015]
  - $J^{PC} \to 3\pi$ [JPAC 2019, Stamen *et.al.* 2022]
  - $\pi_1(1600) \to 3\pi$ [JPAC *in progress*] → this talk

JPAC: Joint Physics Analysis Center

- Work in **theoretical/experimental/phenomenological** analysis
- Light/heavy meson **spectroscopy**
- Interaction with many **experimental collaborations** (BESIII, CLAS, GlueX, KLOE, LHCb, MAMI,...) and LQCD groups
- Website: https://www.jpac-physics.org



Three-body: definitions

- Decay amplitude for $V \to \pi^+\pi^-\pi^0$ ($V = \omega, \phi, J/\psi$)

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu} p_+^{\nu} p_-^{\alpha} p_0^{\beta} \mathcal{F}(s,t,u)\,,$$

- Mandelstam variable, s-channel scattering angle and momenta

$$s = (p_+ + p_-)^2\,, \quad t = (p_0 + p_+)^2\,, \quad u = (p_0 + p_-)^2\,,$$

$$\cos\theta_s(s,t,u) = \frac{t-u}{4\,p(s)\,q(s)}\,, \quad \sin\theta_s(s,t,u) = \frac{\sqrt{\phi(s,t,u)}}{2\sqrt{s}\,p(s)\,q(s)}\,,$$

$$p(s) = \frac{\lambda^{\frac{1}{2}}(s, m_{\pi^+}^2, m_{\pi^-}^2)}{2\sqrt{s}}\,, \quad q(s) = \frac{\lambda^{\frac{1}{2}}(s, m_{J/\psi}^2, m_{\pi^0}^2)}{2\sqrt{s}}\,,$$

$$|\mathcal{M}(s,t,u)|^2 = \frac{1}{4}\left(2\sqrt{s}\sin\theta_s p(s) q(s)\right)^2 |\mathcal{F}(s,t,u)|^2\,,$$

Khuri-Treiman representation of the amplitude

- s-channel partial-wave (pw) expansion of the amplitude

$$F(s,t,u) = \sum_{J=0}^{\infty} (p(s)\,q(s))^{J-1}\; P'_J(z_s)\; f_J(s)\,,$$

- Khuri-Treiman/isobar decomposition of the amplitude

$$F(s,t,u) = \sum_{J=0}^{J_{\max}} (p(s)\,q(s))^{J-1}\; P'_J(z_s)\; F_J(s) + (s \leftrightarrow t) + (s \leftrightarrow u)\,,$$

- Consider only $J = 1$ isobars

$$F(s,t,u) = F_1(s) + F_1(t) + F_1(u)\,,$$

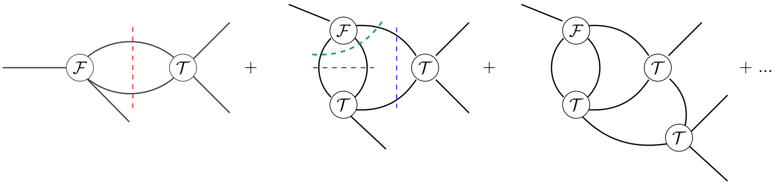
- pw projection of the KT decomposition

$$f_1(s) = F_1(s) + \hat{F}_1(s)\,, \quad \hat{F}_1(s) = 3\int_{-1}^{1} \frac{dz_s}{2}(1 - z_s^2)F_1(t(s,z_s))\,,$$

  - $F_1(s)$: right-hand cut (RHC)
  - $\hat{F}_1(s)$: left-hand cut (given by the RHC of the crossed channels $F_1(t), F_1(u)$)

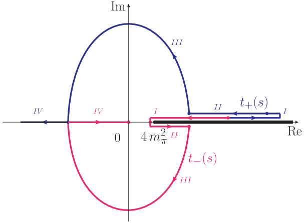
Unitarity and analyticity

- Unitarity relation for the isobar amplitude $F_1(s)$



$$\mathrm{disc}F_1(s) = 2i\left(F_1(s) + \hat{F}_1(s)\right)\sin\delta(s)e^{-i\delta(s)}\theta(s-4m_\pi^2)\,,$$

