Recent Developments in Lattice QCD Calculations of the Muon $g-2$ HVP

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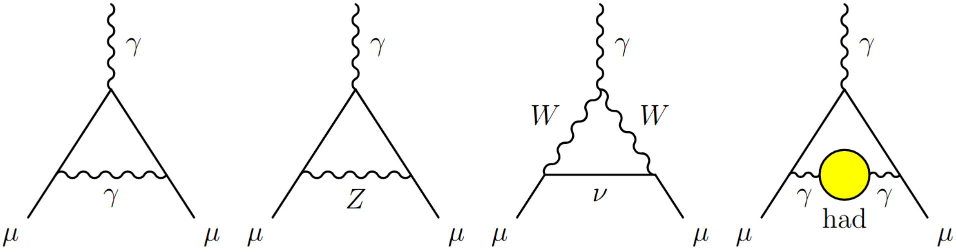
The muon $g-2$: A probe for new physics

Leading contributions to a_μ: [PDG]



- SM prediction from QED, electroweak and hadronic contributions:

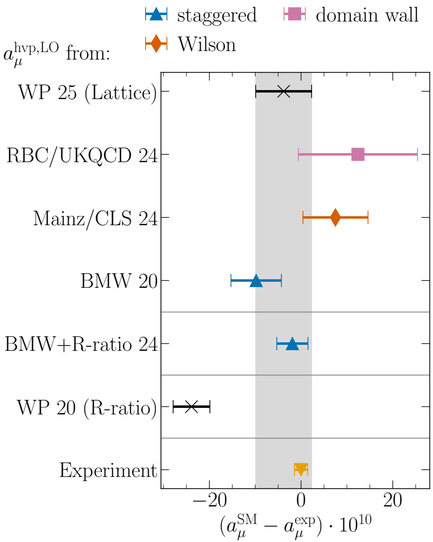
$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{EW}} + a_\mu^{\mathrm{had}} \quad \text{where} \quad a_\mu^{\mathrm{had}} = a_\mu^{\mathrm{hvp}} + a_\mu^{\mathrm{HLbL}} .$$

- Let's focus on a_μ^{hvp} in the following.

CONTRIBUTIONS TO a_μ

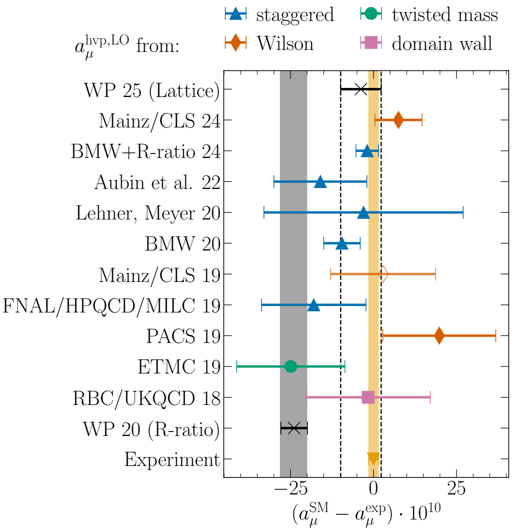
Contribution	Value $\times 10^{11}$
Experiment (E821 + E989)	$116\,592\,071.5(14.5)$
QED	$116\,584\,718.8(\ 2)$
Electroweak	$154.4(\ 4)$
HVP	$7045\ \ (61)$
HLbL	$115.5(\ 9.9)$
Total SM value	$116\,592\,033\ \ (62)$
Difference: $\Delta a_\mu = a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}}$	$38\ \ (63)$
Δa_μ(WP 20)	$249\ \ (48)$

- SM central value dominated by QED contributions.
- SM uncertainty dominated by hadronic contributions.
- These require to evaluate QCD in the low-energy region.



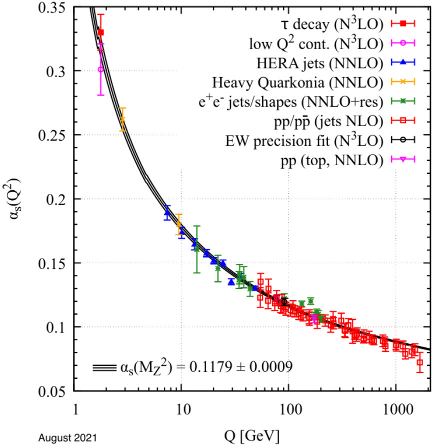
- Muon $g-2$ collaboration reached their precision target [Muon $g-2$, 2506.03069] [Quinn].
- Muon $g-2$ Theory Initiative: Second White Paper published in 2025 [2505.21476, Aliberti et al.] [Lehner].
- Significant shift of the SM prediction compared to 2020: Hadronic vacuum polarization from lattice QCD.
- Theory still needs to improve by a factor of 4…

