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Microcausality Bounds on Goldstones

with M. Mancini, W. P. McBlain, in progress

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Positivity: Lorentz-invariant case

Implications of **fundamental principles** (microcausality, unitarity...) **on EFTs**

(if there is a “standard” UV completion)

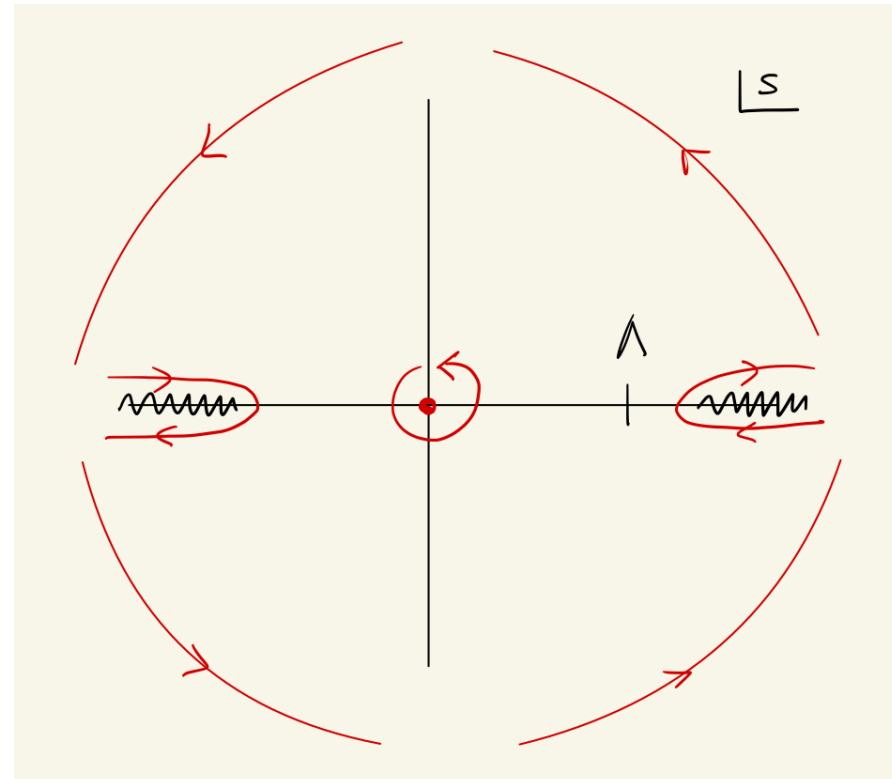
Adams, Arkani-Hamed,
Dubovsky, Nicolis, Rattazzi, 06

$$\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 + \frac{c}{\Lambda^4}(\partial\pi)^4 \quad c \geq 0$$

$$\mathcal{A}(s) \equiv \mathcal{M}(s, t \rightarrow 0)$$

$$\mathcal{A}(s) = c \frac{s^2}{\Lambda^4} + \dots$$

Froissart bound: $|\mathcal{A}(s)| < s \log^2 s$



Similar bounds for non-LI theories?

In many interesting cases Lorentz-invariance is **spontaneously broken**

1. **Cosmology**. Inflation, Dark Energy, EFTofLSS...

LI broken by the presence of “stuff”, and on large scales by curvature

Interested in “peculiar” theories (Galileon, NEC violation...): consistent?

2. **Condensed Matter**. Can we deduce general inequalities for a system?

3. **QFT at finite T or finite Q**

4. Compact objects: **worldline EFT (Love numbers)**

Lessons learned so far

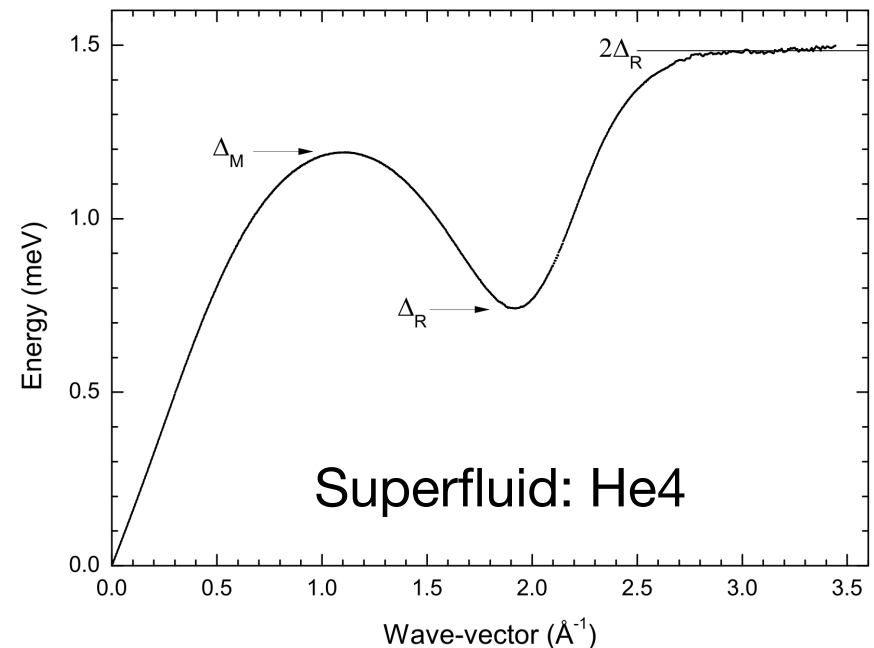
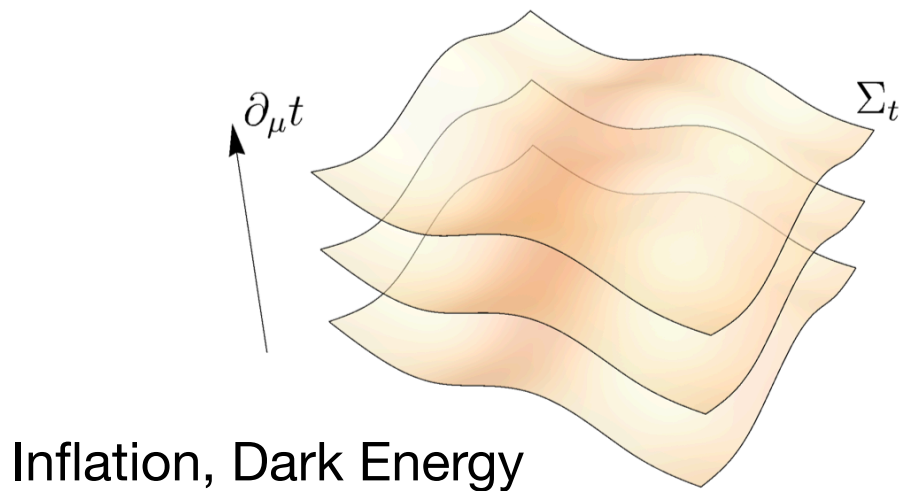
- The EFT is **defined** with non-linearly realised Lorentz

Cannot be “extrapolated” from a LI invariant theory: think about a fluid

- The **S-matrix is not a nice object** without LI $\rightarrow$ Correlation functions of J^μ
- **UV behaviour is system dependent** (vs Froissart bound)
- Without LI is **very different**: many things must be rederived from scratch
- Even 2pf $G_R(\omega, k)$ is rich

Goldstone dispersion relation

Arguably the simplest question: **what are the constraints on $\omega(k)$?**



- Weakly-coupled UV theory: no loops in EFT
- $\omega(k)$ real for real k —imaginary part neglected i.e. no dissipation
- Boosts broken, rotational invariance

Simple model: superfluid

$$S = \int d^4x \left[-|\partial\Phi|^2 - m^2|\Phi|^2 - \lambda|\Phi|^4 \right]$$

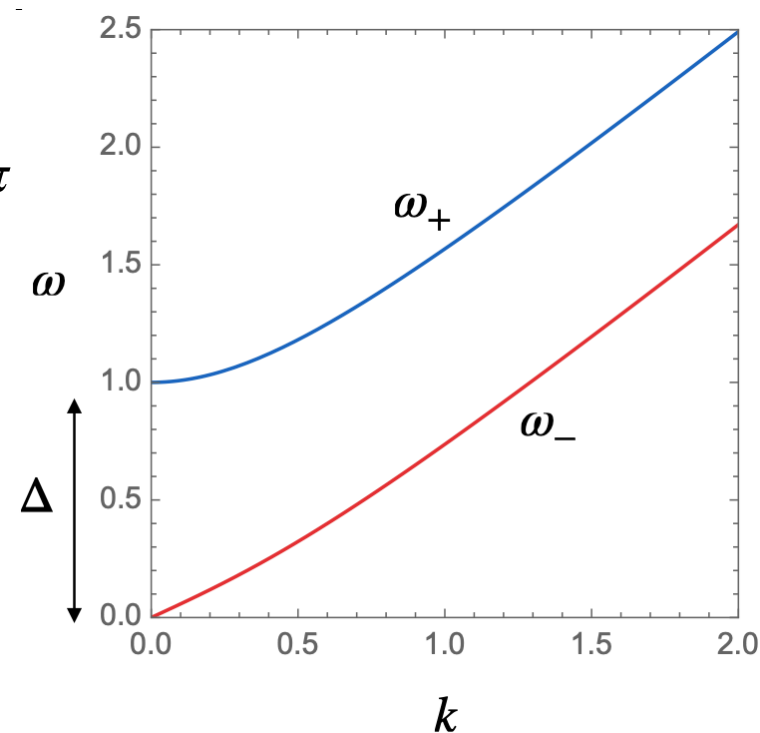
Finite chemical potential: $\bar{\Phi} = e^{i\mu t}$

Hui et al 23, Creminelli et al 23

Gapped mode (+) and a gapless mode (−)

