Lecture 2: Simplicial Homology: From Simplicial Complexes to Vector Spaces

Claudia Landi

UNIMORE
UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA

Italian-Japanese summer school in Mathematics - Pisa, 2026

Motivation

- Homology provides a mathematical language for holes in a topological space.
- It captures holes indirectly, by focusing on what surrounds them.
- Ingredients are vector spaces and linear maps
- For computational reasons, it is convenient to work with vector spaces with scalars in $\mathbb{Z}_2$, called **modulo 2 coefficients**.
- Coefficients could, however, be more complicated like elements of some other field, or of a ring, like the integers.

Free vector spaces

Let S be a finite set and $\mathbb{F}$ a field.

Definition. The **free** $\mathbb{F}$-vector space generated by S is the vector space $F(S)$ with elements given by formal linear combinations $\sum_{s\in S} a_s s$ where each $a_s \in \mathbb{F}$

- addition and multiplication by a scalar are defined component-wise in the obvious way:

$$\sum_{s\in S} a_s s + \sum_{s\in S} b_s s := \sum_{s\in S} (a_s + b_s) s$$

$$\lambda \sum_{s\in S} a_s s := \sum_{s\in S} \lambda a_s s$$

- the neutral element is $0 = \sum_i 0\sigma_i$.
- the inverse of c is $-c$.

If $S = \emptyset$, then we set $F(S) = 0$.

The resulting $\mathbb{F}$-vector space has a canonical vector space basis given by the elements in S.

Example. Let $\mathbb{F} = \mathbb{Z}_2$ or $\mathbb{F} = \mathbb{Z}_3$ and $S = \{a, b\}$.

Free vector spaces generated by ordered simplices

- Let K be an (abstract) simplicial complex and $\mathbb{F}$ a field.
- Fix a total order $\leq$ for the vertices of K.
- If a simplex σ in K is spanned by vertices $v_0, \ldots, v_p$, we can assume $v_i \leq v_j$ for $i \leq j$ and we write $\sigma = [v_0, \ldots v_p]$ to signify that vertices of σ are ordered.

Definition

For a given integer $p \geq 0$, the free $\mathbb{F}$-vector space generated by the ordered p-simplices of K is called the vector space of ***p*-chains** in K, and denoted $C_p(K; \mathbb{F})$ or simply $C_p(K)$

If K has n_p simplices of dimension p, then $C_p(K) \cong \mathbb{F}^{n_p}$.

If $\mathbb{F} = \mathbb{Z}_2$, we can think of a p-chain $c = \sum_i a_i \sigma_i$ as a set of p-simplices, namely those σ_i with $a_i = 1$.

Boundary of an ordered simplex

The **boundary** of an ordered p-simplex is a linear combination of its $(p-1)$-dimensional ordered faces: for $\sigma = [u_0, u_1, \ldots, u_p]$ its boundary is

$$\partial_p \sigma = \sum_{j=0}^{p} (-1)^j [u_0, \ldots, \hat{u}_j, \ldots, u_p],$$

where the hat indicates that u_j is omitted (so $[u_0, \ldots, \hat{u}_j, \ldots, u_p]$ is the face opposite to vertex u_j in $[u_0, \ldots, u_j, \ldots, u_p]$).

Example:

- $\partial_0[x] = [\emptyset]$
- $\partial_1[x, y] = [y] - [x]$
- $\partial_2[x, y, z] = [y, z] - [x, z] + [x, y]$
- $\partial_3[w, x, y, z] = [x, y, z] - [w, y, z] + [w, x, z] - [w, x, y]$

Boundary map

Linearly extending ∂_p we obtain the **p-boundary map**

$$\partial_p \colon C_p(K) \to C_{p-1}(K)$$

For a p-chain, $c = \sum_i a_i \sigma_i$, its boundary is the sum of the boundaries of its simplices, i.e. $\partial_p c = \sum_i a_i \partial_p \sigma_i$

Example: $\partial_1([x,y]+[y,z]) = [z]-[x]$

Being a linear map of vector spaces, ∂_p can be represented as a $n_{p-1} \times n_p$ matrix.
For example, for a tetrahedron with vertices a,b,c,d:

$\mathbb{F} = \mathbb{R}$

$$\partial_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \left[\begin{array}{ccccc} -1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

$\mathbb{F} = \mathbb{Z}_2$

$$\partial_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} ab & bc & cd & ad & ac \end{array} \\ \left[\begin{array}{ccccc} +1 & 0 & 0 & +1 & +1 \\ 1 & +1 & 0 & 0 & 0 \\ 0 & 1 & +1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

- The chains of the vector subspace $Z_p := \ker \partial_p$ are called *p*-**cycles**
- The chains of the vector subspace $B_p := \operatorname{im} \partial_{p+1}$ are called *p*-**boundaries**

Fundamental lemma of homology

$$\partial_{p-1}\partial_p c = 0 \text{ for every integer } p \text{ and every } p\text{-chain } c \in C_p(K).$$

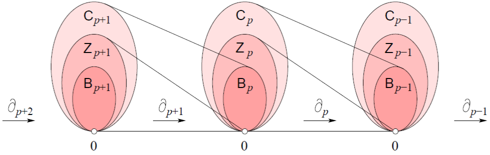
Proof. By linearity, it is sufficient to prove the claim for $c = [u_0, \ldots, u_p]$:

$$\partial_{p-1}\partial_p([u_0, \ldots, u_p]) = \partial_{p-1} \sum_{i=0}^{p} (-1)^i [u_0, \ldots, \hat{u}_i, \ldots, u_p]$$

$$= \sum_{j<i} (-1)^{i+j} [u_0, \ldots, \hat{u}_j, \ldots, \hat{u}_i, \ldots, u_p] + \sum_{j>i} (-1)^{i+j-1} [u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_p] = 0$$

□

- The condition $\partial_p \circ \partial_{p+1} = 0$ ensures that, for every p, B_p is a subspace Z_p justifying the following definition.



