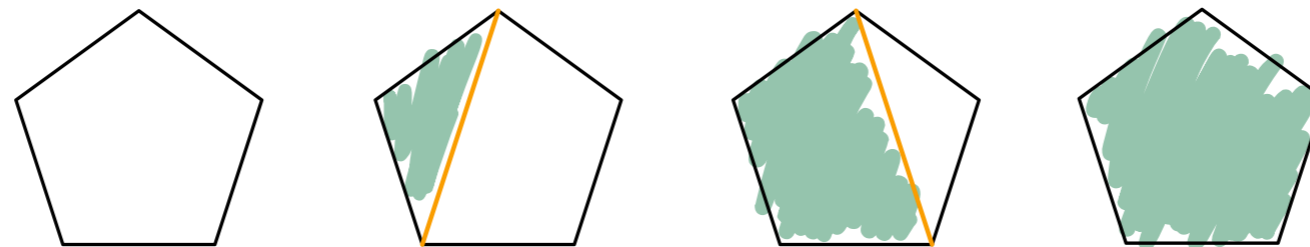


# A New Twist on Time:

## Differential Equations for Cosmological Correlators



Hayden Lee

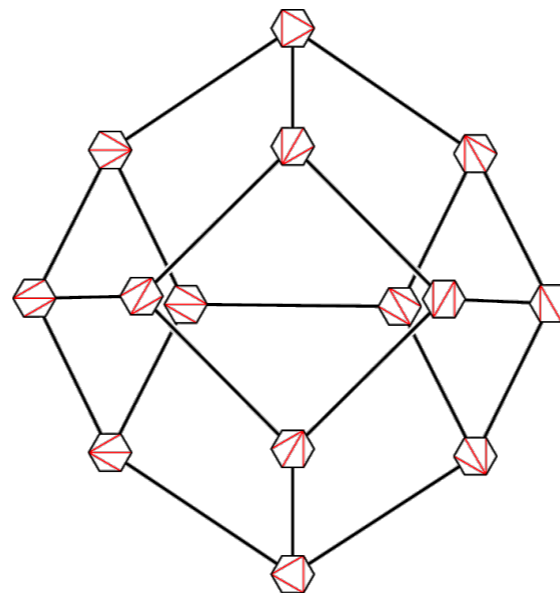
University of Chicago

w/ N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, G. Pimentel [to appear]

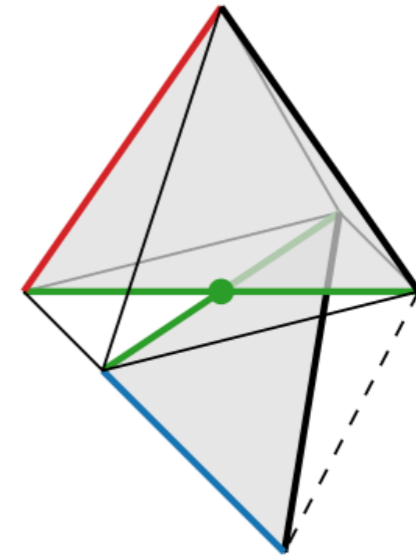
Over the past decade, we have seen scattering amplitudes emerge from new mathematical structures in boundary kinematic space.