IR: $S = \int d^4x \frac{(X - m^2)^2}{4\lambda} + \text{higher der.}$ $X \equiv \partial^\mu \pi \partial_\mu \pi$

$$\omega_{\pm}(k) = \sqrt{k^2 + \frac{1}{2}\Delta^2 \left(1 \pm \sqrt{1 + 4\frac{(1 - c_s^2)k^2}{\Delta^2}} \right)}$$



$$\omega^2 = c_s^2 k^2 + \frac{k^4}{\Delta^2} + \dots$$

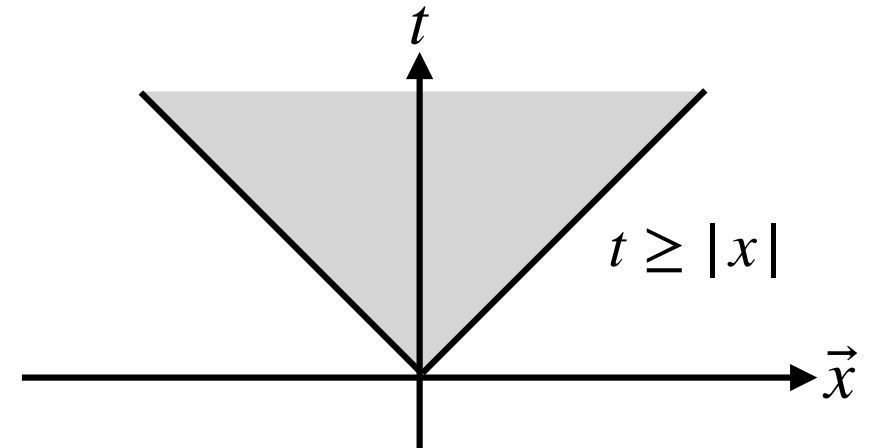
Bogolubov spectrum for $c_s \ll 1$

Microcausality and Analyticity

$[\hat{\mathcal{O}}(x), \hat{\mathcal{O}}(x')] = 0 \quad (x - x')^2 < 0$ **Operator statement:** holds also in Lorentz-breaking state

E.g Dubovsky etal 07

Retarded Green's: $G^R(x) = i\theta(t)\langle[\hat{\mathcal{O}}(x), \hat{\mathcal{O}}(0)]\rangle$



In Fourier space:

$$\tilde{G}^R(p) = \int d^d x \, G^R(x) e^{-ipx} = \int d^d x \, G^R(0; x) e^{-i\text{Re}(p)x + \text{Im}(p)x}$$

Converges for $-\text{Im}(\omega)t + \text{Im}(\vec{k}) \cdot \vec{x} < 0$

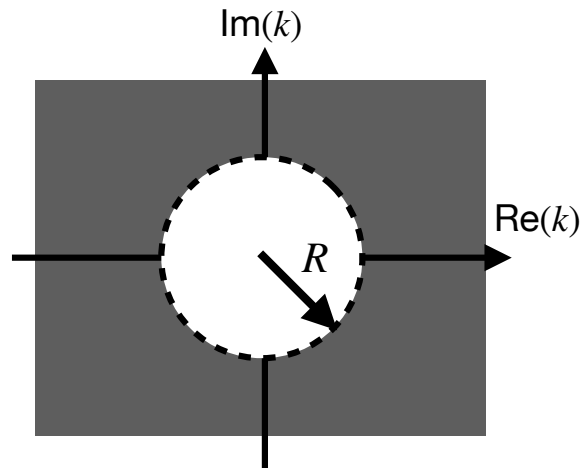
$\tilde{G}^R(\omega, k)$ analytic in $\text{Im}(\omega) > |\text{Im}(k)|$

See Creminelli, Longo, Salehian, Zahed etal 25

Singularities of $\tilde{G}^R(\omega, k)$ must be in: $\text{Im}(\omega) \leq |\text{Im}(k)|$

Goldstone dispersion relation

$$\tilde{G}^{-1}(\omega, k) = 0 \longrightarrow \omega^2 = f(k^2) \longrightarrow \omega(k) = \pm \sum_{n=1}^{\infty} \alpha_n k^n \quad \alpha_n = \frac{a_n}{R^{n-1}}$$

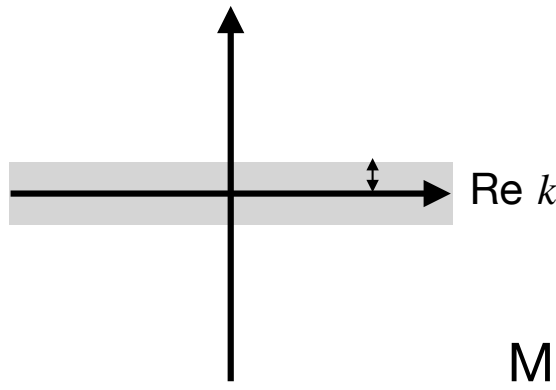


R is radius of convergence
'momentum cut-off' of Goldstone EFT

See Heller, Serantes,
Spalinski, Withers 23 & 24

Question: Under microcausality, what are the optimal bounds on α_n ?

Group velocity ≤ 1 ?



$$\text{Im } \omega(k) \approx \frac{d\omega}{dk} \text{Im } k \quad \xrightarrow{\text{Im}(\omega) \leq |\text{Im}(k)|} \quad \left| \frac{d\omega}{dk} \right| \leq 1$$

Much more than bound on group velocity

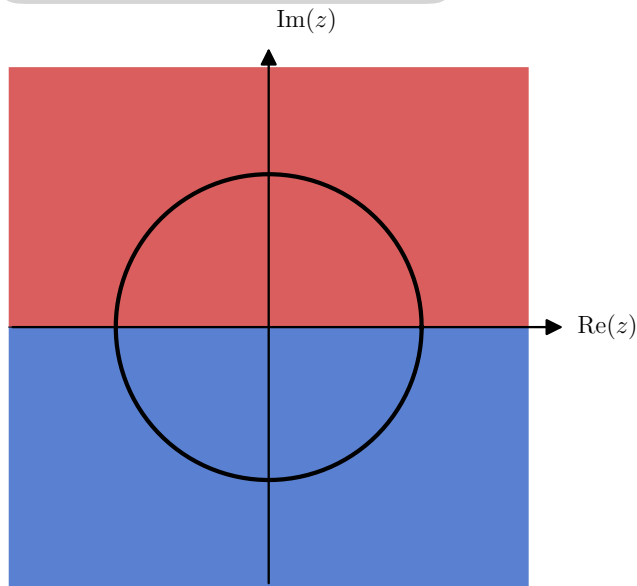
Hui et al 25

Typically-real functions

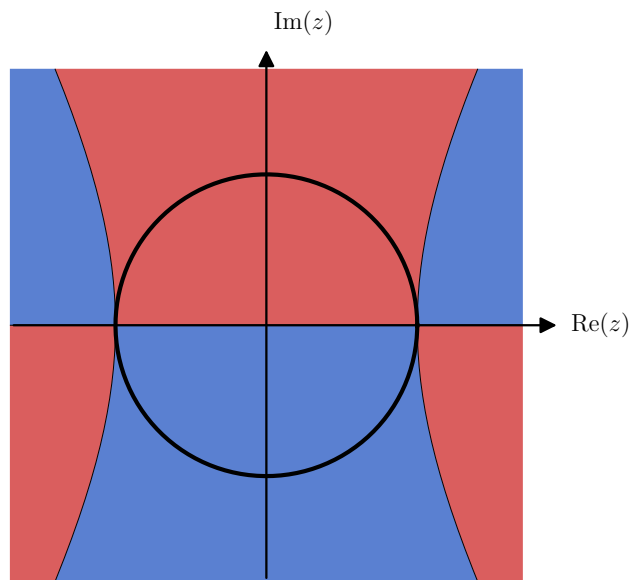
$$\operatorname{Im} f(z) \operatorname{Im} z \geq 0$$

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

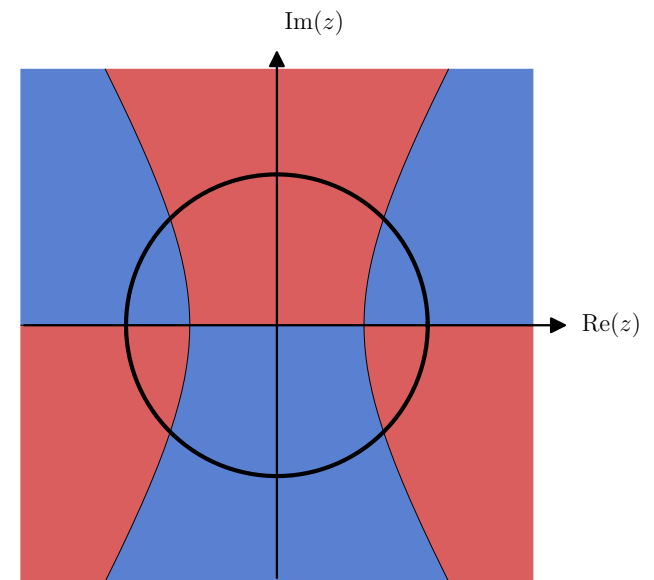
Rogosinski 1932



z



$z - \frac{1}{3}z^3$



$z - z^3$

Typically-real functions must satisfy 2-sided bounds on coefficients

Typically-real in dispersion relations

Consider dispersion relation:

$$\omega(k) = \pm \left(c_s k + \sum_{n=1}^{\infty} \alpha_{2n+1} k^{2n+1} \right) \Delta\omega$$

$$\text{Im}(k) > 0$$

$$\text{Im}(\omega) \leq |\text{Im}(k)| \longrightarrow \pm \text{Im}(c_s k + \Delta\omega) \leq \text{Im}(k)$$

