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Banach manifolds of weights and quantization of mechanical systems whose phase space is a complex manifold

Abstract. Let $\Omega \subset \mathbb{C}^n$ be an open set. A Lebesgue measurable function $\gamma : \Omega \to (0, +\infty)$ is a *weight* on Ω. The set $W(\Omega)$ of all weights on Ω is an infinite dimensional Banach manifold modeled on $L^\infty(\Omega)$. Let $L^2 H(\Omega, \gamma)$ be the space of all holomorphic functions in $L^2(\Omega, \gamma)$. A weight $\gamma \in W(\Omega)$ is *admissible* if i) the evaluation functional $\delta_z : L^2 H(\Omega, \gamma) \to \mathbb{C}$, $\delta_z(f) = f(z)$, is continuous for any $z \in \Omega$, and ii) $L^2 H(\Omega, \gamma)$ is a closed subspace of $L^2(\Omega, \gamma)$. The set $AW(\Omega)$ of admissible weights on Ω is an open subset in $W(\Omega)$. To every admissible weight $\gamma \in AW(\Omega)$ one associates a kernel function $K_\gamma(z, \zeta)$ organizing $L^2 H(\Omega, \gamma)$ as a RKH space (cf. [2]). The interest in weighted kernels comes from quantization theory, for given a mechanical system whose phase space is Ω (or more generally a complex manifold admitting globally defined Kähler metrics) one may quantize classical states $z \in \Omega$ (besides from quantizing observables) by building an embedding

$$\Omega \hookrightarrow \mathbb{CP}(\mathcal{M}), \tag{0.1}$$

$$\mathcal{M} = \Big\{ s \in H^0\big(\Omega, \mathcal{O}\big(T^{*(n,0)}(\Omega) \otimes E\big)\big) \, : \, \langle s \, , \, s \rangle < \infty \Big\},$$

$$\langle s \, , \, t \rangle = i^{n^2} \int_\Omega H(s, \, t) \, , \quad s, t \in \mathcal{M}.$$

Here $E = \Omega \times \mathbb{C}$ (the trivial complex line bundle). Using the embedding (0.1) one can (cf. [6]) calculate the transition probability amplitude from one point of Ω to another, and actually provide the interpretation of the normalized reproducing kernel function as the transition probability amplitude between two points of the complex phase space Ω. The above interpretation is possible when the holomorphic and metric structures of the line bundle $E \to \Omega$ are tied by the requirement that the weight $\gamma \in AW(\Omega)$ satisfies the complex Monge-Ampère equation

$$\det \Big[\frac{\partial^2 \gamma}{\partial z_j \, \partial \overline{z}_k}(z) \Big] = (-1)^{n(n+1)/2} \, C \, \frac{1}{n!} \, \gamma(z) \, K_\gamma(z, \, z).$$

Let $\Omega = \{\varphi < 0\} \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain. A notable class of admissible weights is $\gamma_m(z) = |\varphi|^m$, $m \in \{0, \, 1, \, 2, \, \cdots\}$. Let $K_{\gamma_m}(\zeta, z)$ be the reproducing kernel for $L^2 H(\Omega, \, \gamma_m)$. By a result of M.M. Peloso (cf. [8])

$$K_{\gamma_m}(\zeta, z) = C_\Omega \, \big| \nabla \varphi(z) \big|^2 \, \cdot \, \det L_\varphi(z) \, \cdot \, \Psi(\zeta, z)^{-(n+1+m)} + E(\zeta, z), \tag{0.2}$$

$$E \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \Delta\big),$$

$$\big|E(\zeta, z)\big| \leq C'_\Omega \, \big|\Psi(\zeta, z)\big|^{-(n+1+m)+1/2} \, \big| \log |\Psi(\zeta, z)| \big|.$$

For $m = 0$ this is Fefferman's asymptotic expansion formula for the ordinary Bergman kernel, and Peloso recovers that for the points of the curve

$$C : \big(-1, \, +\infty\big) \to W(\Omega), \quad C(\alpha) = \big|\varphi\big|^\alpha \in AW(\Omega), \quad \alpha > -1, \tag{0.3}$$

corresponding to the integer values of the parameter. Extending (0.2) to all weights $\gamma \in AW(\Omega)$ is so far an open problem. By a result in [3] the curve (0.3) is discontinuous and every point of C is an isolated point in $W(\Omega)$. The result may be looked at as a measure of the amount of job [deriving an asymptotic expansion formula for $K_\gamma(z, \, \zeta)$] left unsolved. We report on results extending (0.2) to ampler classes of weights (cf. [3], and M. Englis, [5]). There are significant classes of admissible weights going back as far as the more romantic times of the work by G. Cimmino (cf. [4]) on the Dirichlet problem with L^2 boundary data, and the classical work by A. Andreotti & E. Vesentini (cf. [1]) who proved Carleman type estimates [to the purpose of establishing vanishing results for

the cohomology with compact supports $H^q_k\big(\Omega\,,\,\Omega^p(E)\big) = 0$] in which admissible weights spring from the (many possible) choices of Hermitian metrics on E.

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