amplituhedron



associahedron

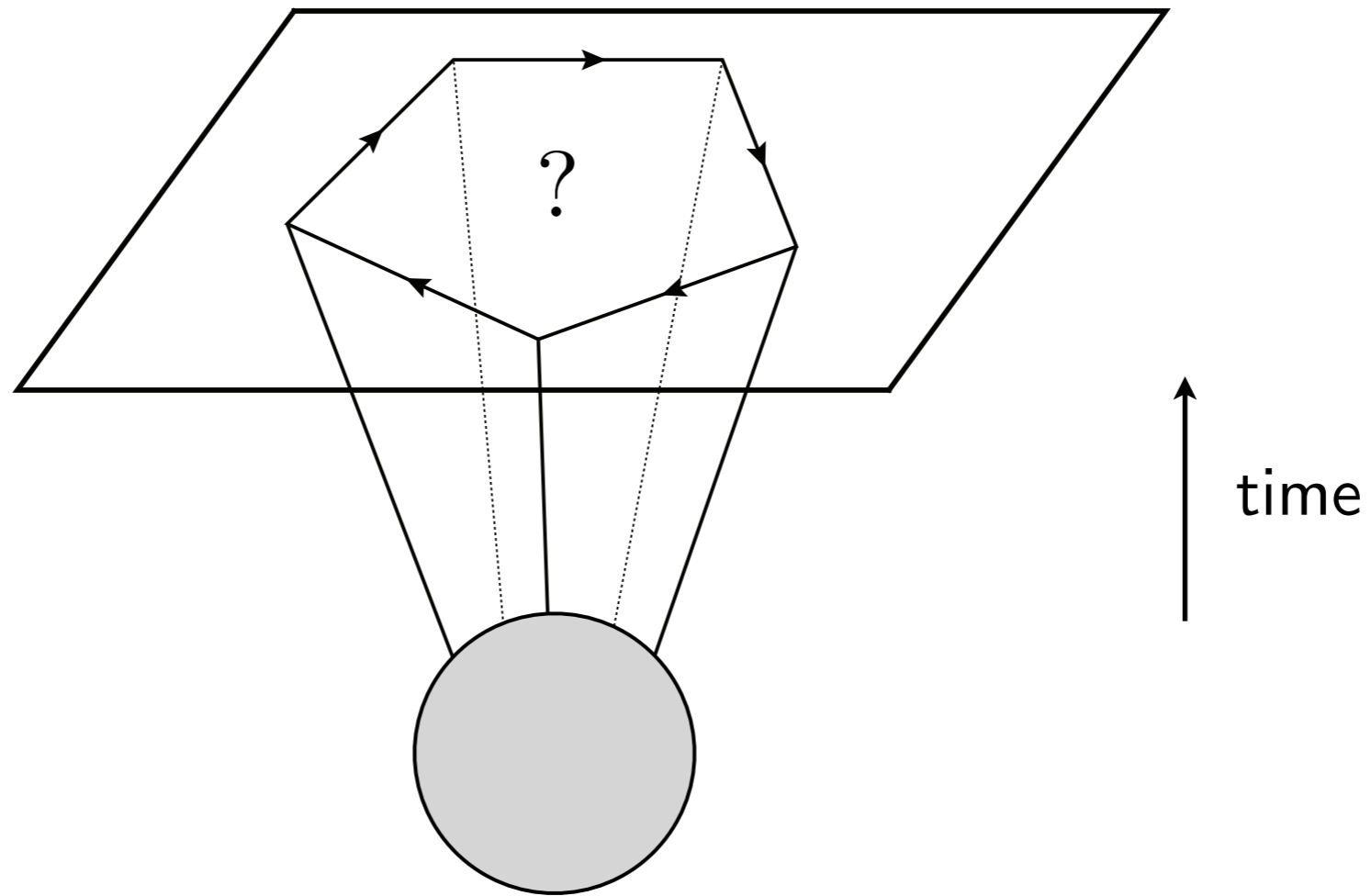


cosmological polytopes

**conceptual advantage:** focuses directly on boundary observables

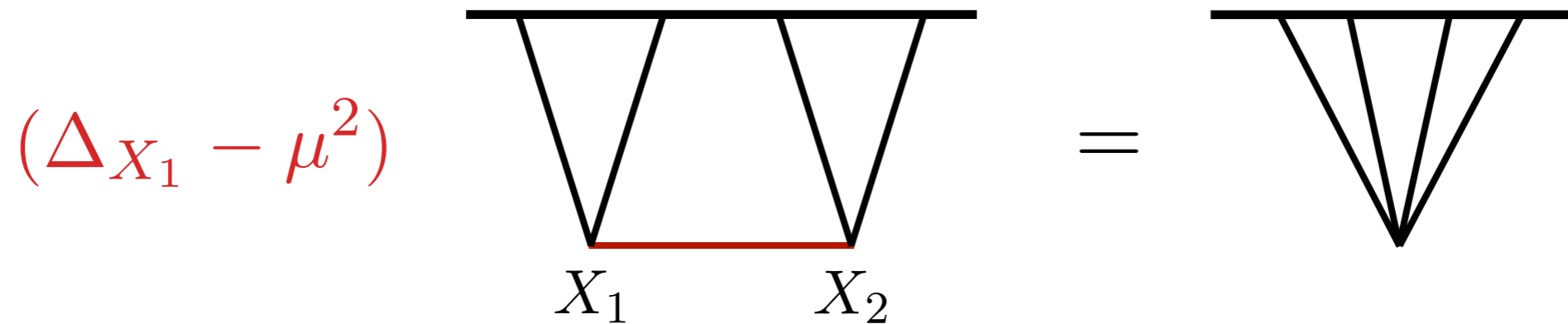
**practical advantage:** simplifies calculations

We can't directly observe the pre-Big Bang evolution of the universe, but instead must infer it from spatial correlations on the future boundary.



*How can we see “time evolution” from boundary correlators?*

In de Sitter space, boundary **conformal** correlators satisfy local differential equations that reflect bulk time evolution.



$$\Delta_X = (X^2 - 1)\partial_X^2 + 2X\partial_X$$

Arkani-Hamed, Maldacena [2015]  
Arkani-Hamed, Baumann, HL, Pimentel [2018]

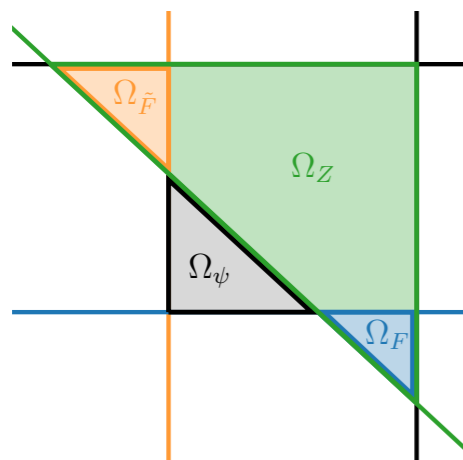
*Is there a deeper reason for their existence beyond de Sitter?*



# Outline

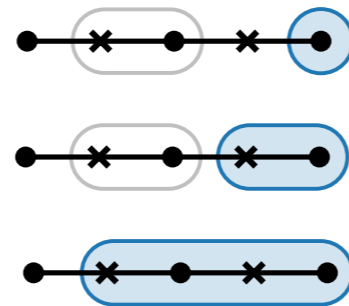
I.

Correlators as  
Twisted Integrals



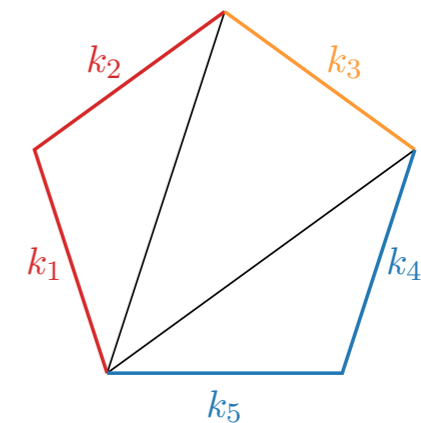
II.

Time Evolution as  
Kinematic Flow



III.

Beyond Single Graphs

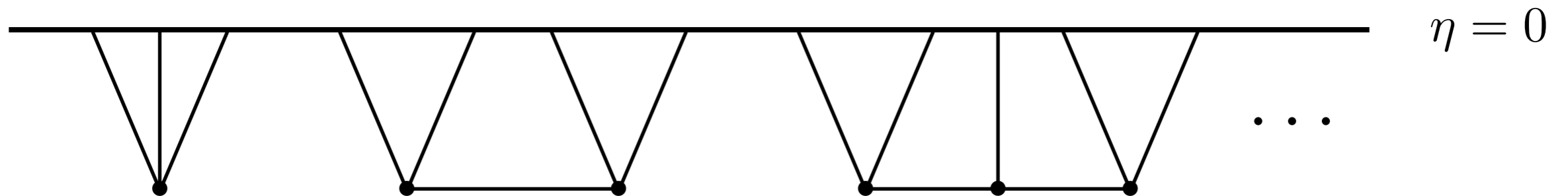


*Today, I'll present a new mathematical perspective on cosmological time evolution.*

# **I. Correlators as Twisted Integrals**

# Cosmological Wavefunction

We'll be interested in the wavefunction (coefficients) in FRW cosmology.



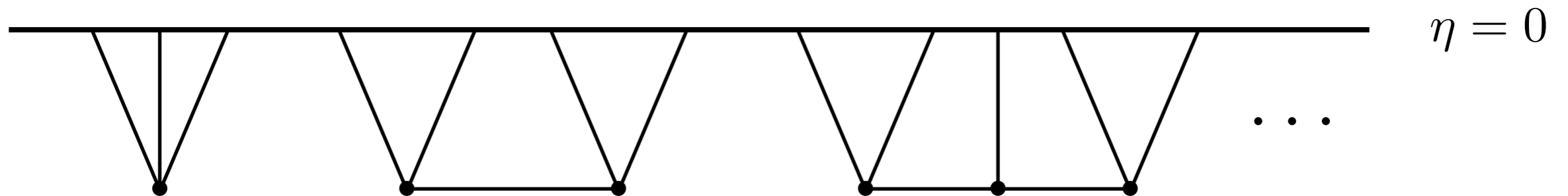
Specifically, we'll consider conformally-coupled scalars with (non-conformal) polynomial interactions and a power-law scale factor.