$a_\mu^{\rm hvp}$ FROM LATTICE QCD (LEADING ORDER)



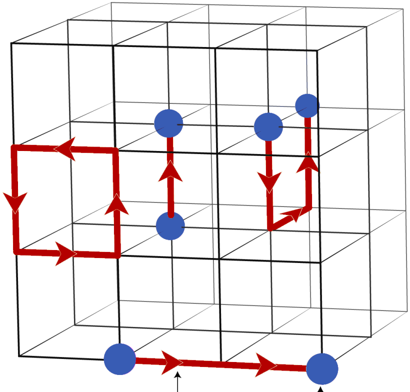
- Two sub-percent determinations of $a_\mu^{\rm hvp}$ from the lattice: [BMWc, 2002.12347], [Mainz/CLS, 2411.07969].
- One hybrid lattice QCD + R-ratio determination at 0.5% precision [BMW+DMZ, 2407.10913] [Lupo].
- Many further lattice results for sub-contributions to $a_\mu^{\rm hvp}$.
- The basis for the WP 25 SM prediction.

a_μ^{hvp} ON THE LATTICE



- QCD is a strongly coupled theory in the hadronic regime at $Q \sim 300\,\mathrm{MeV}$.
- Perturbative expansion fails below $1\,\mathrm{GeV}$.

[1] [PDG, PTEP **2022** (2022), 083C01]



- QCD is a strongly coupled theory in the hadronic regime at $Q \sim 300\,\mathrm{MeV}$.
- Perturbative expansion fails below $1\,\mathrm{GeV}$.
- Formulate the theory
  - on a finite grid $\rightarrow$ regulator Λ_{UV}.
  - in finite volume $\rightarrow \Lambda_{\mathrm{IR}}$.
  - in Euclidean space-time
  - as a Boltzmann distribution
- Compute expectation values $\langle \mathrm{O} \rangle$ by sampling the QCD path integral with Markov Chain Monte Carlo methods.

[2] http://www.jicfus.jp/en/promotion/pr/mj/guido-cossu/

The QCD Lagrange density

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}_f(\not{D} + m_f)\psi_f + \frac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$$

- Contains $N_f + 1$ **bare** parameters (gauge coupling and N_f quark masses)
- Renormalize the theory from hadronic input, e.g., m_Ω, m_π, m_K, m_{D_s}, m_{B_s}.
  $\rightarrow$ All other observables are **predictions**.
- Freedom of choice on how to discretize $\mathcal{L}_{\mathrm{QCD}}$:
  Wilson, twisted mass, staggered, domain wall, overlap, ...
- *Ab initio* predictions after lifting the cutoffs:
  - Λ_{IR}: Infinite-volume limit.
  - Λ_{UV}: Continuum limit.

$a_\mu^{\rm hvp}$ ON THE LATTICE

- Compute $a_\mu^{\rm hvp}$ via [Laurup et al.] [Blum, hep-lat/0212018]

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2 f(Q^2)\hat{\Pi}(Q^2)\,, \qquad \text{with} \quad \hat{\Pi}(Q^2) = 4\pi^2\left[\Pi(Q^2) - \Pi(0)\right]$$

from a known QED kernel function $f(Q^2)$ and the polarization tensor

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x\, \mathrm{e}^{iQ\cdot x}\langle j_\mu^{\rm em}(x)\, j_\nu^{\rm em}(0)\rangle = (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(Q^2)\,.$$

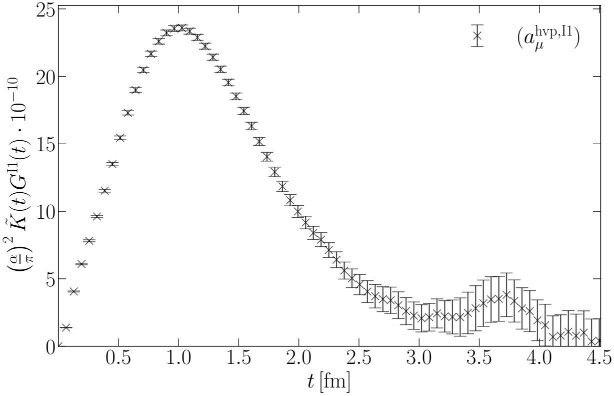
- $a_\mu^{\rm hvp}$ in the time-momentum representation (TMR) [Bernecker, Meyer, 1107.4388],

$$a_\mu^{\rm hvp} := \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t) \quad \text{with the known QED kernel function } \widetilde{K}(t)\,,$$