- Complications: integration contour for $\hat{F}_1(s)$
- Classic' strategy (Khuri-Treiman, 1960)
  — deform path of angular integral
  to avoid crossing branch cuts
- Alternative approach (Gasser and Rusetsky, 2018)
  — deform path of dispersion relation

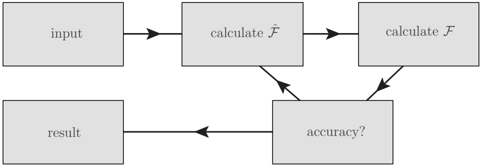


KT equations: DR, solutions, subtractions

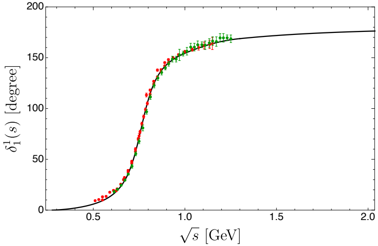
- Unsubtracted dispersion relation:

$$F_1(s) = \frac{1}{2\pi i}\int_{4m_\pi^2}^{\infty} ds' \, \frac{\text{disc}\, F_1(s')}{s'-s}\,, \quad F_1(s) = \Omega_1(s)\left(a + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1(s')\,\hat{F}_1(s')}{|\Omega_1(s')|\,(s'-s)}\right),$$

- $\delta_1(s)$: $\pi\pi$ phase taken as input:
- Solution by numerical iteration
  - Initial input: $F_1 = \Omega_1(s)$



Garcia-Martin, *et al.* PRD 83, 074004 (2011)



KT equations: solutions for $\omega \to 3\pi$

- Once-subtracted dispersion relations

$$F_1(s) = \Omega_1(s)\left(a + b'\,s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2}\frac{\sin\delta_1(s')\,\hat{F}_1(s')}{|\Omega_1(s')|\,(s'-s)}\right),$$

- Sum rule:

$$b_{\rm sr} \equiv b'/a = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2}\frac{\sin\delta_1(s')\,\hat{F}_1(s')/a}{|\Omega_1(s')|}\,.$$

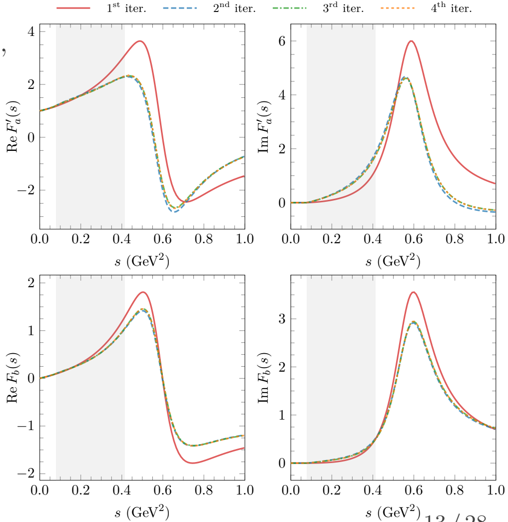
- Solution

$$F_1(s) = a\left[F_a(s) + b\,F_b(s)\right],$$

$$F_a(s) = \Omega_1(s)\left[1 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1(s')\,\hat{F}_a(s')}{|\Omega_1(s')|(s'-s)}\right],$$