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (\partial\phi)^2 - \frac{1}{12} R\phi^2 - \frac{\lambda}{3!} \phi^3 \right] \quad a(\eta) \propto \frac{1}{\eta^{1+\varepsilon}}$$

[ $\varepsilon = 0$  : dS,  $\varepsilon = -1$  : flat,  $\varepsilon = -2$  : radiation,  $\varepsilon = -3$  : matter]

# Cosmological Wavefunction

We'll be interested in the wavefunction (coefficients) in FRW cosmology.



This can be mapped to the action of massless scalars with time-dependent couplings.

$$S = \int d^4x \left[ -\frac{1}{2} (\partial\phi)^2 - \frac{\tilde{\lambda}(\eta)}{3!} \phi^3 \right] \quad \tilde{\lambda}(\eta) \propto \frac{1}{\eta^{1+\varepsilon}} \propto \int_0^\infty d\omega \omega^\varepsilon e^{i\omega\eta}$$

[ $\varepsilon = 0$  : dS,  $\varepsilon = -1$  : flat,  $\varepsilon = -2$  : radiation,  $\varepsilon = -3$  : matter]

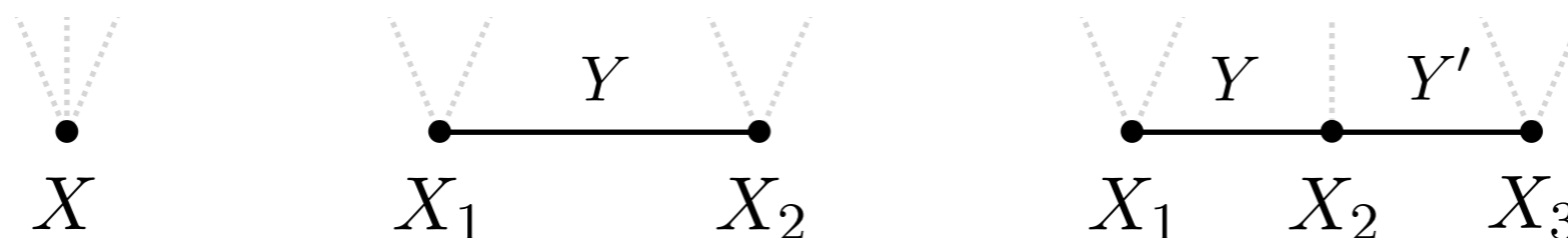
# Cosmological Wavefunction

The wavefunction in FRW is related to the flat-space one as

$$\psi_{\text{FRW}}^{(n)}(X_i) = \int_0^\infty dx_1 \cdots dx_n (x_1 \cdots x_n)^\varepsilon \psi_{\text{flat}}^{(n)}(X_i \rightarrow X_i + x_i)$$

“twist” rational function (tree level)

It's convenient to consider graphs with all external propagators truncated.



# Wavefunction in Flat Space

Flat-space wavefunction is described by rational functions with simple poles.

$$\psi_{\text{flat}}^{(2)} = \text{Diagram} = \frac{1}{(X_1 + X_2)(X_1 + Y)(X_2 + Y)}$$

The diagram shows a horizontal oval containing two dots connected by a line. The left dot is enclosed in a red circle, and the right dot is enclosed in a blue circle.

$$\psi_{\text{flat}}^{(3)} = \text{Diagram 1} + \text{Diagram 2} = \frac{1}{(X_1 + X_2 + X_3)(X_1 + Y)(X_2 + Y + Y')(X_3 + Y')} \left( \frac{1}{X_1 + X_2 + Y'} + \frac{1}{X_2 + X_3 + Y} \right)$$

The first diagram shows a horizontal oval with three dots connected by a line. The left dot is in a red circle, the middle dot is in a green circle, and the right dot is in a blue circle. The second diagram is similar but the middle dot is in an orange circle.

There is a simple combinatorial prescription for constructing the flat-space wavefunction.

# Case Study: Two-Site Chain

The simplest nontrivial example in FRW is the two-site chain graph:

$$\psi_{\text{FRW}}^{(2)}(X_1, X_2) = \int_0^\infty dx_1 dx_2 \frac{(x_1 x_2)^\varepsilon}{(x_1 + x_2 + X_1 + X_2)(x_1 + X_1 + 1)(x_2 + X_2 + 1)}$$

Modern approaches to compute integrals of this type include

- ▶ the method of differential equations
- ▶ twisted cohomology

Similar techniques can be applied to our cosmology problem.

# Family of Integrals

Consider a family of integrals with the same singularities.

$$I_{\vec{n}} = \int (x_1 x_2)^\varepsilon \Omega_{\vec{n}}, \quad \Omega_{\vec{n}} = \frac{dx_1 \wedge dx_2}{T_1^{m_1} T_2^{m_2} B_1^{n_1} B_2^{n_2} B_3^{n_3}}$$