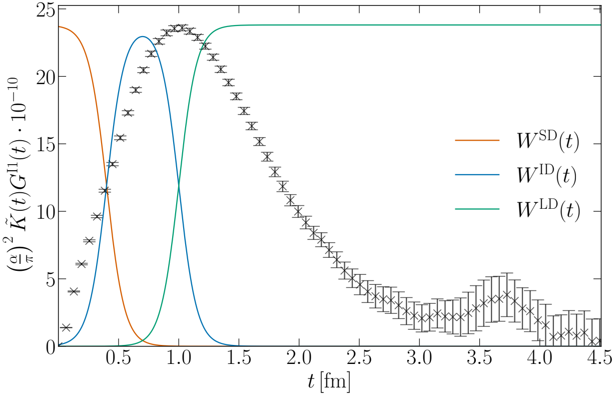
in terms of the zero-momentum vector correlator $G(t)$ (de facto standard).

- Alternative: coordinate space method [Meyer, 1706.01139] [Chao et al., 2211.15581].

$$(a_\mu^{\rm hvp}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t), \qquad G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle j_k^{\rm em}(t,\vec{x})\, j_k^{\rm em}(0)\rangle$$



$$(a_\mu^{\text{hvp}})^i = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t)\;\; W^i(t;t_0;t_1)\,, \qquad G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle j_k^{\text{em}}(t,\vec{x})\, j_k^{\text{em}}(0)\rangle$$



- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].

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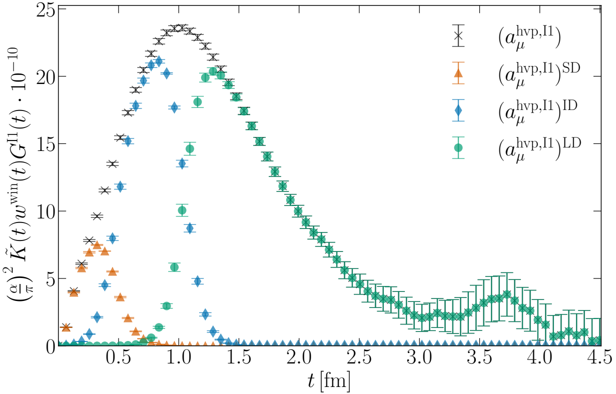


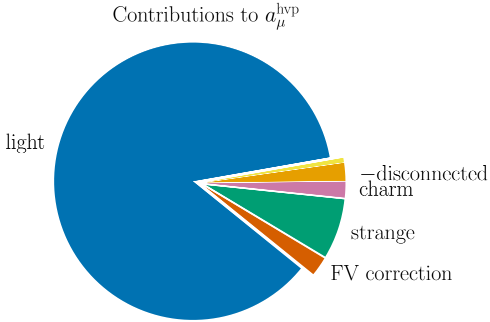
- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].
- Intermediate window $(a_\mu^{\rm hvp,I1})^{\rm ID}$
  - Cutoff effects suppressed.
  - No signal-to-noise problem.
  - Finite-volume effects small.

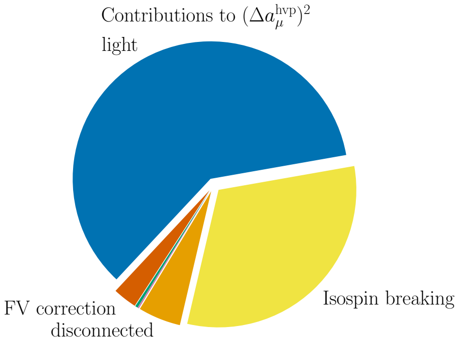
The electromagnetic current

$$j_\mu^{\rm em} = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \tfrac{2}{3}\bar{c}\gamma_\mu c + \ldots = j_\mu^{I=1} + j_\mu^{I=0}$$

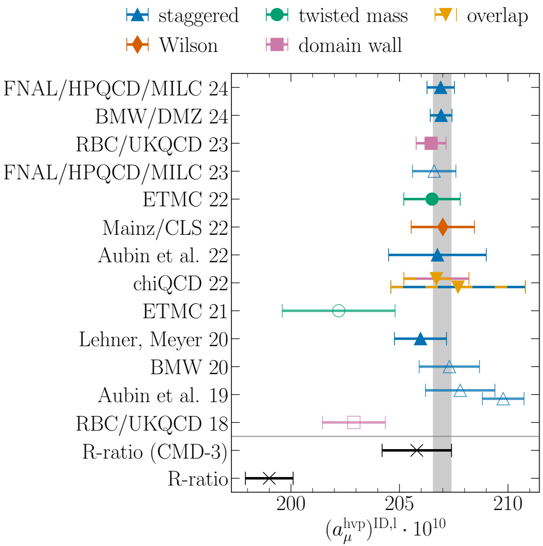
from zero-momentum vector-vector correlation functions

$$G^{\rm isoQCD}(t) = \tfrac{5}{9}G^{\rm light}(t) + \tfrac{1}{9}G^{\rm strange}(t) + \tfrac{4}{9}G^{\rm charm}(t) + G^{\rm disc}(t) + \ldots$$