$$(+) : \text{Im} \left[k - \frac{\Delta\omega}{1 - c_s} \right] \geq 0$$

$$(-) : \text{Im} \left[k + \frac{\Delta\omega}{1 + c_s} \right] \geq 0$$

$$\text{Im}(k) < 0$$

Opposite signs

Two **coupled** typically-real functions:

$$\omega_P(k) = k + \frac{\Delta\omega}{1 + c_s}$$

$$\omega_M(k) = k - \frac{\Delta\omega}{1 - c_s}$$

Rogosinski bounds

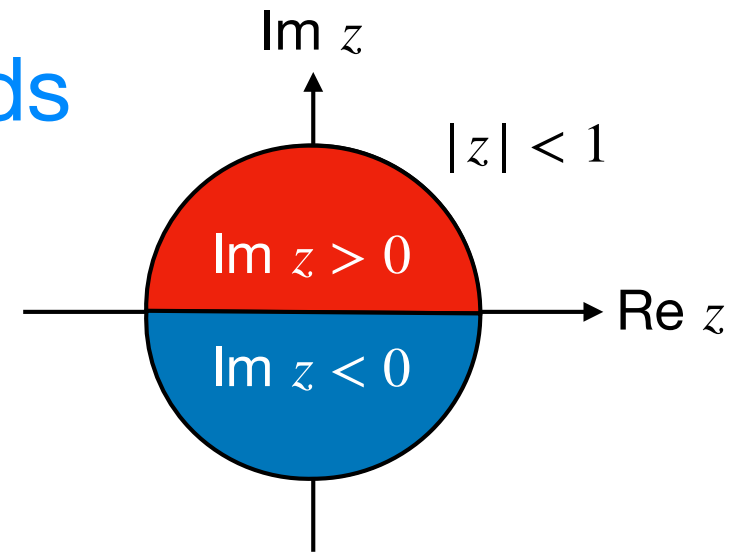
(Odd)
Robertson
representation:

$$f(z) = \int_0^1 \frac{z(1+z^2)}{(1+z^2)^2 - 4z^2t^2} d\mu(t)$$

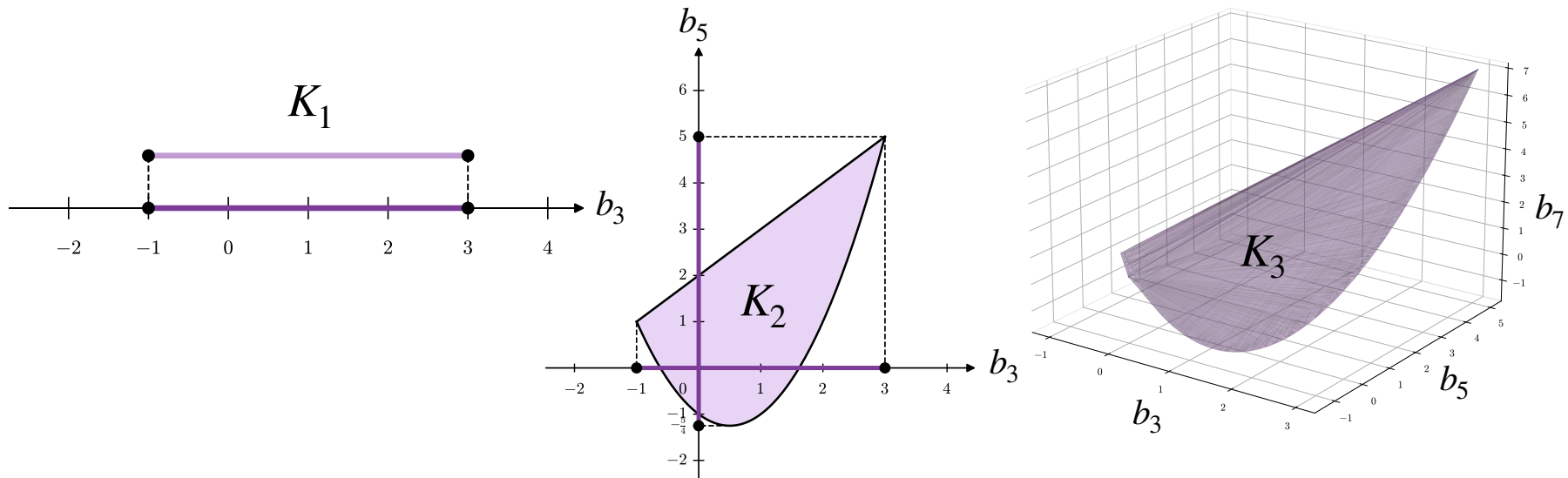
$$f(z) = z + \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1}$$

$$-\kappa_{2n+1} \leq b_{2n+1} \leq 2n+1$$

$$\kappa_{2n+1} = - \min_{0 < \theta \leq \frac{\pi}{2}} \left[\frac{\sin(2n+1)\theta}{\sin \theta} \right]$$