$$T_1 = x_1, \quad B_1 = x_1 + X_1 + 1$$

$$T_2 = x_2, \quad B_2 = x_2 + X_2 + 1, \quad B_3 = x_1 + x_2 + X_1 + X_2$$

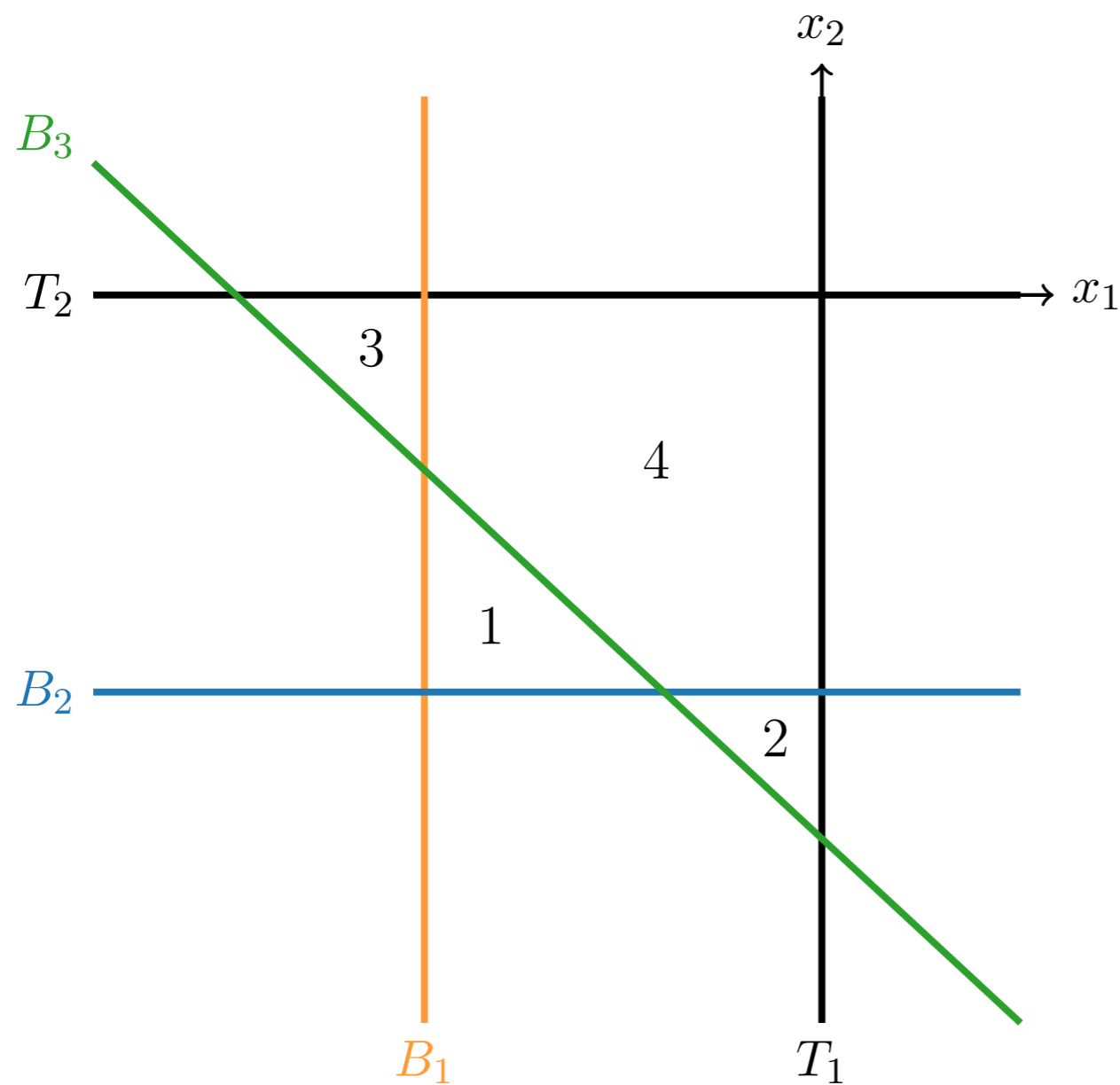
These integrals form a finite-dimensional vector space.

Twisted cohomology provides a geometric way to determine the size of this vector space.



# Master Integrals

The number of master integrals equals the number of bounded regions defined by the singular factors of the integrand.



$$T_1 = x_1$$

$$T_2 = x_2$$

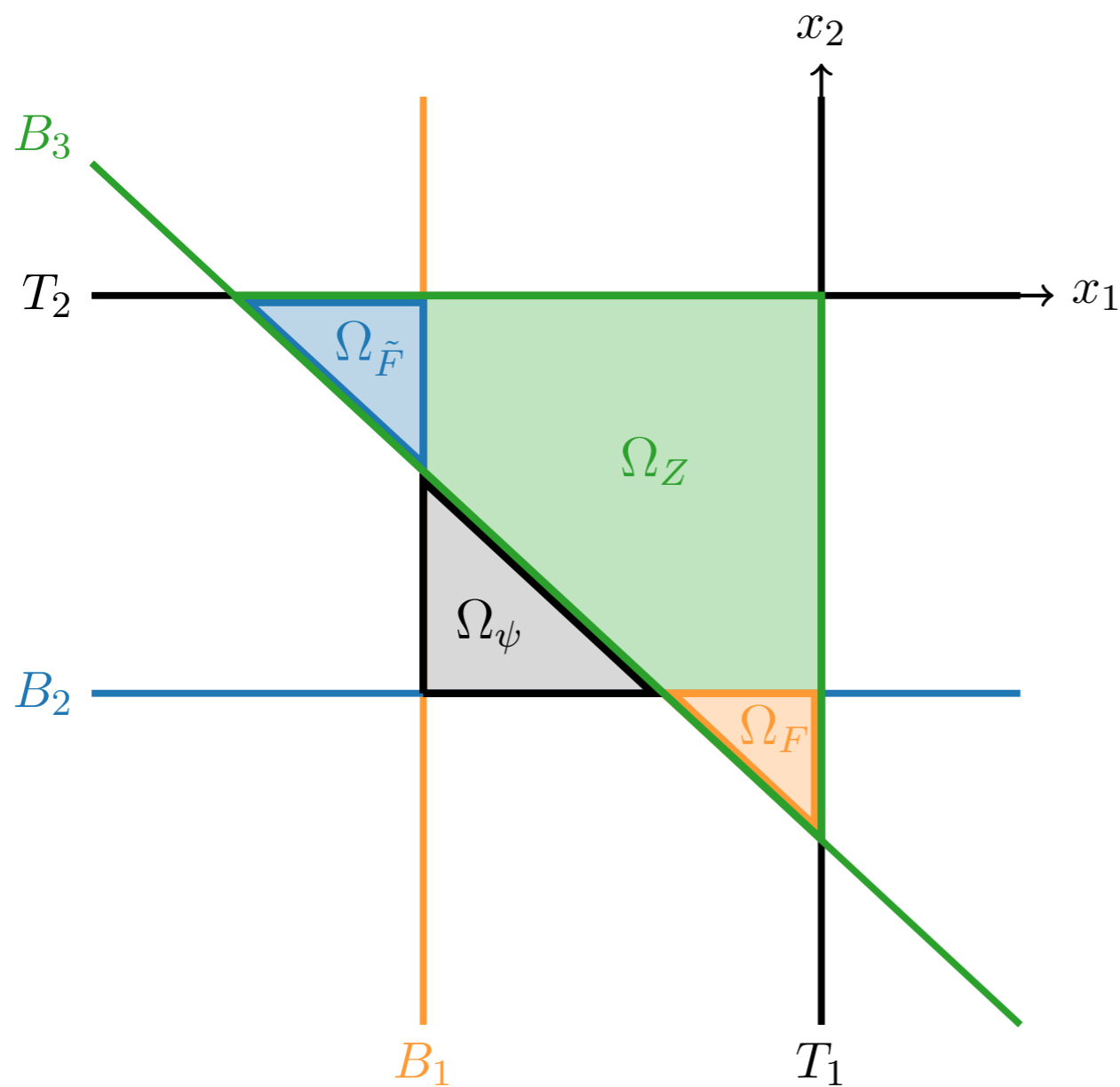
$$B_1 = x_1 + X_1 + 1$$

$$B_2 = x_2 + X_2 + 1$$

$$B_3 = x_1 + x_2 + X_1 + X_2$$

# Master Integrals

A good choice for the basis of integrals is given by the canonical forms of bounded regions.