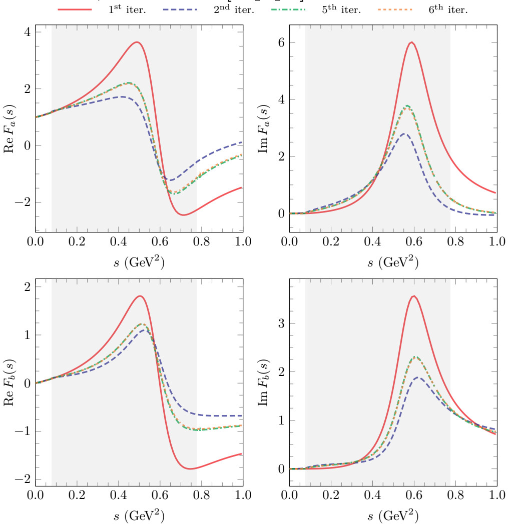
$$F_b(s) = \Omega_1(s)\left[s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1(s')\,\hat{F}_b(s')}{|\Omega_1(s')|(s'-s)}\right].$$

JPAC, Eur.Phys.J.C 80 (2020), 1107

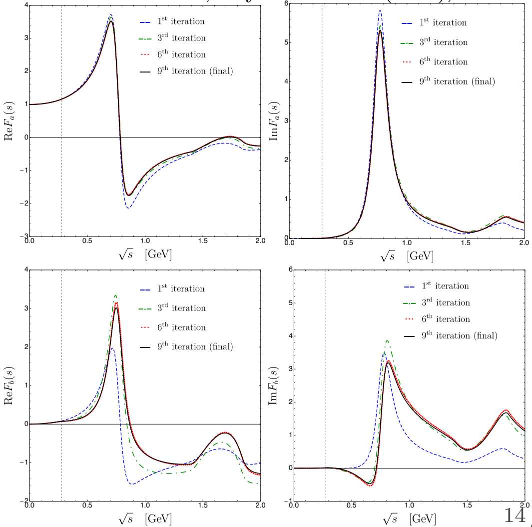


KT equations: solutions for $\phi \to 3\pi$ and $J/\psi \to 3\pi$

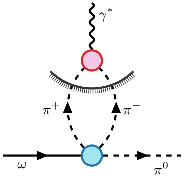
JPAC, 2505.15309 [hep-ph]



JPAC, Phys.Rev.D 108 (2023), 014035



Combined analysis of $\omega \to 3\pi$ and $\omega \to \pi^0\gamma^*$



- Dispersive $\omega \to \pi^0\gamma^*$ representation:

$$f_{\omega\pi^0}(s) = |f_{\omega\pi^0}(0)|\, e^{i\phi_{\omega\pi^0}(0)} + \frac{s}{12\pi^2}\int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^{3/2}} \frac{p^3(s')\, F_\pi^{V*}(s')\, f_1^{\omega\to 3\pi}(s')}{(s'-s)},$$

- $\omega \to 3\pi$ Dalitz parameters:

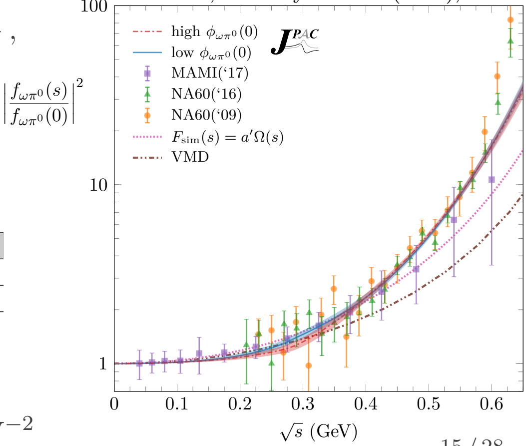
$$|\mathcal{F}(Z,\phi)|^2 = |N|^2 \left[1 + 2\alpha Z + 2\beta Z^{3/2}\sin 3\phi + 2\gamma Z^2\right],$$

Dalitz parameters (DP)	$\alpha \times 10^3$	$\beta \times 10^3$	$\gamma \times 10^3$
BESIII [PRD98, 112007 (2018)]	111(18)	25(10)	22(29)
This work	109(14)	26(6)	19(5)

- Subtraction constant from DP and TFF:

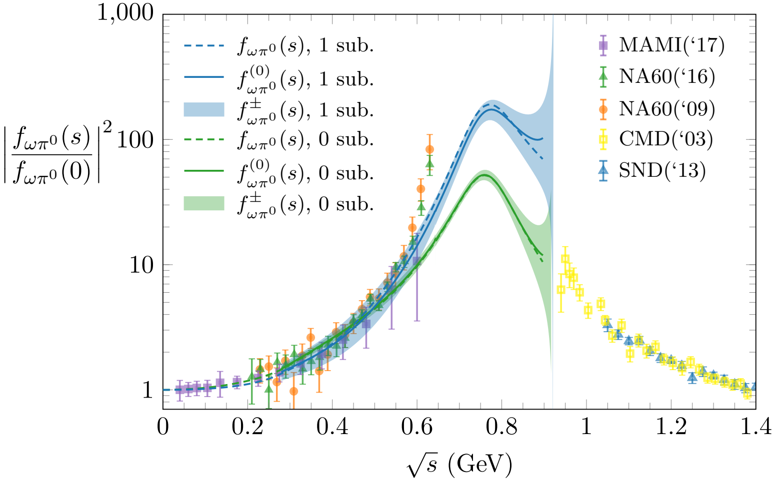
$$b_{\rm Fit} \simeq 2.65(35)e^{1.70(27)i}\,\text{GeV}^{-2} \text{ vs } b_{\rm sr} = 0.55e^{0.15i}\,\text{GeV}^{-2}$$

JPAC, Eur.Phys.J.C 80 (2020), 1107

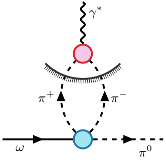


$\omega \to \pi^0\gamma^*$ form factor

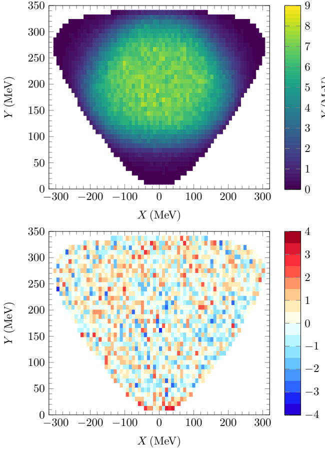
- Tension with $e^+e^- \to \pi^0\gamma$ data



Combined analysis of $\phi \to 3\pi$ and $\phi \to \pi^0\gamma^*$

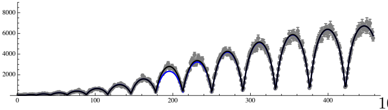


- Fits to Dalitz plot [KLOE PLB 757, 362 (2016)] and TFF:

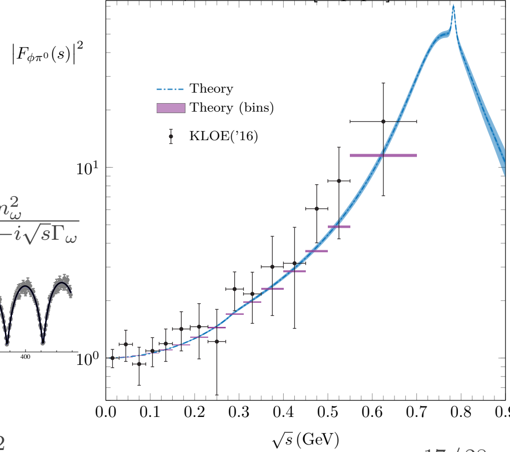


$\rho - \omega$ mixing:

$$F_1(s) \to F_1(s) + a\frac{m_\omega^2}{m_\omega^2 - s - i\sqrt{s}\Gamma_\omega}$$



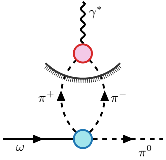
JPAC, 2505.15309 [hep-ph]



