Nonlinear and Structured Optimization: Theory and Algorithms

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Course Description

This is an introductory course in optimization theory and algorithms. We will cover basic concepts of convex analysis, generalized convexity, smooth and nonsmooth optimality conditions, duality, and algorithmic methods, including first-order and splitting techniques. The course also addresses multiobjective optimization and concludes with selected topics in global and structured optimization.

Course Syllabus

1 Foundations of Optimization

1.1 Optimization Problems and Modeling

- Linear and nonlinear optimization problems
- Aspects of good problem formulation
- Existence and basic optimality concepts

1.2 Basic Concepts of Convexity

- Convex sets and convex functions
- Operations preserving convexity
- Separating hyperplane theorem

1.3 Generalized Convexity

- Generalized convexity in the noncontinuous case
- Generalized convexity in the continuous case
- Generalized convexity in the differentiable case
- Generalized linearity

2 Smooth and Nonsmooth Optimality Conditions and Duality

2.1 Differentiability and Smooth Optimality Conditions

- Fréchet, Hadamard, Gâteaux and Dini differentiability
- Necessary optimality conditions: from Fermat's lemma to KKT
- Sufficient optimality conditions under generalized convexity assumptions
- Duality in the differentiable setting: Wolfe, Mond–Weir and Lagrangian duality

2.2 Nonsmooth Optimality and Convex Duality

- Subgradients and subdifferentials
- Fermat's lemma in the nonsmooth setting
- Fenchel conjugates and Fenchel duality

3 Algorithmic Implications of Optimality and Duality

3.1 Fixed-Point Methods and First-Order Algorithms

- Contractions and Banach fixed-point theorem
- Nonexpansive and averaged operators
- Krasnosel'skiĭ–Mann iteration
- Proximal operators and regularization techniques

3.2 Splitting and Decomposition Methods

- Gradient and subgradient methods
- Proximal gradient method and forward–backward splitting
- Douglas–Rachford splitting
- ADMM and dual proximal gradient methods

4 Multiobjective and Vector Optimization

- Vector-valued objective functions and cone-induced orderings
- Pareto efficiency and Pareto geometry
- Convexity and generalized convexity for Pareto optimality
- Necessary and sufficient optimality conditions for Pareto efficiency
- Scalarization techniques and efficient frontiers

5 Selected Topics in Global and Structured Optimization

- Linear fractional optimization and Data Envelopment Analysis
- Rank-2 optimization problems
- Bilevel optimization problems

Recommended References

Optimization Theory

- R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.
- J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer, 2001.
- M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*, 3rd Edition, Wiley-Interscience, 2006.
- A. Cambini and L. Martein, *Generalized Convexity and Optimization: Theory and Applications*, Springer, 2009.
- M. Avriel, W. E. Diewert, S. Schaible, and I. Zang, *Generalized Concavity*, SIAM Classics in Applied Mathematics, Vol. 63, 2010.
- O. L. Mangasarian, *Nonlinear Programming*, SIAM Classics in Applied Mathematics, Vol. 10, 1987.

Modeling and Algorithms

- S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.
- A. Beck, *First-Order Methods in Optimization*, SIAM, 2017.
- D. P. Bertsekas, *Convex Optimization Algorithms*, 2nd Edition, Athena Scientific, 2015.
- Y. Nesterov, *Lectures on Convex Optimization*, 2nd Edition, Springer, 2018.

Prerequisites

Basic knowledge of **linear algebra** and **multivariate calculus** is required.

Students are encouraged to review the following resources prior to the course:

- Appendices A and C of *S. Boyd and L. Vandenberghe, Convex Optimization*, Cambridge University Press, 2004.
- Appendix A of *D. P. Bertsekas, Nonlinear Programming*, 3rd Edition, Athena Scientific, 2016.

Course timetable

from Monday to Thursday:

9-10.30 Lecture

10.30-11 Coffee Break

11-12.30 Lecture

12.30-14 Lunch

14-16 Lecture

16-18 Individual or group work

Friday:

9-10.30 Lecture

10.30-11 Coffee Break

11-12.30 Lecture

12.30-14 Lunch

14-16 Individual or group work review