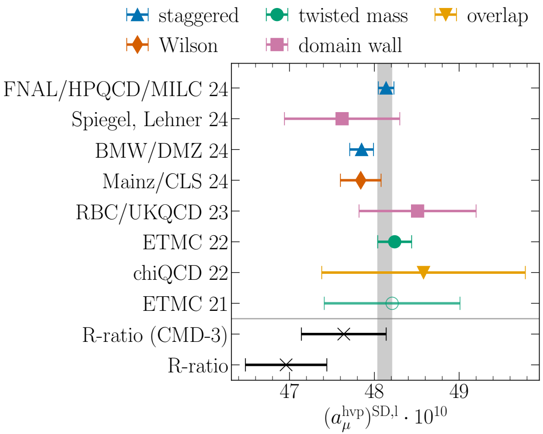


Based on [Mainz/CLS, 2411.07969]: $a_\mu^{\rm hvp} = 724.5\,(7.1)\cdot 10^{-10}$



R-ratio determinations from [Benton et al., 2411.06637]

- Many results for the dominant (light-connected) contribution.
- Good agreement.
  - Statistically uncorrelated.
  - Systematically only weakly correlated.
  - **Blinded analyses** are now standard.
- White Paper estimate for $a_\mu^{\rm hvp}$ by combining window averages:
  - 8 collaborations, 17 papers.
  - 3 precision calculations of $a_\mu^{\rm hvp}$.



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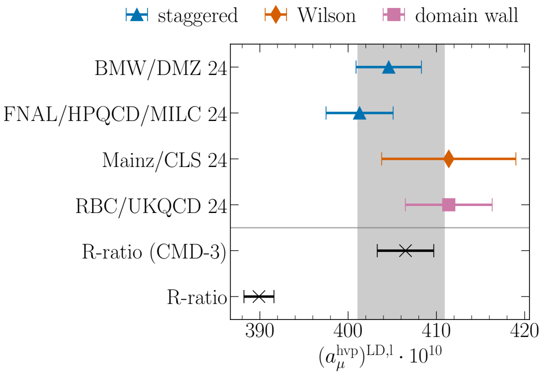

BMW/DMZ 24 did not enter the White Paper average

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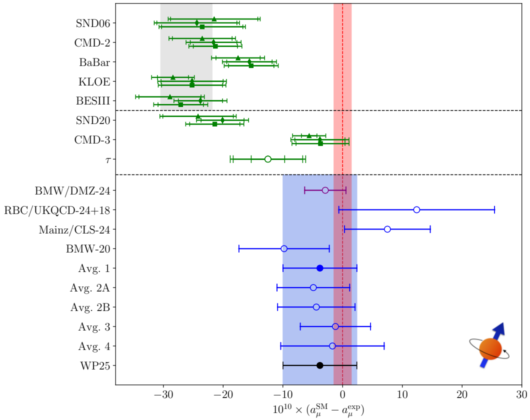
Windows



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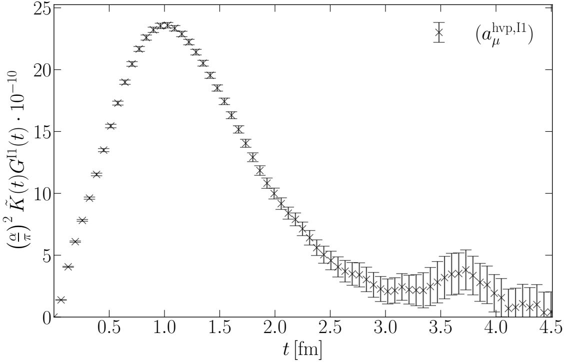
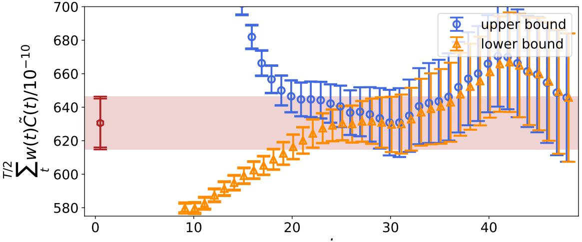
Dominant sources of uncertainty for a_μ^{hvp}



Exponential deterioration of the signal-to-noise ratio.