- Subtraction constant from DP and TFF:

$$b_{\text{Fit}} = 0.76e^{0.43i}\,\text{GeV}^{-2} \quad \text{vs} \quad b_{\text{sr}} = 0.79e^{0.69i}\,\text{GeV}^{-2}$$

Combined analysis of $\phi \to 3\pi$ and $\phi \to \pi^0\gamma^*$



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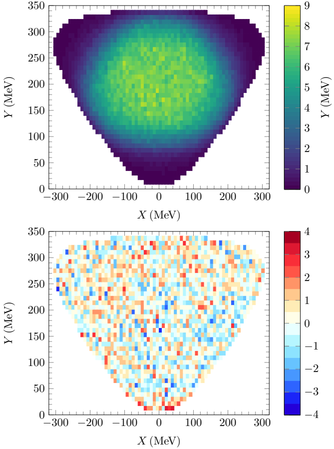

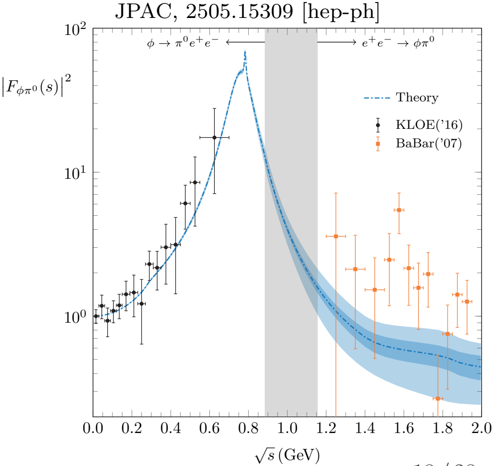

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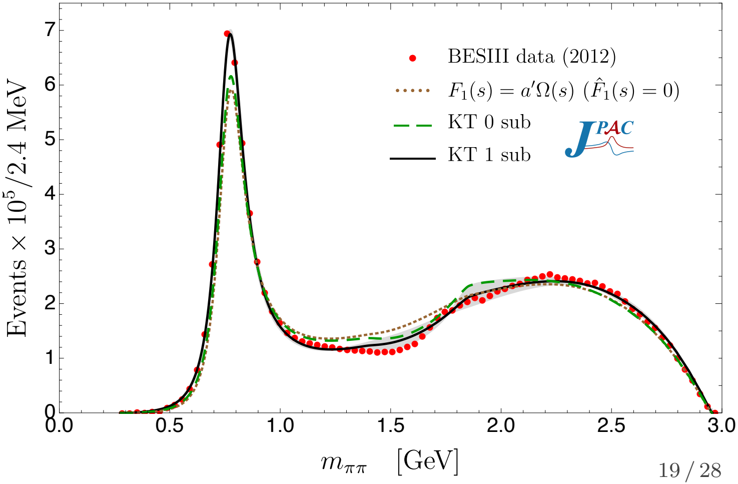
JPAC $J/\psi \to 3\pi$ BESIII data analysis

- Dominated by the ρ
- BESIII data [PLB 710 (2012)]
- JPAC analysis:
  - Two-body
  - KT unsub basic features
  - KT 1-sub improves the description

$$b_{\text{fit}} = 0.198(35)e^{i2.675(300)}\ \text{GeV}^{-2}$$

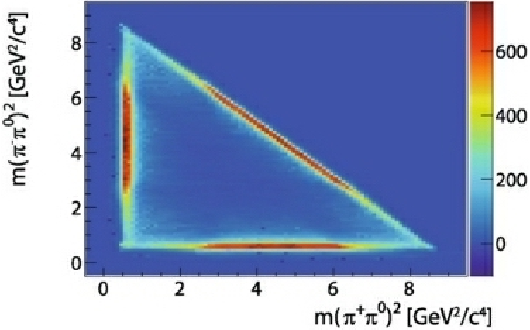
$$b_{\text{sr}} = 0.141e^{i2.32}\ \text{GeV}^{-2}$$