Homology

Definition. The pth **homology** of K is the quotient vector space $H_p(K) = Z_p / B_p$.

- Each element of $H_p(K)$ is the equivalence class, called **homology class**, obtained by adding all p-boundaries to a given p-cycle:

$$c + B_p, \quad \textit{with} \quad c \in Z_p$$

- two p-cycles c, c' in the same homolgy class are called **homologous** and they differ by a boundary:

$$c \sim c' \quad \Leftrightarrow \quad c - c' \in B_p$$

- **Addition of two classes** is defined by $(c + B_p) + (c' + B_p) = (c + c') + B_p$
- **Multiplication by a scalar** $\lambda \in \mathbb{F}$ is defined by $\lambda(c + B_p) = (\lambda c) + B_p$
- Addition an multiplication by a scalar are independent of the representative and are therefore well defined.
- $H_p(K) = (\mathbb{F}, H_p(K), +, \cdot)$ is a vector space.

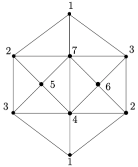
Betti numbers of K

Definition. The dimension of $H_p(K)$, denoted $\beta_p(K)$ is called the pth **Betti number** of K.

- $\beta_p(K) = \dim H_p(K) = z_p - b_p$

For simplicial complexes in $\mathbb{R}^3$ the Betti numbers are independent of $\mathbb{F}$.

Dependence on $\mathbb{F}$ is exemplified in the case of the projective plane:



$\beta_0(\mathbb{R}P^2;\mathbb{Z}_3) = 1$	$\beta_1(\mathbb{R}P^2;\mathbb{Z}_3) = 0$	$\beta_2(\mathbb{R}P^2;\mathbb{Z}_3) = 0$
$\beta_0(\mathbb{R}P^2;\mathbb{Z}_2) = 1$	$\beta_1(\mathbb{R}P^2;\mathbb{Z}_2) = 1$	$\beta_2(\mathbb{R}P^2;\mathbb{Z}_2) = 1.$

The 0th Betti number

Proposition $\beta_0 =$ number of connected components of the 1-skeleton of K.

Proof. It is enough to consider the 1-skeleton of K. Assume it is connected. Its vertex set $\{p_1, \ldots p_m\}$ is a basis for Z_0.

- By a change of basis, $\{p_1, p_1 - p_2, \ldots p_1 - p_m\}$ is also so.
- If every two vertices of are connected by an edge path, $p_1 - p_2, \ldots p_1 - p_m$ are boundaries.
- So, $\beta_0 = z_0 - b_0 \leq m - (m-1) = 1$.
- The boundary of a 1-chain is the linear combination of an even number of vertices, so p_1 cannot be a boundary
- $\beta_0 \geq 1$

If the 1-skeleton of K is the union of connected components $K^1, \ldots, K^s$, ∂_1 is a block diagonal matrix with one block ∂_1^i for each connected component. So, letting m_i be the number of vertices of each K^i.

$$\beta_0(K) = m - \mathrm{rk}\partial_1 = \sum_i m_i - \sum_i \mathrm{rk}\partial_1^i = \sum_i (m_i - \mathrm{rk}\partial_1^i) = \sum_i \beta_0(K^i) = s$$

Maps induced in homology

Let K and L be abstract simplicial complexes with vertex sets V_K and V_L, respectively.

Associated with a simplicial map $f\colon K \to L$, we get an induced linear map of chains in all dimensions $p \in \mathbb{Z}$:

$$f_\bullet\colon C_p(K) \to C_p(L),\ f_\bullet(\sigma) = \begin{cases} f(\sigma) & \text{if } \dim f(\sigma) = \dim \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Lemma (proof for $\mathbb{F} = \mathbb{Z}_2$) for simplicity)

It holds that $\partial_p \circ f_\bullet = f_\bullet \circ \partial_p$ for every $p \geq 0$.

Proposition

A simplicial map $f\colon K \to L$ induces a well-defined homological linear map $f_*\colon H_p(K) \to H_p(L)$ defined by $f_*([c]) = [f_\bullet(c)]$, for every $p \geq 0$.

- if $f = id_K$, then $f_* = id_{H(K)}$
- $(gf)_* = g_* f_*$

Invariance of Homology

Let K, L be two simplicial complexes and let $|K|, |L|$ be their geometric realizations.

Theorem (without proof)

If $|K|$ and $|L|$ are homotopy equivalent, then $H_*(K) \cong H_*(L)$.

To see this, one needs another homology theory, called singular homology, which is defined for topological spaces, is a homotopy invariant, and is such that $H_*(K) \cong H_*^{\text{sing}}(|K|)$.

With more work, the Nerve Lemma lemma thus becomes:

Persistent Nerve Lemma

Let P be a point cloud. If $r \leq r'$, the following diagram commutes, where the vertical arrows are isomorphisms and the horizontal arrows are induced by inclusions:

$$\begin{tikzcd}
H_*^{\text{sing}}(\cup_{p \in P} B_r(p)) \arrow[r] \arrow[d, "\cong"] & H_*^{\text{sing}}(\cup_{p \in P} B_{r'}(p)) \arrow[d, "\cong"] \\
H_*(\check{C}_r(P)) \arrow[r] & H_*(\check{C}_{r'}(P))
\end{tikzcd}$$