$$\vec{I} = \begin{bmatrix} \psi \\ F \\ \tilde{F} \\ Z \end{bmatrix} = \int (x_1 x_2)^\varepsilon \begin{bmatrix} \Omega_\psi \\ \Omega_F \\ \Omega_{\tilde{F}} \\ \Omega_Z \end{bmatrix}$$

$$\Omega_{\text{can}}[\triangle_{L_1 L_2 L_3}] = d \log \left( \frac{L_1}{L_3} \right) \wedge d \log \left( \frac{L_2}{L_3} \right)$$

# Differential Equations

The basis of master integrals satisfy differential equations.

For instance, taking an  $X_1$  derivative of the wavefunction gives

$$\partial_{X_1} \psi = \int (x_1 x_2)^\varepsilon \partial_{x_1} \Omega_\psi \qquad (\partial_{X_1} \Omega_\psi = \partial_{x_1} \Omega_\psi)$$

# Differential Equations

The basis of master integrals satisfy differential equations.

For instance, taking an  $X_1$  derivative of the wavefunction gives

$$\begin{aligned}\partial_{X_1}\psi &= \int (x_1x_2)^\varepsilon \partial_{x_1}\Omega_\psi && (\partial_{X_1}\Omega_\psi = \partial_{x_1}\Omega_\psi) \\ &= \varepsilon \int (x_1x_2)^\varepsilon \left[ -\frac{1}{T_1}\Omega_\psi \right] && \text{(IBP)}\end{aligned}$$

# Differential Equations

The basis of master integrals satisfy differential equations.

For instance, taking an  $X_1$  derivative of the wavefunction gives

$$\begin{aligned}\partial_{X_1} \psi &= \int (x_1 x_2)^\varepsilon \partial_{x_1} \Omega_\psi && (\partial_{X_1} \Omega_\psi = \partial_{x_1} \Omega_\psi) \\ &= \varepsilon \int (x_1 x_2)^\varepsilon \left[ -\frac{1}{T_1} \Omega_\psi \right] && \text{(IBP)} \\ &= \varepsilon \int (x_1 x_2)^\varepsilon \left[ \frac{1}{X_1 + 1} (\Omega_\psi - \Omega_F) + \frac{1}{X_1 - 1} \Omega_F \right] \\ &= \varepsilon \left[ \frac{1}{X_1 + 1} (\psi - F) + \frac{1}{X_1 - 1} F \right]\end{aligned}$$

# Differential Equations

The basis of master integrals satisfy differential equations.

For instance, taking an  $X_1$  derivative of the wavefunction gives

$$\partial_{X_1} \psi = \varepsilon \left[ \frac{1}{X_1 + 1} (\psi - F) + \frac{1}{X_1 - 1} F \right]$$

This can be expressed in terms of the total differential as

$$\begin{aligned} d\psi = \varepsilon & \left[ d \log(X_1 + 1) (\psi - F) + d \log(X_1 - 1) F \right. \\ & \left. + d \log(X_3 + 1) (\psi - \tilde{F}) + d \log(X_3 - 1) \tilde{F} \right] \end{aligned}$$

# Differential Equations

Taking the total differential of the basis vector gives

$$d = \sum_i dX_i \frac{\partial}{\partial X_i} \quad \boxed{d\vec{I} = \varepsilon A \vec{I}} \quad d^2 = 0 \Rightarrow dA = A \wedge A = 0$$

with

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} d \log(X_1 + X_2) + \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} d \log(X_1 + 1) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} d \log(X_1 - 1) \\ + \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} d \log(X_2 + 1) + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} d \log(X_2 - 1)$$

letters

# Locality and Time Evolution

The equations can also be combined into a **local** second-order equation.

$$\left[ (X_1^2 - 1) \partial_{X_1}^2 + 2(1 - \varepsilon) X_1 \partial_{X_1} - \varepsilon(1 - \varepsilon) \right] \psi = \frac{1}{(X_1 + X_2)^{1-2\varepsilon}}$$

—————  
depends only on  $X_1$

The solution after imposing appropriate boundary conditions is

$$\psi = c_A(\varepsilon)(1 + X_1)^\varepsilon(1 + X_2)^\varepsilon \\ + c_B(\varepsilon)(X_1 + X_2)^{2\varepsilon} \left( 1 - {}_2F_1 \left[ \begin{matrix} 1, \varepsilon \\ 1 - \varepsilon \end{matrix} \middle| \frac{1 - X_2}{1 + X_1} \right] - {}_2F_1 \left[ \begin{matrix} 1, \varepsilon \\ 1 - \varepsilon \end{matrix} \middle| \frac{1 - X_1}{1 + X_2} \right] \right)$$



# General Tree Graphs

There are several challenges for the conventional approach:

(!) Hard to visualize higher-dimensional planes.

(!!) Finding an optimal basis is a bit of an art.

(!!!) The derivatives are not automatically expressible in terms of the original basis.

*Remarkably, there are hidden **combinatorial** and **geometric** structures underlying these differential equations that allow us to bypass these challenges.*

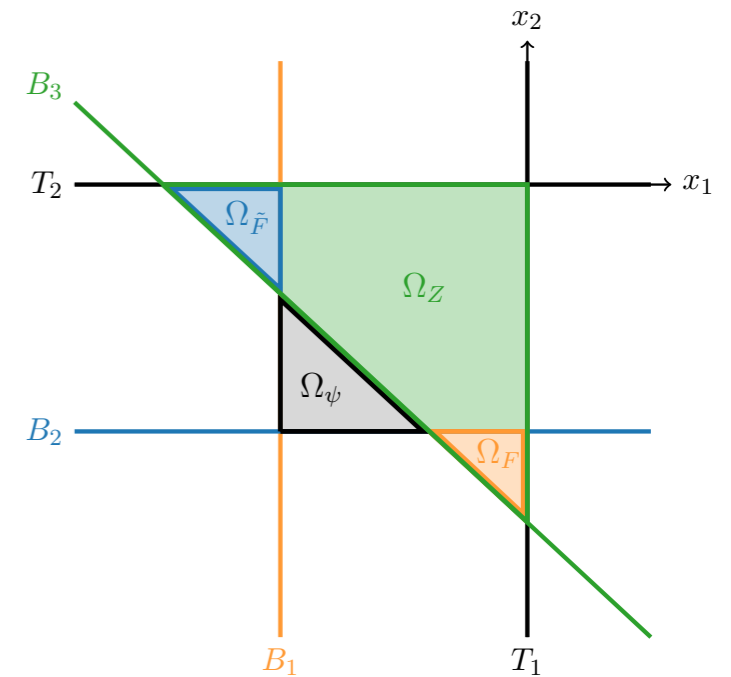
## **II. Time Evolution as Kinematic Flow**