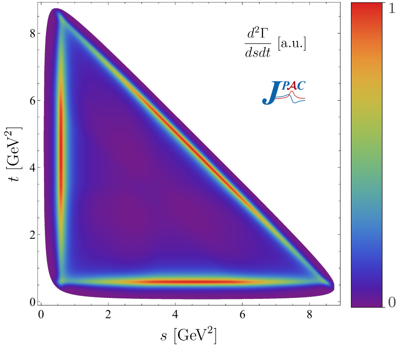
JPAC Coll, Phys.Rev. D108, 014035 (2023)



JPAC $J/\psi \to 3\pi$ BESIII data analysis

- Dominated by the ρ
- BESIII data [PLB 710 (2012)]
- JPAC analysis:
  - Dalitz plot distribution similar to experimental one





Contribution of the F-wave

- Isobar decomposition of the amplitude including F-wave:

$$F(s,t,u) = F_1(s) + F_1(t) + F_1(u) \\ + \big(p(s)q(s)\big)^2 P_3'(z_s) F_3(s) + \big(p(t)q(t)\big)^2 P_3'(z_t) F_3(t) + \big(p(u)q(u)\big)^2 P_3'(z_u) F_3(u)\,,$$

- $F_{1,3}(s)$ is the P, F-wave isobar amplitude
- Discontinuity of the F-wave:

$$\operatorname{disc} F_3(s) = 2i\left(F_3(s) + \hat{F}_3(s)\right)\, \sin\delta_3(s)\, e^{-i\delta_3(s)}\, \theta(s - 4m_\pi^2)\,,$$

- We neglect $\hat{F}_3(s)$ (for simplicity)

$$F_3(s) = p_3(s)\Omega_3(s)\,, \quad \Omega_3(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'}\frac{\delta_3(s')}{s'-s}\right]\,,$$

Model of the F-wave: $\rho_3(1690)$ exchange

- Exchange of a $\rho_3(1690)$ in the s-channel with an energy-dependent width:

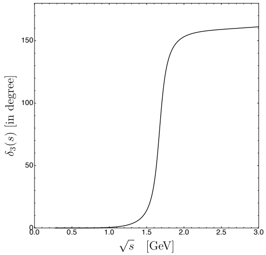
$$F_3(s)|_{\text{BW}} = \frac{m_{\rho_3}^2}{m_{\rho_3}^2 - s - i m_{\rho_3} \Gamma_{\rho_3}^{\ell=3}(s)}\,, \qquad \Gamma_R^\ell(s) = \frac{\Gamma_R m_R}{\sqrt{s}} \left(\frac{p(s)}{p(m_R^2)}\right)^{2\ell+1} \left(F_R^\ell(s)\right)^2\,,$$

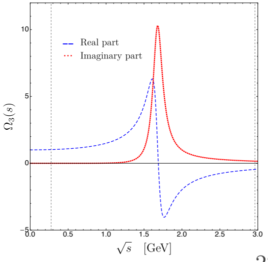
$$p(s) = \frac{\sqrt{s}}{2}\sigma_\pi(s)\,, \quad F_R^{\ell=3}(s) = \sqrt{\frac{z_0(z_0-15)^2 + 9(2z_0-5)^2}{z(z-15)^2 + 9(2z-5)^2}}\,, \quad z = r_R^2 p^2(s)\,, \quad z_0 = r_R^2 p^2(m_{\rho_3}^2)\,,$$

- Extraction of the F-wave phase and Omnès F-wave function:

$$\tan\delta_3(s) = \frac{\text{Im}F_3(s)|_{\text{BW}}}{\text{Re}F_3(s)|_{\text{BW}}}\,,$$

$$\Omega_3(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^\infty \frac{ds'}{s'}\frac{\delta_3(s')}{s'-s}\right]\,,$$