Improve the signal at large t via algorithmic advancements and smart ideas:

- Bounds on the correlator.
- Noise reduction methods
- Spectral reconstruction of the $\pi\pi$ contributions.



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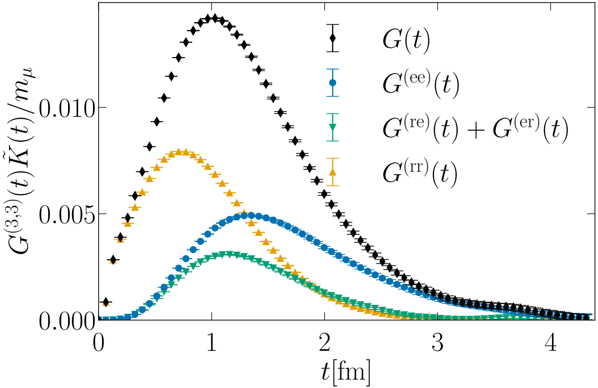
[RBC/UKQCD, 1910.11745]



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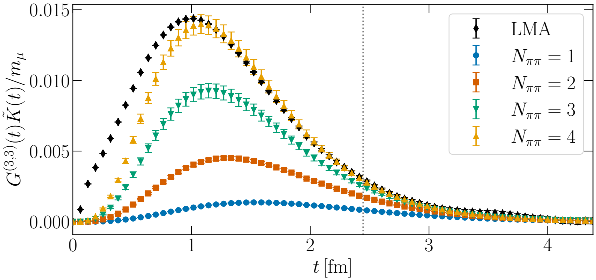

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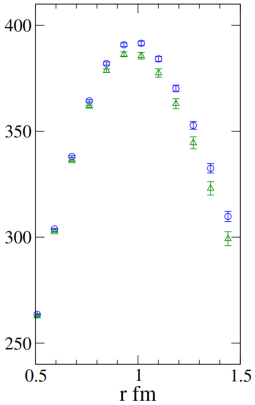
[Mainz, 2411.07969]

3% finite-L corrections for a_μ^{hvp} at $m_\pi L = 4$, mostly in the **isovector channel**.

- EFT and model calculations.
  - NNLO χPT $\rightarrow$ first steps toward 3NLO [Lellouch et al., 2510.12885]
  - Two-pion spectrum in finite-volume and the timelike pion form factor [Meyer, 1105.1892] [Lellouch and Lüscher, hep-lat/0003023] [Giusti et al., 1808.00887].
  - Pions winding around the torus and the electromagnetic pion form factor [Hansen, Patella, 1904.10010, 2004.03935].
  - Rho-pion-gamma model [Sakurai] [Jegerlehner, Szafron, 1101.2872] [HPQCD, 1601.03071].
- Simulations at $L > 10\,\text{fm}$ [PACS, 1902.00885] [BMWc, 2002.12347].
  - Uncertainty statistics dominated.
  - Show good consistency with models.



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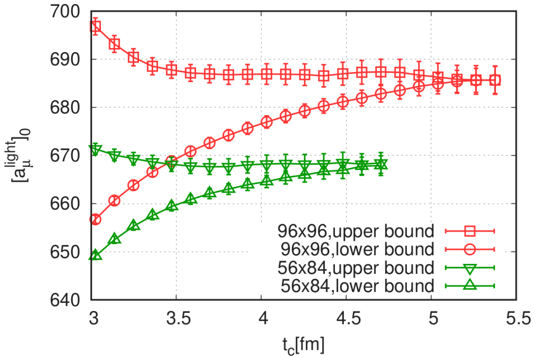

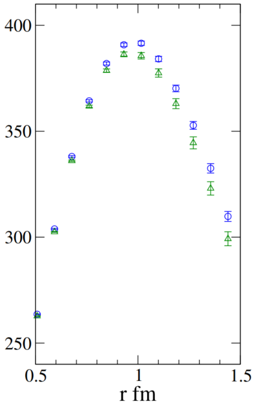

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Cutoff effects

Systematic uncertainties from the continuum extrapolation become dominant.

- Extrapolation to the continuum limit guided by Symanzik effective theory.
- Cutoff effects start at $\mathrm{O}(a^2)$ in modern lattice calculations.[1]
- Mandatory to
  - include ≥ 4 resolutions to constrain higher order cutoff effects.
  - include fine resolutions $a \leq 0.05\,\mathrm{fm}$ for per-mil uncertainties.