$\{b_{2n+1}\}$ lie within a bounded, convex, infinite-dim. region K_{∞}

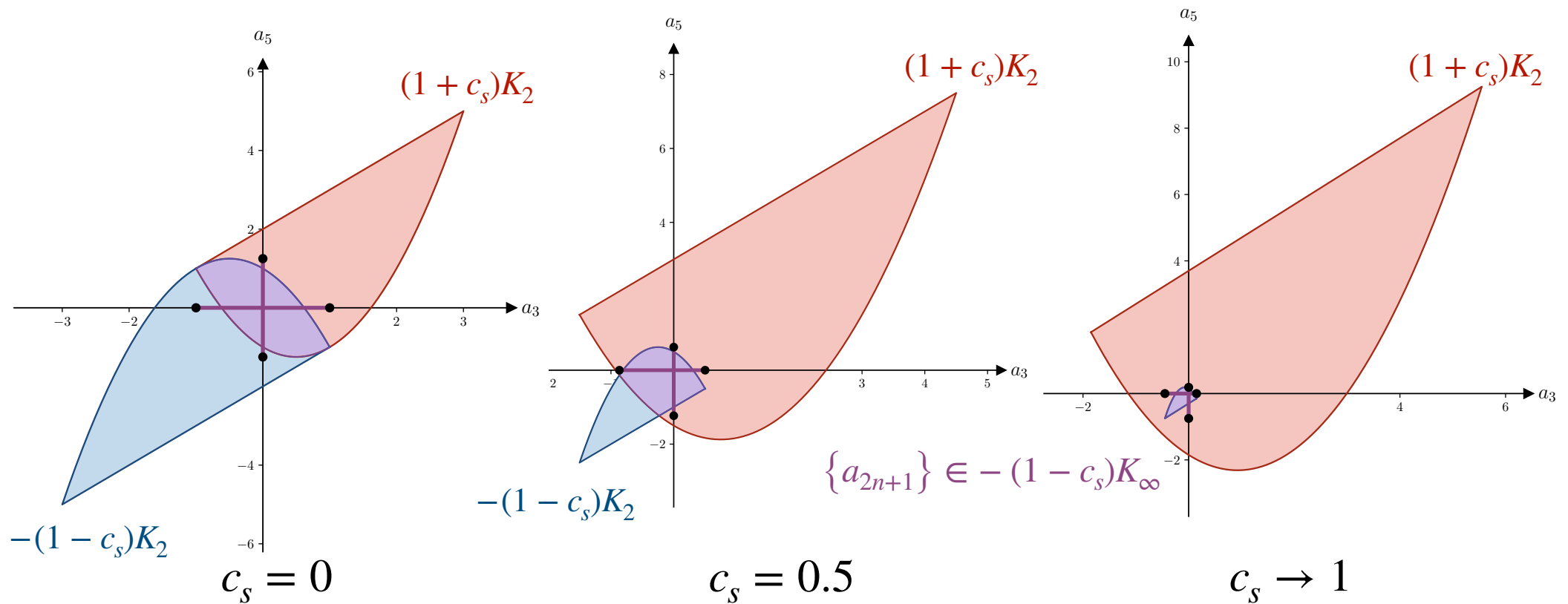


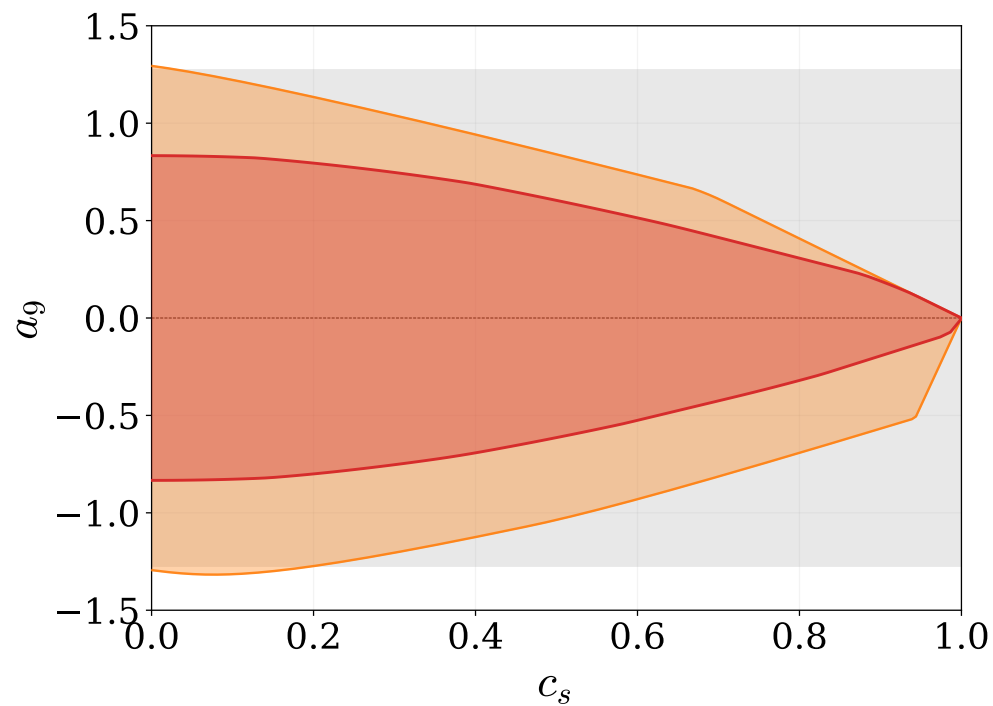
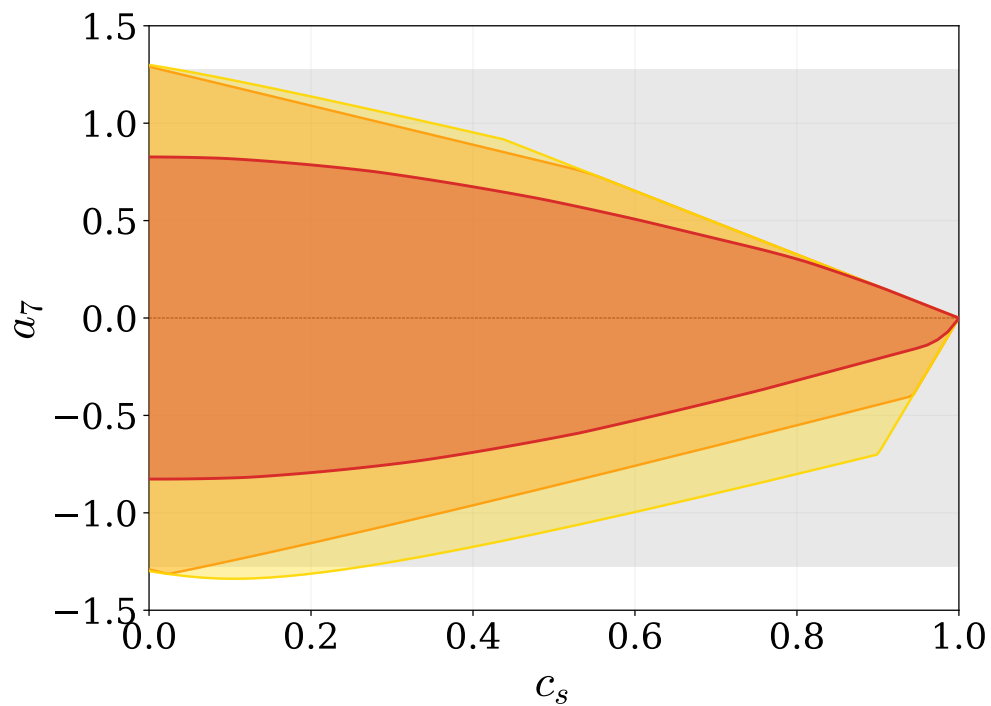
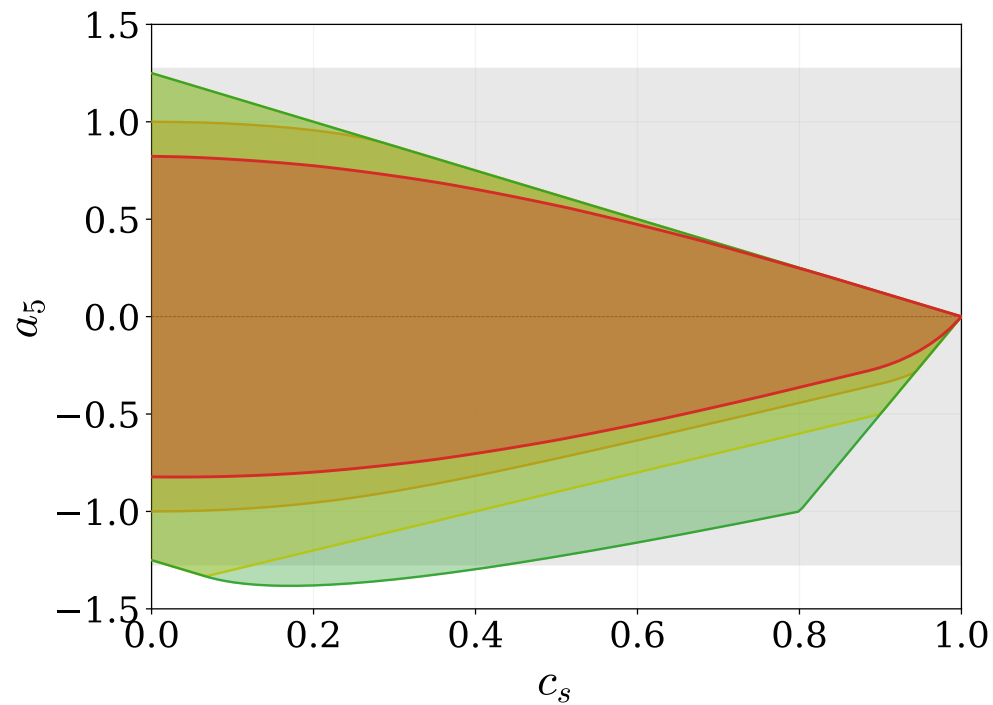
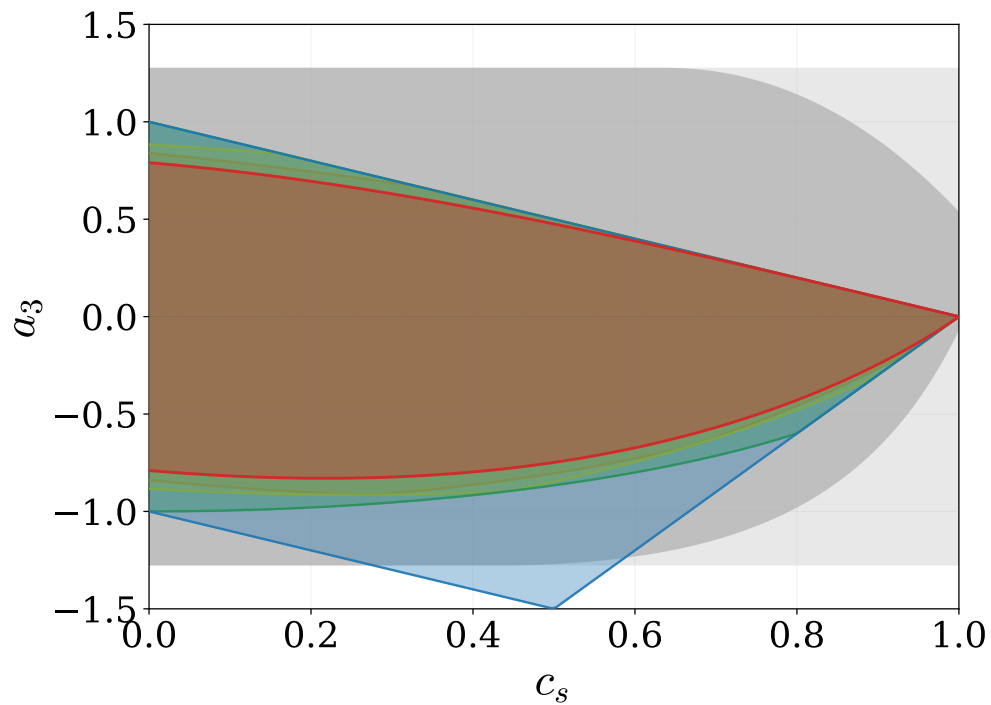
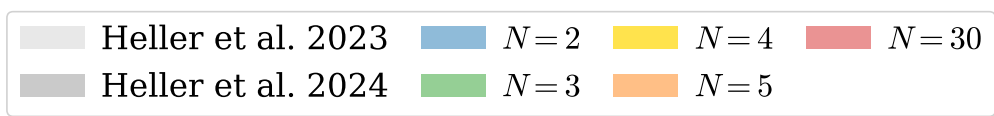
Bounds on dispersion relations

$$f_P(k) = z + \sum_{n=1}^{\infty} \frac{a_{2n+1}}{1 + c_s} z^{2n+1}$$

$$f_M(k) = z - \sum_{n=1}^{\infty} \frac{a_{2n+1}}{1 - c_s} z^{2n+1}$$

$$\{a_{2n+1}\} = \{\alpha_{2n+1} R^{2n+1}\} \in -(1 - c_s)K_{\infty} \cap (1 + c_s)K_{\infty}$$





Upper bounds on cut-off

$$\omega = c_s k \pm \frac{k^3}{M^2} + \dots \quad \Rightarrow \quad R^2 < \#(1 - c_s)M^2$$

For $c_s \rightarrow 1$, $-(1 - c_s)K_\infty$ dominates

$$-(2n + 1)(1 - c_s) \leq \alpha_{2n+1} R^{2n} \leq \kappa_{2n+1}(1 - c_s)$$

$$\begin{aligned} \kappa_3 &= 1 \\ \kappa_5 &= \frac{5}{4} \\ &\vdots \end{aligned}$$