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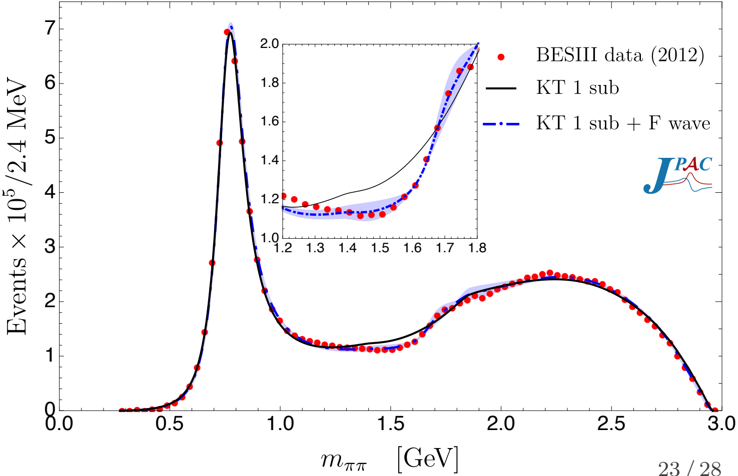
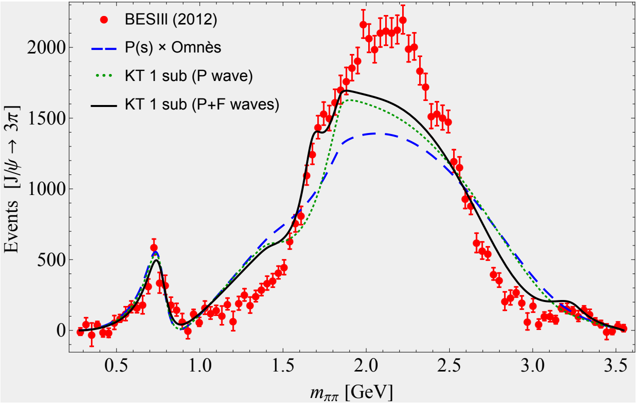
- JPAC analysis:
  - Two-body
  - KT unsub basic features
  - KT 1-sub improves the description
  - KT 1-sub+F-wave describe better $m_{\pi\pi} \sim 1.5$ GeV.

JPAC Coll, Phys.Rev. D108, 014035 (2023)



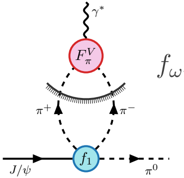
$\psi(2S) \to \pi^+\pi^-\pi^0$ decays

- Completely analogous formalism
- Larger phase space, $m_{\psi(2S)} = 3.68610$ GeV
- BESIII data [PLB 710 (2012)]
- Why does the $\psi(2S) \to \pi^+\pi^-\pi^0$ Dalitz plot differ so dramatically from $J/\psi \to \pi^+\pi^-\pi^0$?
  - — $\rho(2150)$?
  - — Coupled-channels?



$J/\psi \to \pi^0\gamma^*$ transition form factor

- Dispersive representation (once-subtracted)

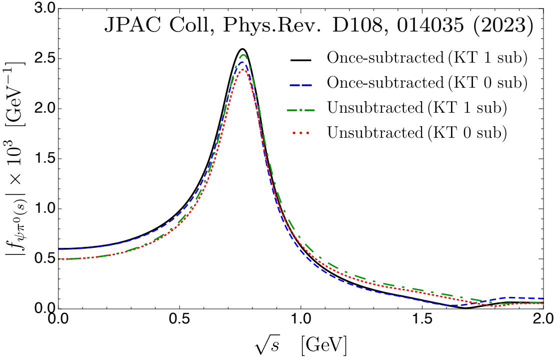


$$f_{\omega\pi^0}(s) = |f_{J/\psi\pi^0}(0)|\, e^{i\phi_{J/\psi\pi^0}(0)} + \frac{s}{12\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^{3/2}}\frac{p^3(s')\, F_\pi^{V\,*}(s')\, f_1^{J/\psi\to3\pi}(s')}{(s'-s)},$$