# Two-Site Chain Revisited

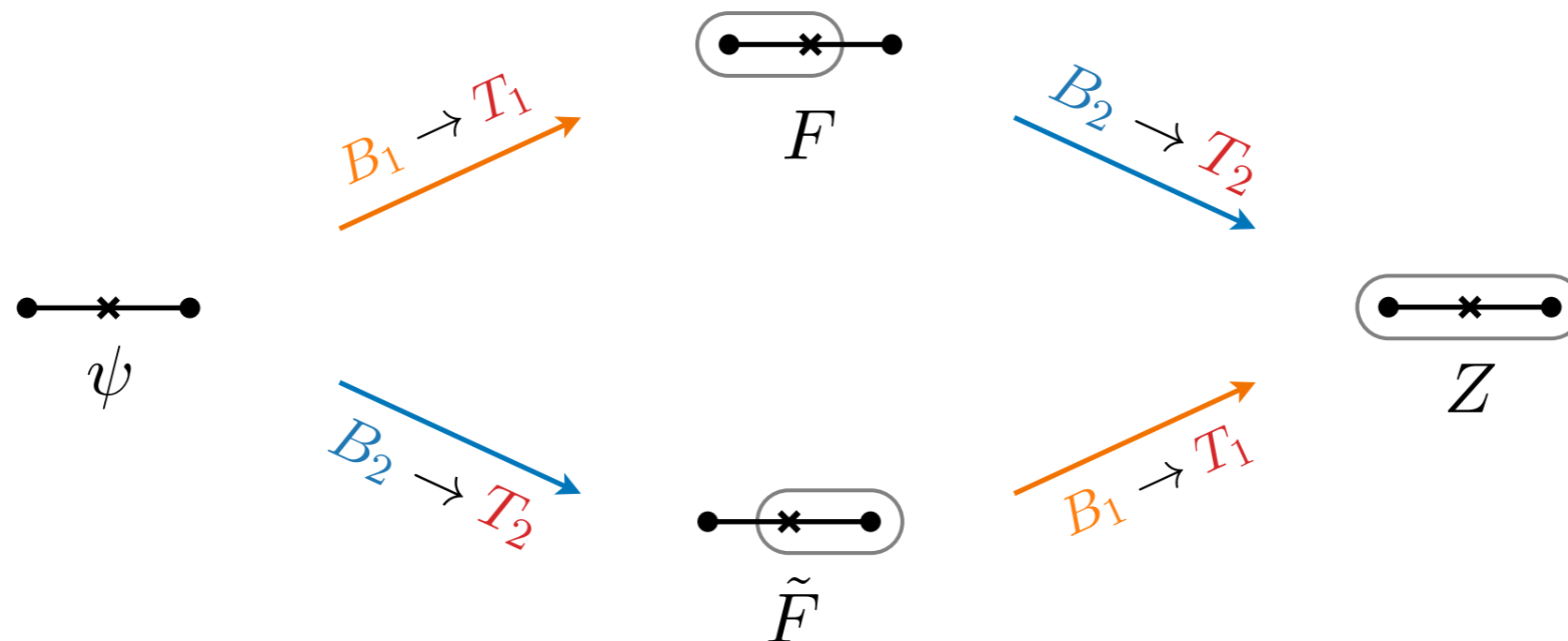
For the two-site chain, we had

$$\Omega_\psi = \Omega_{\text{can}}[\Delta_{B_1 B_2 B_3}] \quad \Omega_F = \Omega_{\text{can}}[\Delta_{B_2 B_3 T_1}]$$

$$\Omega_Z = \Omega_{\text{can}}[\Delta_{B_3 T_1 T_2}] \quad \Omega_{\tilde{F}} = \Omega_{\text{can}}[\Delta_{B_3 B_1 T_2}]$$



Notice that the source functions are obtained by substituting **twisted** planes.

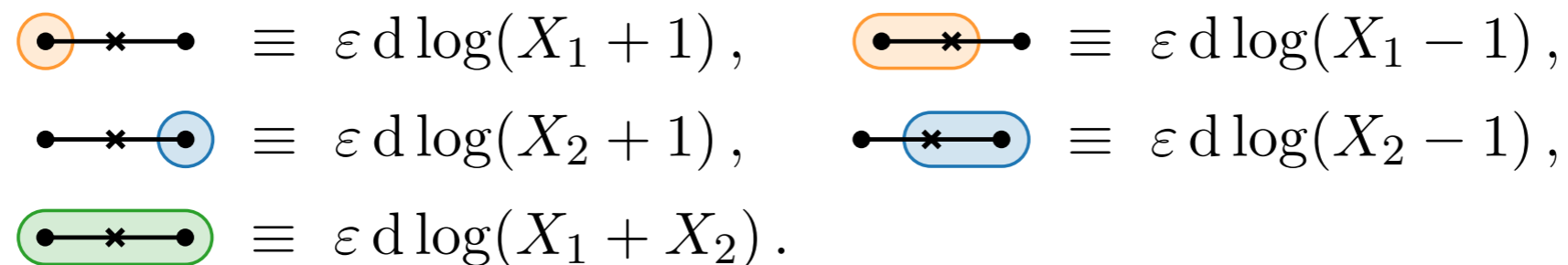


# Graphical Representation

It's natural to represent the basis integrals by (disconnected) tubings that enclose at least one cross



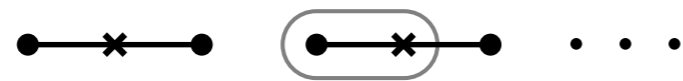
and the letters by connected tubings as



*Under the action of “d”, these tubings grow according to simple graphical rules.*

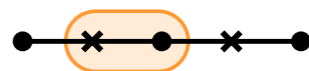
# Graphical Rules for All Trees

- ▶ Enumerate all possible tubings for the basis functions.