Microcausality forces all α_{2n+1} to vanish in this limit: **both signs**

Hui et al 25

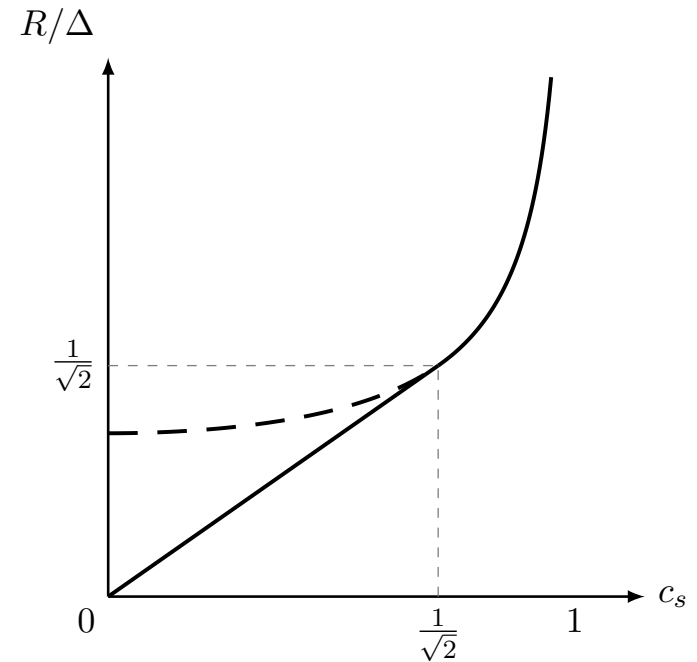
Baumann, Lee, Green, Porto 15

Superfluid

Gapless mode: $\omega_-(k) = \pm \sqrt{k^2 + \frac{1}{2}\Delta^2 \left(1 - \sqrt{1 + 4\frac{(1 - c_s^2)k^2}{\Delta^2}} \right)}$

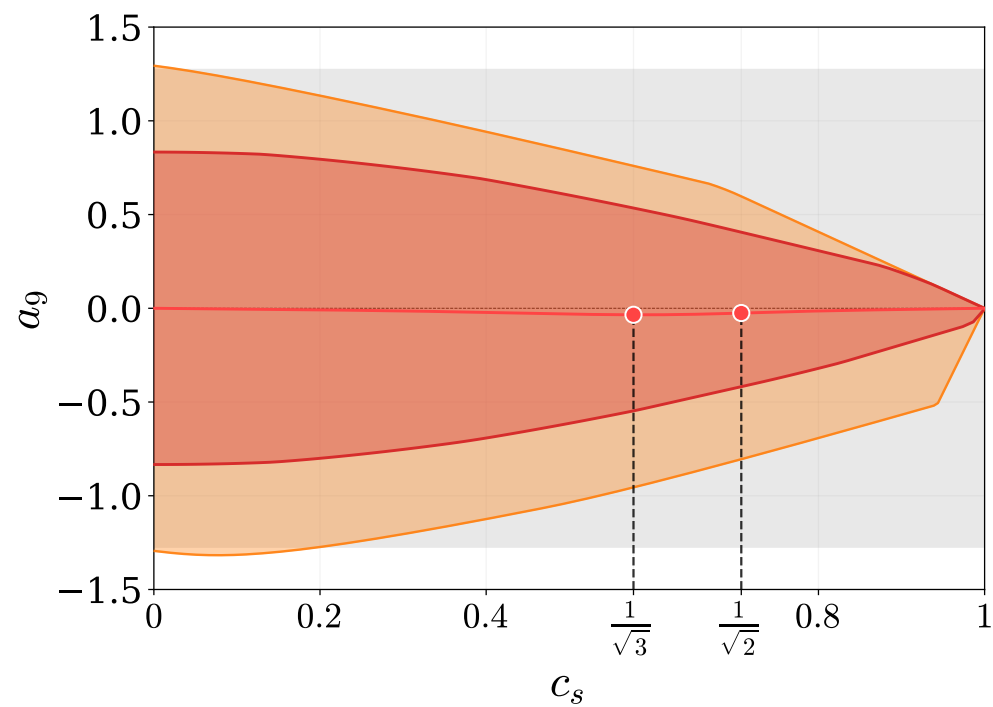
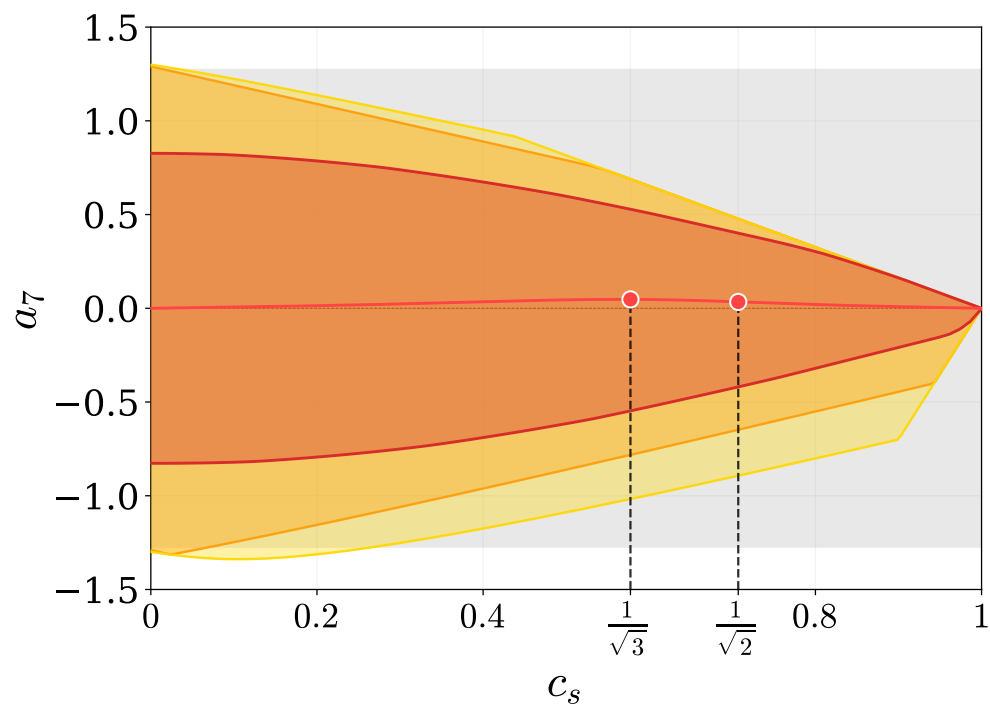
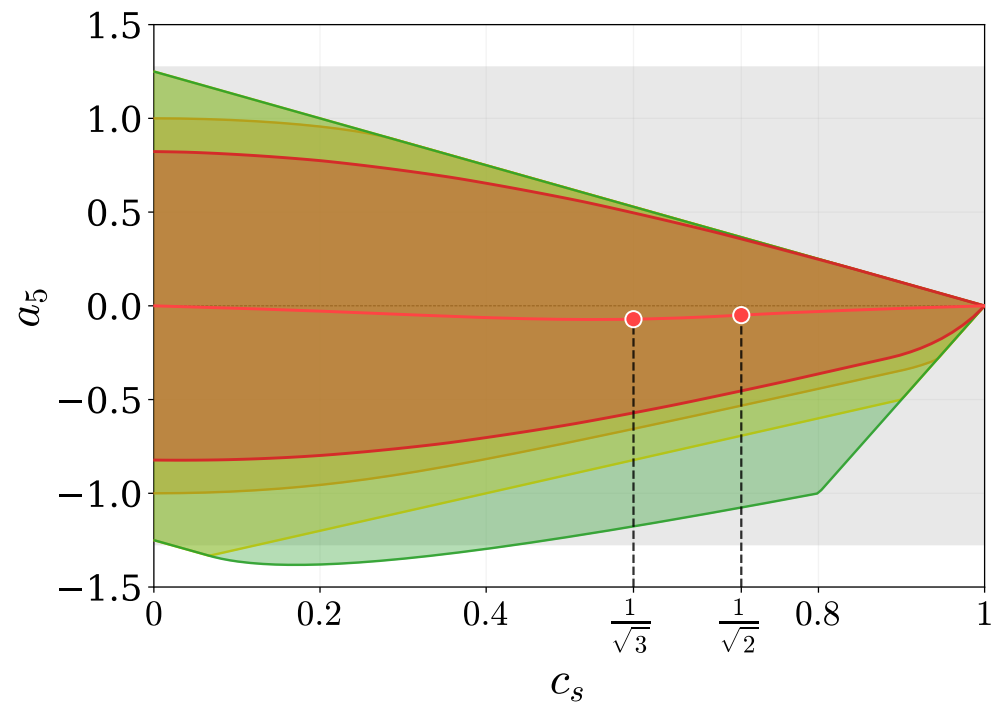
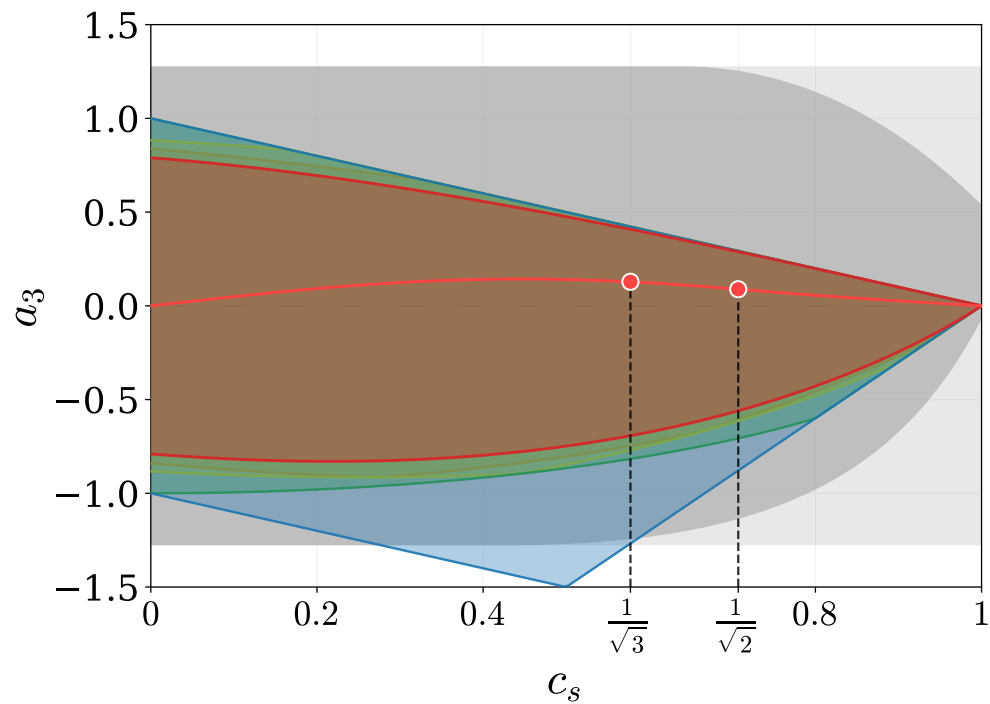
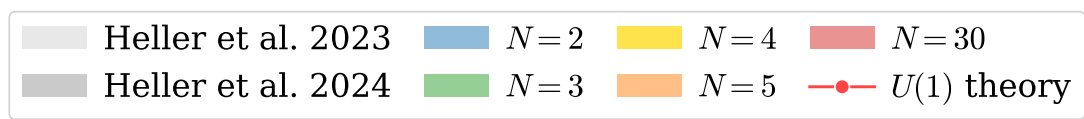
Radius of Convergence:

$$R_- = \begin{cases} c_s \Delta, & 0 < c_s < \frac{1}{\sqrt{2}} \\ \frac{\Delta}{2\sqrt{1 - c_s^2}}, & \frac{1}{\sqrt{2}} < c_s < 1 \end{cases}$$



Expand ω_- around $k = 0$:

$$\omega_-(k) = \pm \left[c_s k + \underbrace{\frac{(1 - c_s^2)^2}{2c_s}}_{a_3 R^2} \frac{k^3}{\Delta^2} - \underbrace{\frac{(1 - c_s^2)^3(1 + 7c_s^2)}{8c_s^3}}_{a_5 R^4} \frac{k^5}{\Delta^4} + \dots \right]$$



Two cutoffs

- Singularities are always **branch-points**: Goldstone touches new solutions

Heller et al 23

- $\omega = \pm \sqrt{\omega^2}$ introduces BP, not necessarily associated to extra states

- E.g. $\omega^2 = c_s^2 k^2 + \frac{k^4}{\Delta^2} + \dots$ Bogolubov spectrum for $c_s \ll 1$

1. $R \simeq c_s \Delta$ sets the breaking of Goldstone $\omega = c_s k$, due to $\omega = \pm \sqrt{\omega^2}$ (Healing length)
2. $\tilde{R} \simeq \Delta$, scale of new (radial) mode: radius of convergence of ω^2

Bounds in units of $\tilde{R}$

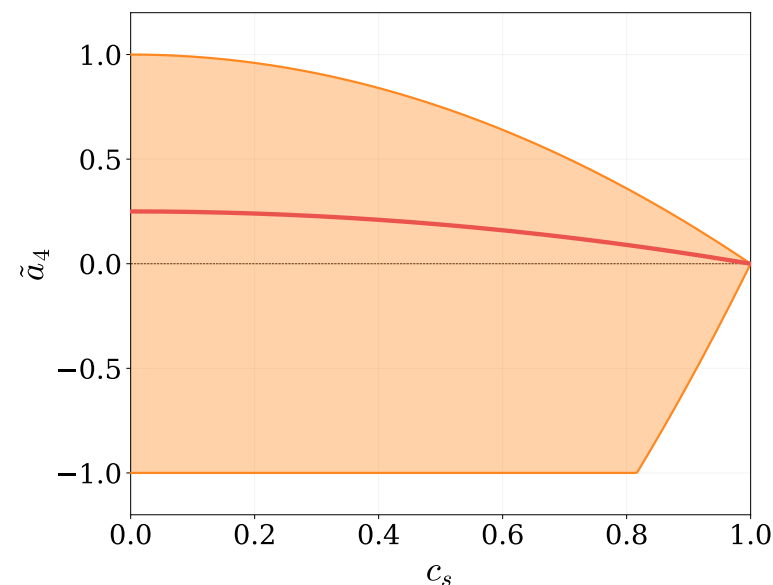
$$\omega^2(k) = \sum_{n=1}^{\infty} \tilde{\alpha}_{2n} k^{2n} \quad \tilde{\alpha}_{2n} = \frac{\tilde{a}_{2n}}{\tilde{R}^{2(n-1)}}$$

$$\pm \operatorname{Im} \sqrt{\omega^2(k)} \leq |\operatorname{Im} k| \quad \Rightarrow \quad \frac{|\omega^2| - \operatorname{Re}(\omega^2)}{2} \leq (\operatorname{Im} k)^2$$

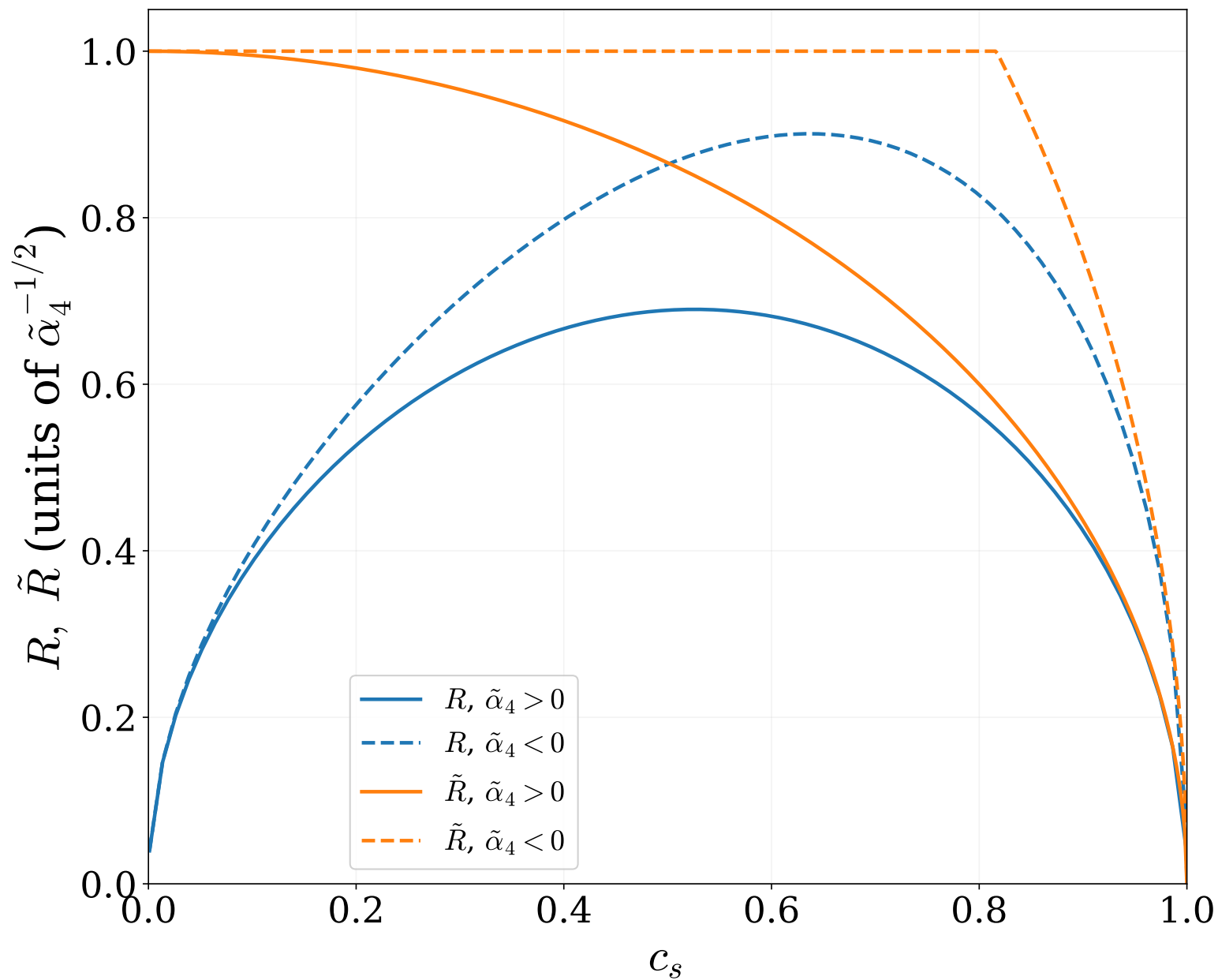
$$F(k) \equiv z - \sum_{n=2}^{\infty} \frac{\tilde{a}_{2n}}{1 - c_s^2} z^{2n-1} \text{ is typically-real (but here loss of optimality)}$$

Rogosinski:

$$-(2n-1)(1 - c_s^2) \leq \tilde{a}_{2n} \tilde{R}^{2(n-1)} \leq \kappa_{2n-1}(1 - c_s^2)$$