[1] Up to logarithmic corrections [Husung et al., 2111.02347].

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- Staggered quarks: taste violations distort the pion spectrum.
  - This is a cutoff effect: Vanishes in the continuum limit.
  - Taste breaking introduces modifications to $\sim a^2$ scaling.

  $\rightarrow$ Model dependent corrections applied at finite lattice spacing.

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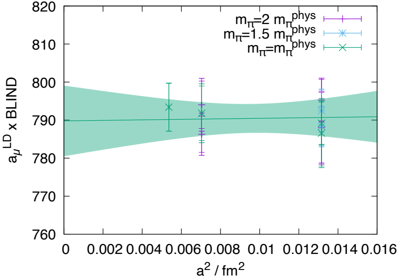
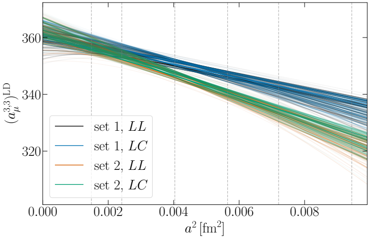
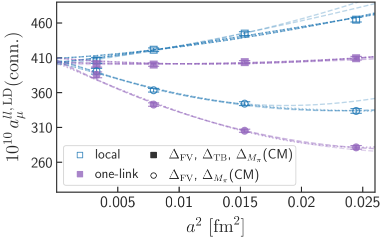
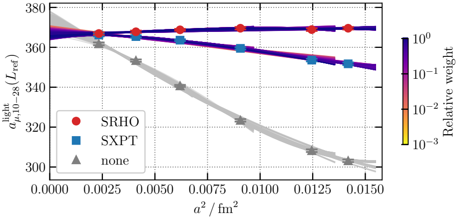
Four sets of extrapolations, based on different fermion actions.









[RBC/UKQCD, 2410.20590]

[Mainz, 2411.07969]

[FNAL, HPQCD, MILC, 2412.18491]

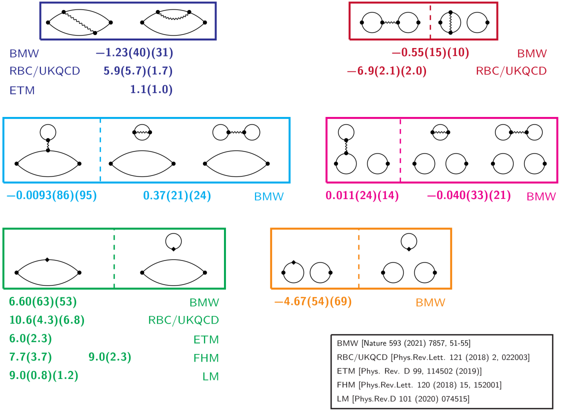
[BMWc, 2407.10913]

Isospin-breaking effects

Need to include $\mathrm{O}\left(\frac{m_\mathrm{u}-m_\mathrm{d}}{\Lambda_\mathrm{QCD}}\right)$ and $\mathrm{O}(\alpha)$ effects for per-mil precision.

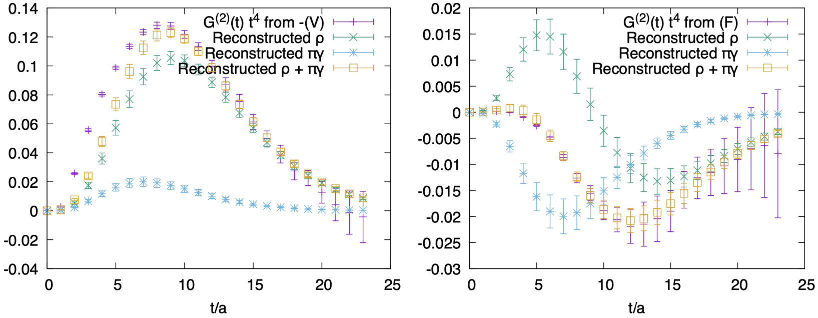
Need to include $\mathrm{O}\left(\frac{m_u - m_d}{\Lambda_{QCD}}\right)$ and $\mathrm{O}(\alpha)$ effects for per-mil precision.

Overview of published results - contributions to $a_\mu \times 10^{10}$



Adapted from [V. Gülpers]

- Method of choice: Perturbative expansion around isospin symmetric QCD [RM123, 1303.4896].
- Many diagrams have to be computed, many of them being very noisy.
- Major challenge: Formulation of QED in a finite box.
- Many results, only one complete calculation [BMWc, 2002.12347].