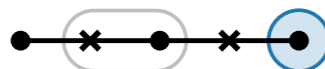
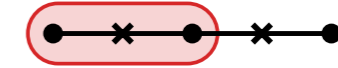
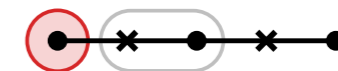
$$N_{\text{basis}} = 4^{n-1}$$

- ▶ Taking the differential, these tubings grow and merge according to four graphical rules:



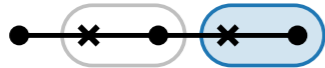
activation

merger



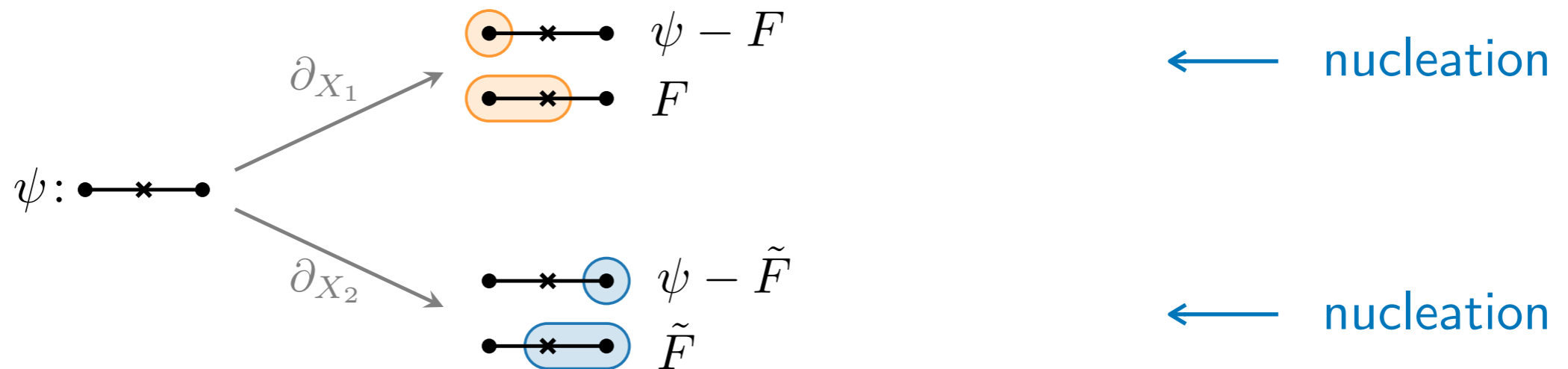
nucleation

absorption



# Growth of Tubings

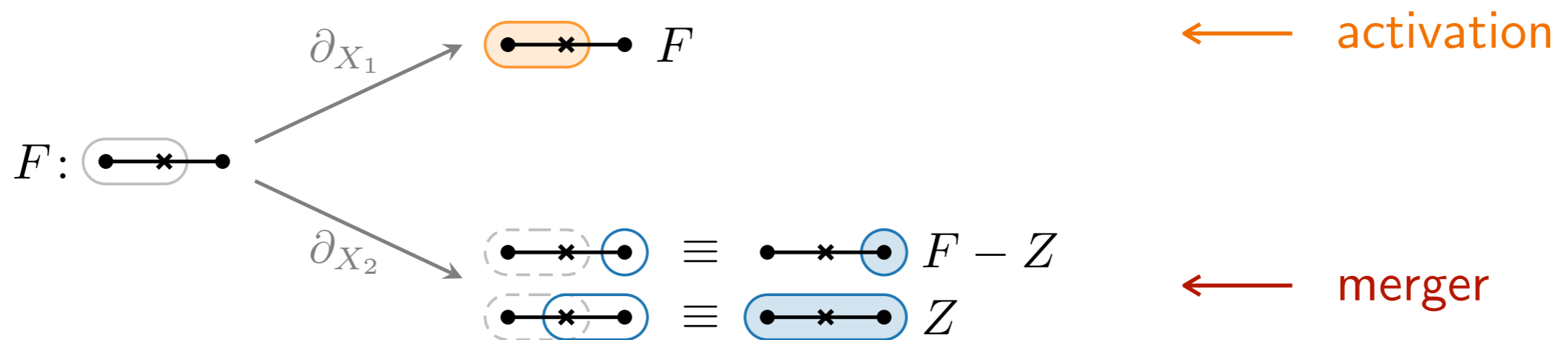
Let's see these rules in action for the two-site chain:



Taking derivatives **nucleates** a pair of tubings around each vertex, which correspond to letters with  $\pm$  sign.

# Growth of Tubings

Let's see these rules in action for the two-site chain:

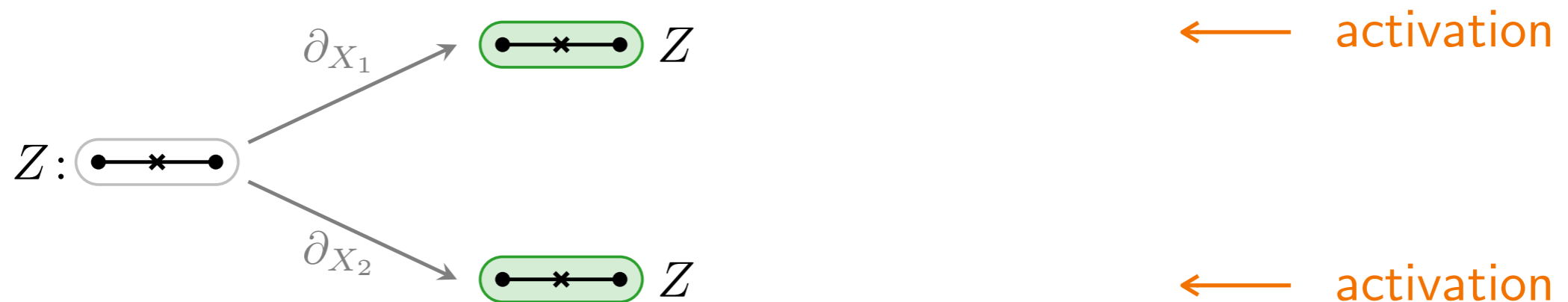


When a vertex is inside the parent tubing, then the tubing gets **activated**.

If two tubings overlap, then they **merge** to form a bigger tubing.

# Growth of Tubings

Let's see these rules in action for the two-site chain:



The growth ends when all vertices are enclosed inside a tubing.

No new source functions appear, and the system closes.