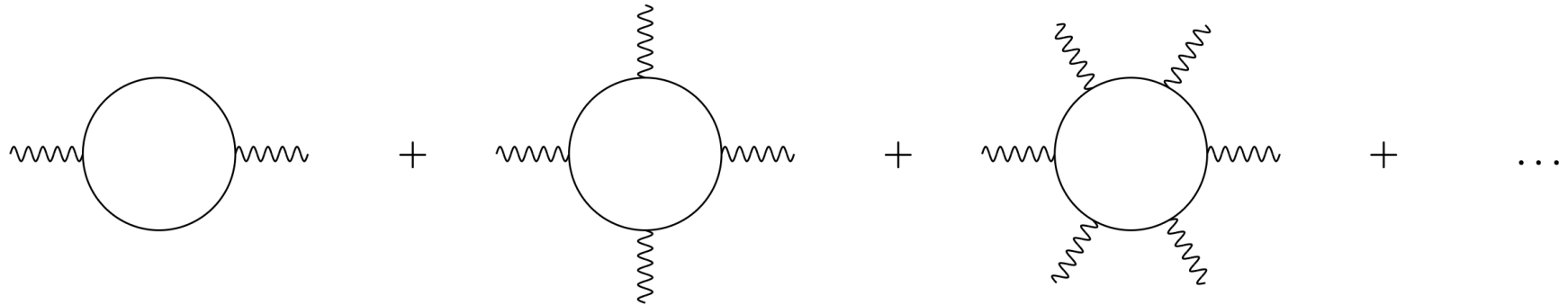
From $c_s^2 k^2 + \alpha_4 k^4$ to R and $\tilde{R}$



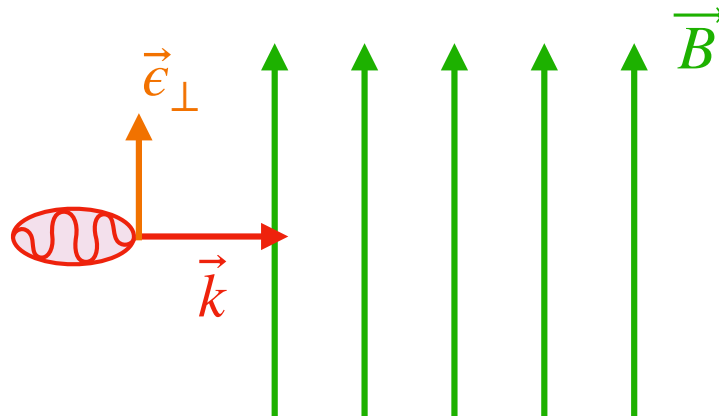
$R = \tilde{R}$ for
 $c_s \rightarrow 1$

Euler-Heisenberg

- One-loop EFT of QED



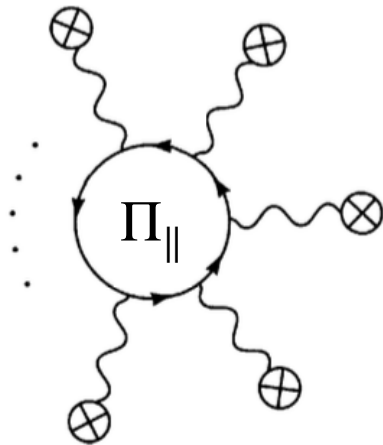
- Photons propagating in magnetic field experiences birefringence



Euler-Heisenberg

Working in the strong field regime: $eB \gg k^2, \quad eB \gg m^2$

Photons polarised parallel to B :



$$\text{Re } \Pi_{\parallel}(\omega, k) \approx \frac{2\alpha eB}{\pi} \left[1 - \frac{\sin^{-1}\left(\frac{\omega}{2m}\right)}{\left(\frac{\omega}{2m}\right) \sqrt{1 - \left(\frac{\omega}{2m}\right)^2}} \right]$$

$$\text{Im } \Pi_{\parallel}(\omega, k) \sim \Theta(\omega^2 - 4m^2)$$

Karbstien 13

Solving for dispersion relation: $-\omega^2 + k^2 + \Pi_{\parallel}(\omega, k) = 0$

$$c_s = 1 - \frac{\alpha eB}{6\pi m^2}$$

$$\alpha_{2n+1} = -\frac{\alpha eB}{\pi} \frac{[(n+1)!]^2}{(2n+3)! m^{2n+2}}$$

Euler-Heisenberg

Radius of Convergence $R = \tilde{R} = 2m$

Express coefficients in terms of c_s and R :

$$\alpha_{2n+1}R^{2n} = -(1 - c_s) \frac{6(4^n)[(n+1)!]^2}{(2n+3)!}$$

Compare to bounds:

$$-(2n+1)(1 - c_s) \leq \alpha_{2n+1}R^{2n} \leq \kappa_{2n+1}(1 - c_s)$$

Bounds satisfied for all n

Similarly for $\tilde{R}$

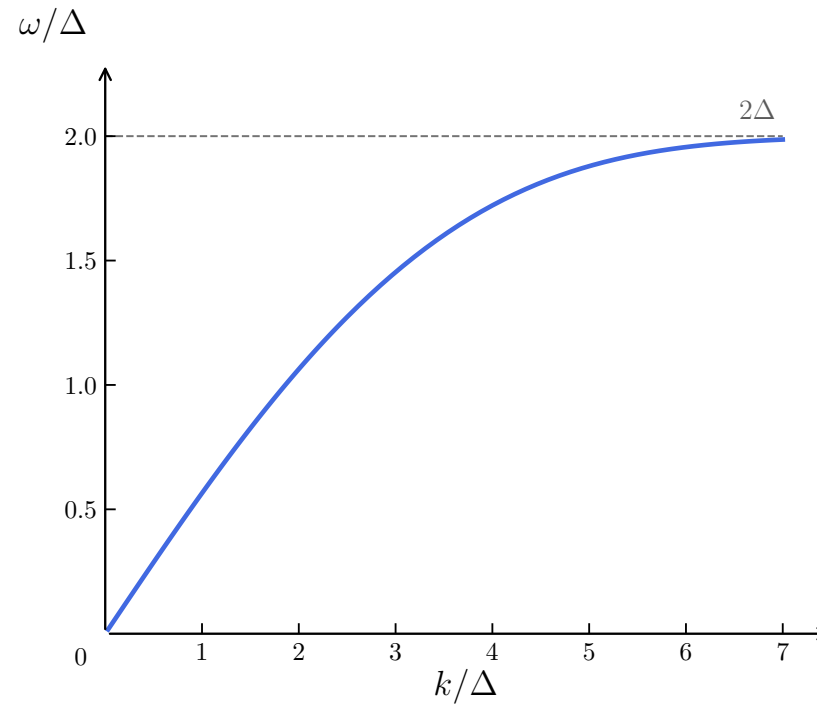
Color superconductor

- Highly dense and cold matter with **large baryon chemical potential μ** (core of Neutron stars?)
- Quarks near the Fermi surface forms a diquark condensate at energy scale $\Delta \ll \mu$: $SU(3)_C \times SU(3)_V \rightarrow SU(3)_D$
- Condensate spontaneously breaks $SU(3)_A \times U(1)_A$, yielding **Goldstones π and η'**

QCD is perturbative at high-energy: dispersion relation can be **calculated**

$$\begin{aligned} \Pi(\omega, k) = & -\frac{\mu^2 \Delta^2}{24\pi^2 k} \int_0^1 dx \int_0^k d\alpha (\alpha^2 - \omega^2) \left[\frac{5}{x(1-x)(\alpha^2 - \omega^2) + \Delta^2} \right. \\ & \left. + \frac{4 - 6x}{x(1-x)(\alpha^2 - \omega^2) + (4 - 3x)\Delta^2} \right], \end{aligned}$$