- $|f_{J/\psi\pi^0}(0)|$ from data:

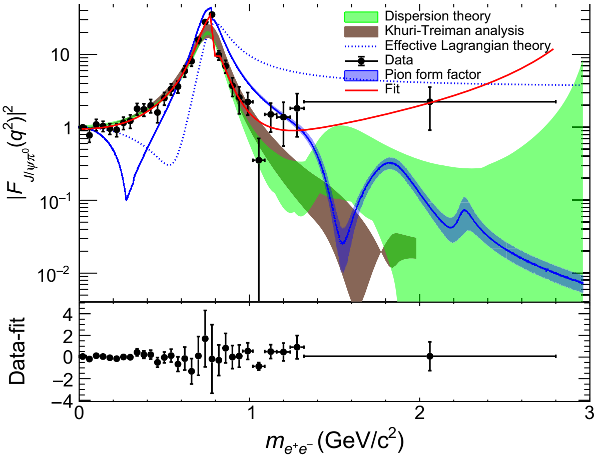
$$\Gamma(J/\psi \to \pi^0\gamma) = \frac{e^2(m_{J/\psi}^2 - m_{\pi^0}^2)^3}{96\pi m_{J/\psi}^3}\,|f_{J/\psi\pi^0}(0)|^2\,,$$

- $\phi_{\omega\pi^0}(0)$ only free parameter
- No data, $\phi_{\omega\pi^0}(0) = 0$



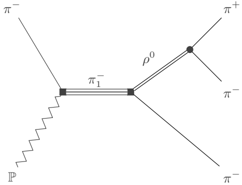
$J/\psi \to \pi^0\gamma^*$ transition form factor

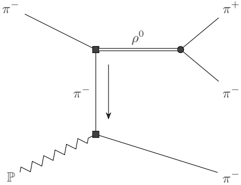
- Recent measurement by BESIII [Phys.Rev.D 112 (2025) 1, L011101]



Exotic $\pi_1(1600) \to 3\pi$ KT analysis (preliminary)

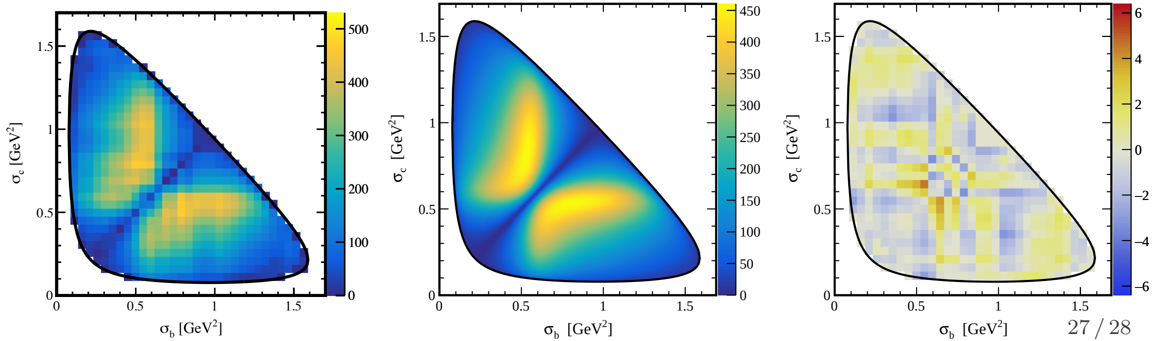
- Exotic with $J^{PC} = 1^{-+}$





COMPASS, 2108.01744

JPAC, in progress



Outlook

- **Khuri-Treiman equations**:
  - Dispersive representation for **3-particle Final-State Interactions**
  - Based on fundamental principles: **analyticity, unitarity and crossing symmetry**
  - Input: $\pi\pi$ scattering **phase shifts**
  - Resonance shape affected by **left-hand cuts / 3-body effects**
  - **Predictive power** (subtraction constants)
  - Experimental **data** well described