↑ Spectral reconstruction for isospin-breaking contributions [Lehner et al., 2508.21685]

- QED_∞ with Pauli-Villars regulator [Parrino et al., 2501.03192][Erb et al., 2505.24344].
- Direct simulation of QCD+QED [Altherr et al., 2506.19770].

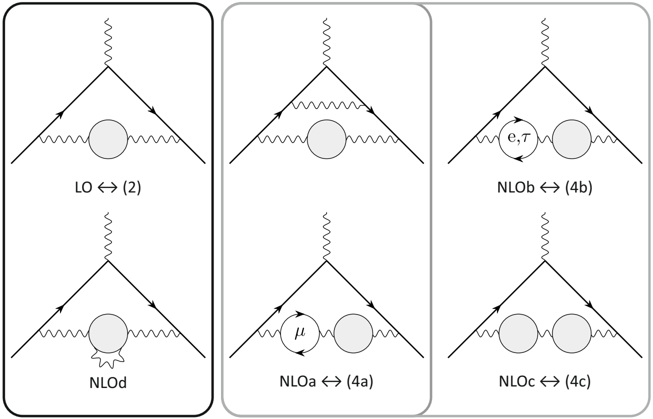
Open questions

- Only few precise results for the subleading isoscalar contribution at long distances, containing the noisy quark disconnected contributions.
- More results for isospin-breaking corrections are needed, especially beyond the *electroquenched approximation*.
- Lattice results are broadly consistent at the current level of precision.
  Small tensions in the LD regime under the variation of the isoQCD scheme have to be better understood.

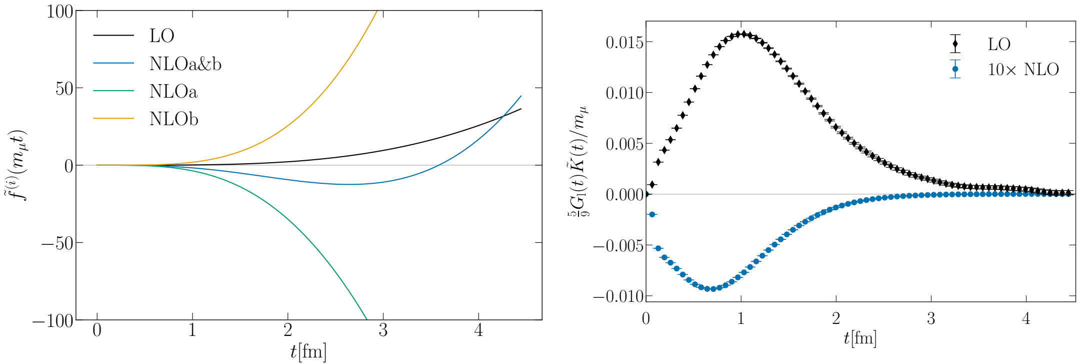
$\rightarrow$ Ongoing work to address these points.

Related observables

Next-to-leading order HVP



- NLO HVP at $\mathrm{O}(\alpha^3)$: 4 additional topologies.
- WP 25 average based on data-driven analyses (only lattice QCD result had 14% uncertainty [Chakraborty et al., 1806.08190]).

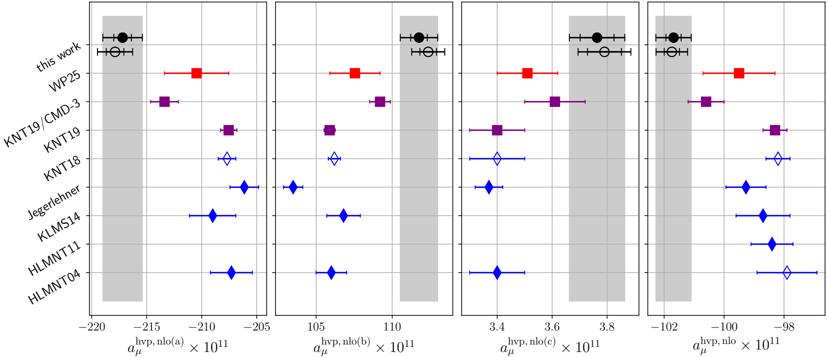


■ TMR kernels for lattice evaluation derived in [Balzani et al., 2406.17940] based on [Balzani et al., 2112.05704], see also [Nesterenko, 2112.05009].

■ Combination of (4a) and (4b) NLO diagrams is more short ranged than LO.

→ Reduction of statistical uncertainties and isospin-breaking corrections.