Color superconductor

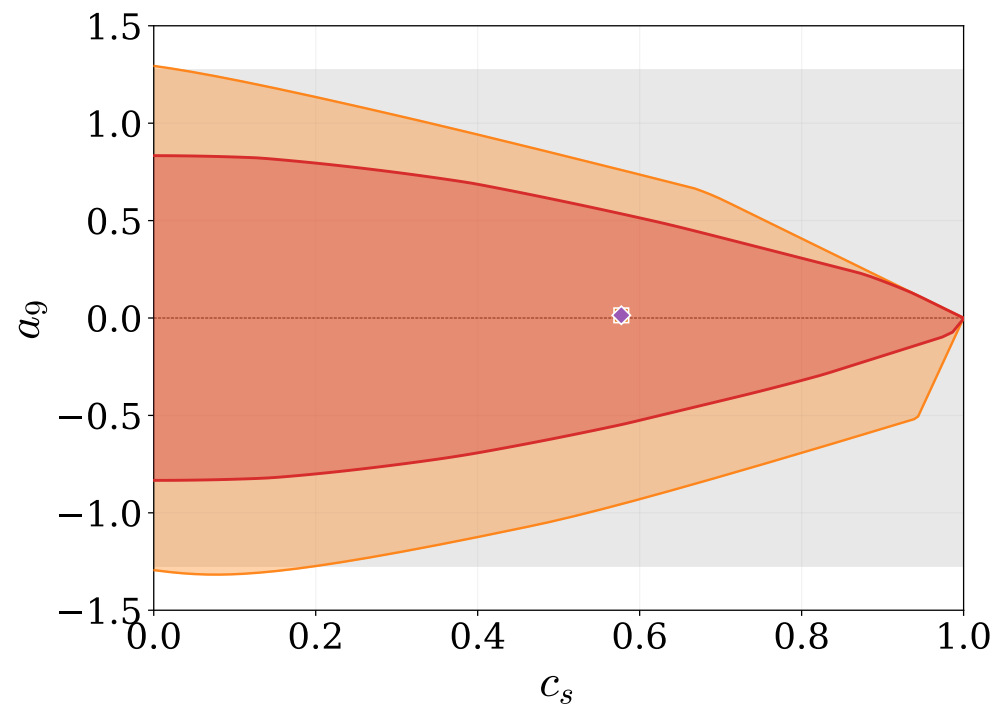
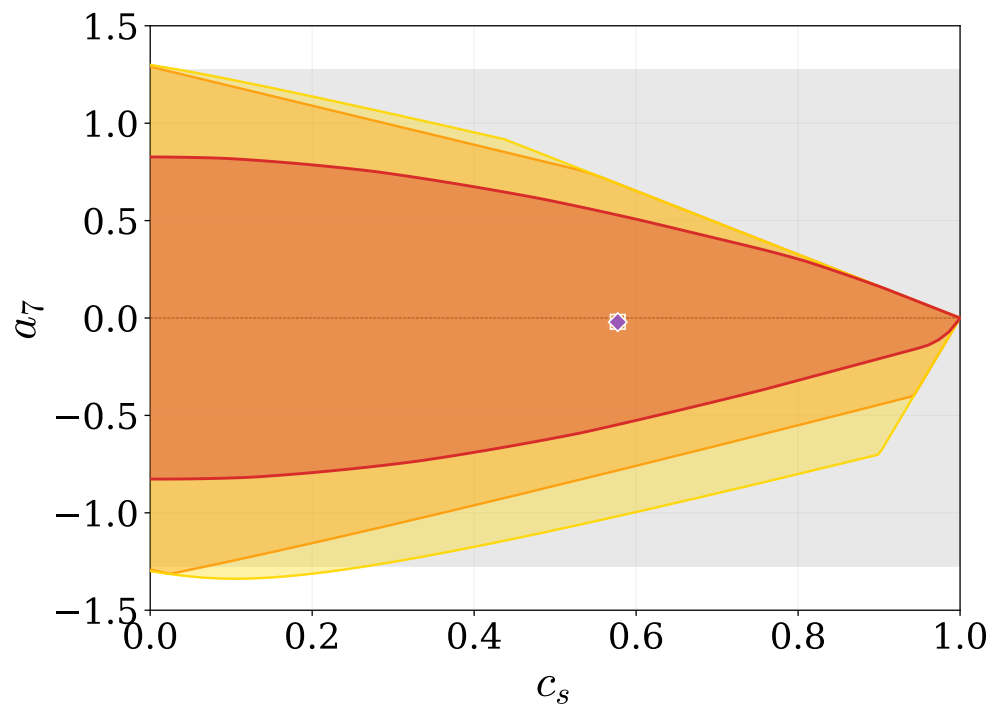
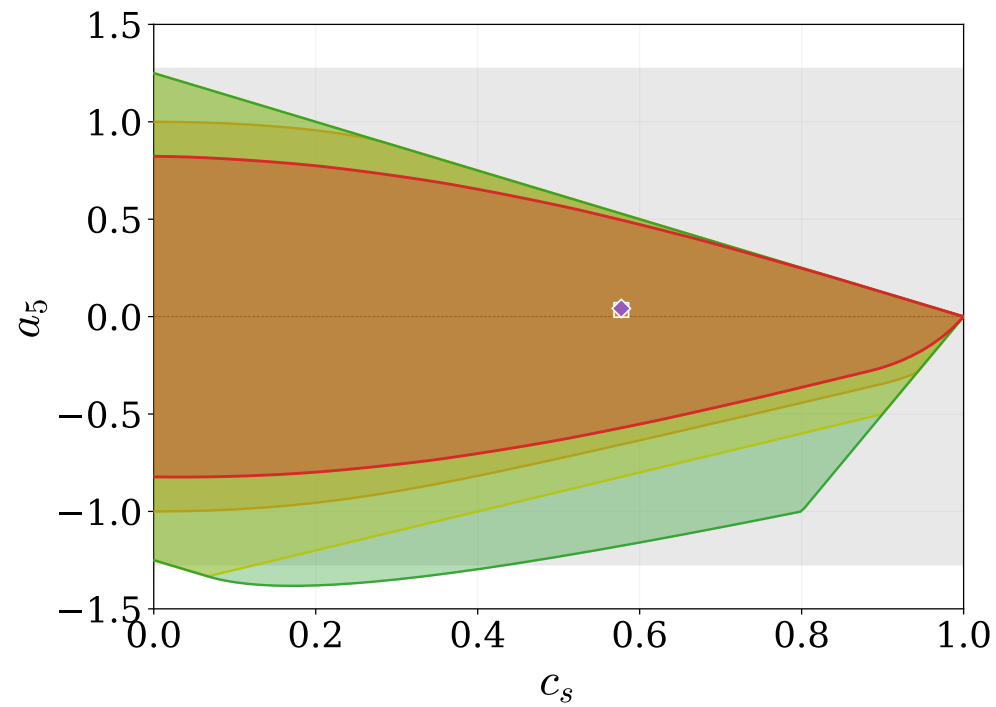
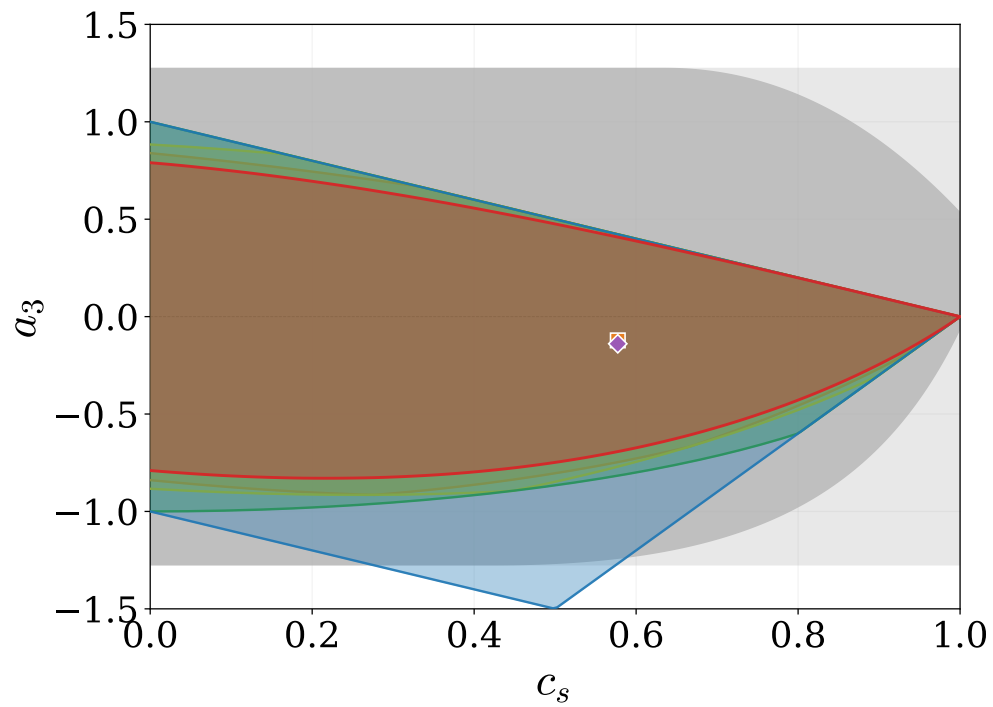
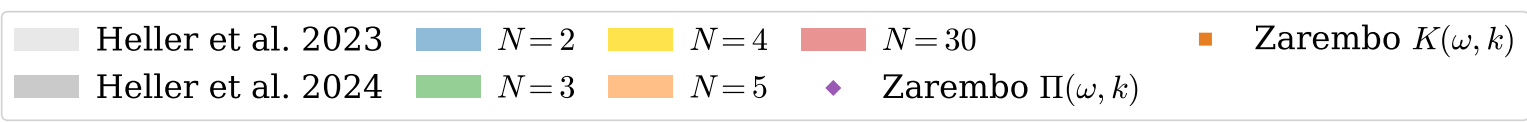


Octet Pions

$$\omega_{\pi}(k) = \frac{1}{\sqrt{3}} k \left[1 - \frac{1}{135} \frac{123 - 112 \ln 2}{21 - 8 \ln 2} \frac{k^2}{\Delta^2} + O\left(\frac{k^4}{\Delta^4}\right) \right] \longrightarrow R_{\pi} = 3.331\Delta$$

$$R^2 = \lim_{n \rightarrow \infty} \frac{\alpha_{2n+1}}{\alpha_{2n+3}}$$

Analogous results for singlet Goldstone



Conclusions and future

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + M_P^2 \dot{H}^2 g^{00} + M_2^4 (g^{00} + 1)^2 + \bar{M}_2^2 \delta K_\mu^\mu \delta K_\mu^\mu + \dots \right]$$

- Extract from $\bar{M}_2^2$ (and all other hd operators) upper bound on cut-off(s)
- For $M_2 \rightarrow 0$, $\bar{M}_2^2$ must turn off too Baumann, Lee, Green, Porto 15
- This also gives information of interactions, given non-linear Lorentz inv.

We are starting to constrain the parameter space of non-LI EFTs

- Include dissipation (e.g. hydro modes) and loops
- Connect $|c_s - 1| \ll 1$ limit with Lorentz-invariant S-matrix
- Lieb-Robinson causality for CM

Backup slides

Microcausality-Analyticity Theorem *A tempered distribution $G(x) \in S'(\mathbb{R}^{d-1,1})$ is retarded and micro-causal if and only if its Fourier transform $G(k^\mu)$, with $k^\mu = k_R^\mu + ik_I^\mu$, is analytic in the domain $\mathbb{R}^{d-1,1} + i\mathring{\text{FLC}}$ and it admits the bound*

$$|G(k^\mu)| \leq A(1 + |k^\mu|)^n \left(1 + \frac{1}{(k_I^0 - |\vec{k}_I|)^m} \right), \quad k^\mu \in \mathbb{R}^{d-1,1} + i\mathring{\text{FLC}}, \quad (3.10)$$

where the positive constant A and the powers $m, n \in \mathbb{N}$ depend on G .

See Creminelli, Longo,
Salehian, Zahed etal 25