Next-to-leading order HVP



- First precision lattice determination by the Mainz group [Beltran et al., 2603.06806]:
  - Full calculation: $-101.57(59) \times 10^{-11}$
  - Isopin-breaking effects estimated phenomenologically: sub per-mille.
  - Final uncertainty of 0.6% is sufficient for all future SM predictions.
- Disagreement with data-driven determinations expected and found.

The hadronic running of $\Delta\alpha$

- The running of the electromagnetic coupling

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha(q^2)}\,, \qquad \Delta\alpha(q^2) = \Delta\alpha_{\rm lep}(q^2) + \Delta\alpha^{(5)}_{\rm had}(q^2) + \Delta\alpha_{\rm top}(q^2)$$

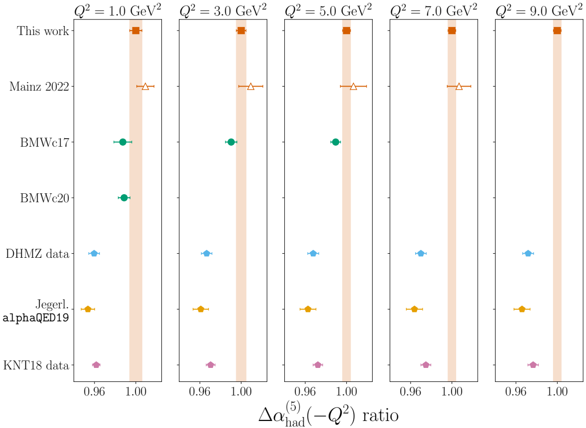
  is closely related to $a_\mu^{\rm hvp}$, also in a lattice QCD calculation.

- Reuse data to determine $\Delta\alpha_{\rm had}(-Q^2)$ at spacelike momenta $Q^2 \lesssim 12\,{\rm GeV}^2$.

- Combination with perturbation theory needed to compute $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$:
  $\rightarrow$ perturbative Adler function, Euclidean split technique [Jegerlehner, 0807.4206].

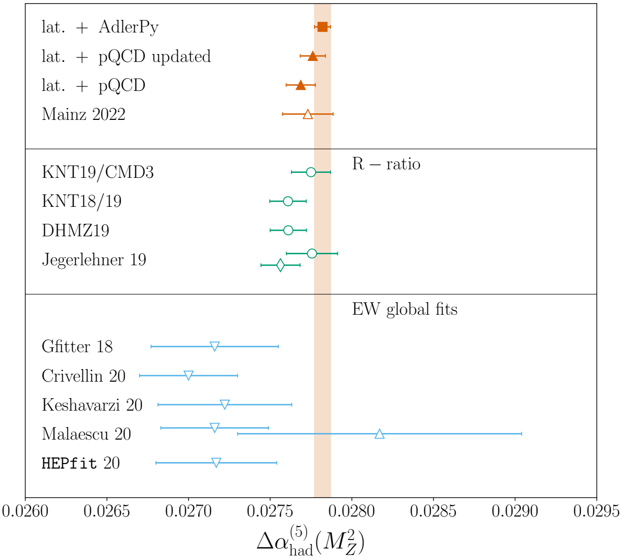
- Challenge:
  - Control over discretization effects on the lattice for large Q^2.
  - Control over perturbative uncertainties for small Q^2.
  - Window problem!

Ab initio calculation of $\Delta\alpha_{\mathrm{had}}$



- Advancements for a_μ^{hvp} enable precision lattice calculation in [Conigli et al, 2511.01623] improving on [Mainz 2022, 2203.08676].
- Further lattice results in [BMW, 1711.04980,2002.12347].
- Same discrepancy with data-driven results as in a_μ^{hvp}.

Ab initio calculation of $\Delta\alpha_{\text{had}}$



- Advancements for a_μ^{hvp} enable precision lattice calculation in [Conigli et al, 2511.01623] improving on [Mainz 2022, 2203.08676].
- Further lattice results in [BMW, 1711.04980,2002.12347].
- Same discrepancy with data-driven results as in a_μ^{hvp}.
- Combine with perturbative Adler function in AdlerPy [Ferro Hernández, 2311.04849] to reach 0.2% precision in $\Delta\alpha^{(5)}_{\text{had}}(M_Z^2)$.

CONCLUSIONS

- SM uncertainty has to be reduced by a factor of 4 to make full use of the astonishing experimental precision $\to 0.2\%$ precision on a_μ^{hvp} needed.
- Pushing the precision of a single lattice QCD calculation of a_μ^{hvp} to better than 0.5% will be very difficult.
- Best scenario: Average of compatible and mostly independent lattice results.
- Ongoing work to reduce the dominant sources of uncertainty by several collaborations.