# Differential Equations

The differential equations for the two-site chain can be represented as

$$d\psi = (\psi - F) \textcircled{\bullet} \text{---} \times \text{---} \bullet + F \textcircled{\bullet \text{---} \times} \text{---} \bullet + (\psi - \tilde{F}) \bullet \text{---} \times \text{---} \textcircled{\bullet} + \tilde{F} \bullet \text{---} \times \text{---} \textcircled{\bullet}$$


---

$$dF = F \textcircled{\bullet \text{---} \times} \text{---} \bullet + (F - Z) \textcircled{\bullet \text{---} \times} \text{---} \textcircled{\bullet} + Z \textcircled{\bullet \text{---} \times} \text{---} \bullet$$

$$d\tilde{F} = \tilde{F} \bullet \text{---} \times \text{---} \textcircled{\bullet} + (\tilde{F} - Z) \textcircled{\bullet} \text{---} \times \text{---} \textcircled{\bullet} + Z \textcircled{\bullet \text{---} \times} \text{---} \bullet$$

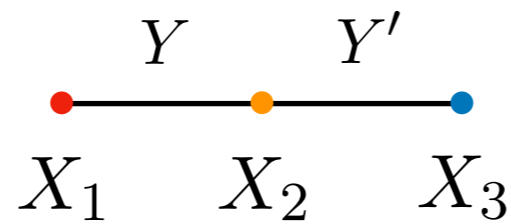

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$$dZ = 2Z \textcircled{\bullet \text{---} \times} \text{---} \bullet$$

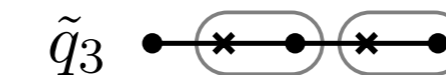
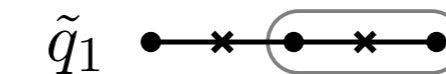
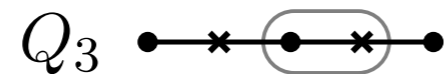
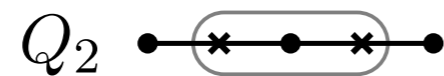
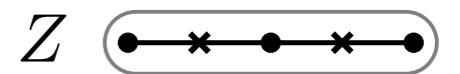
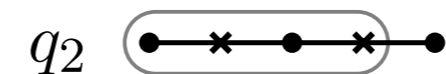
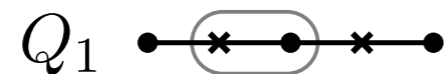
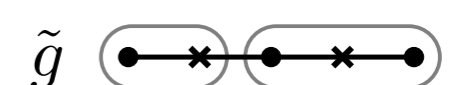
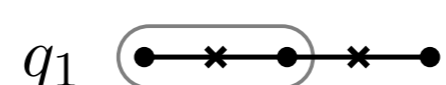
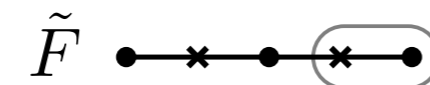
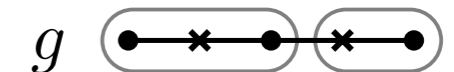
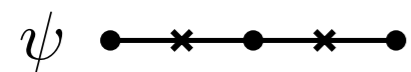
The rules are local and completely general – no more artistic choices of basis integrals and their IBP reduction are needed.

# Three-Site Chain

A similar pattern holds for a three-site graph:

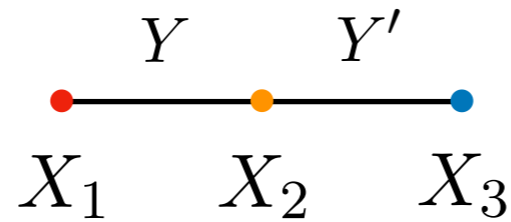


There are 16 basis functions, less than the 25 we get from twisted cohomology.



# Three-Site Chain

A similar pattern holds for a three-site graph:



There are 13 letters, less than the naive counting of 19.

$$\circlearrowleft \bullet \times \bullet \times \bullet \equiv \varepsilon d \log(X_1 + Y)$$

$$\bullet \times \bullet \times \bullet \circlearrowright \equiv \varepsilon d \log(X_3 + Y')$$

$$\bullet \times \bullet \circlearrowright \times \bullet \equiv \varepsilon d \log(X_2 + Y + Y')$$

$$\bullet \times \bullet \bullet \times \bullet \equiv \varepsilon d \log(X_1 - Y)$$

$$\bullet \times \bullet \bullet \times \bullet \equiv \varepsilon d \log(X_3 - Y')$$

$$\bullet \times \bullet \bullet \times \bullet \equiv \varepsilon d \log(X_2 - Y + Y')$$

$$\bullet \times \bullet \times \bullet \bullet \equiv \varepsilon d \log(X_2 - Y - Y')$$

$$\bullet \times \bullet \bullet \times \bullet \equiv \varepsilon d \log(X_2 + Y - Y')$$

$$\bullet \times \bullet \times \bullet \bullet \equiv \varepsilon d \log(X_1 + X_2 + Y')$$

$$\bullet \times \bullet \times \bullet \bullet \equiv \varepsilon d \log(X_1 + X_2 - Y')$$

$$\bullet \times \bullet \times \bullet \bullet \equiv \varepsilon d \log(X_2 + X_3 + Y)$$

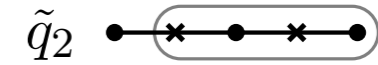
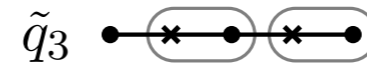
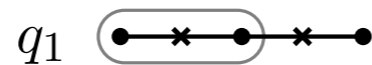
$$\bullet \times \bullet \times \bullet \bullet \equiv \varepsilon d \log(X_2 + X_3 - Y)$$

$$\bullet \times \bullet \times \bullet \bullet \equiv \varepsilon d \log(X_1 + X_2 + X_3)$$

# Graphical Rules

Taking the differential of one of the source functions gives

$$\begin{aligned}
 dQ_1 = & Q_1 \cdot \text{[diagram: line with 3 nodes, middle node circled in orange]} + (Q_1 - q_1) \cdot \text{[diagram: line with 3 nodes, first node circled in red, first two nodes in a grey oval]} \\
 & + (Q_1 - \tilde{q}_3) \cdot \text{[diagram: line with 3 nodes, first two nodes in a grey oval, third node circled in blue]} \\
 & + q_1 \cdot \text{[diagram: line with 3 nodes, first two nodes circled in red]} + (\tilde{q}_3 + \tilde{q}_2) \cdot \text{[diagram: line with 3 nodes, first two nodes in a grey oval, last node circled in blue]} \\
 & - \tilde{q}_2 \cdot \text{[diagram: line with 3 nodes, all three nodes in a blue oval]}
 \end{aligned}$$

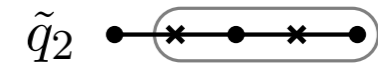
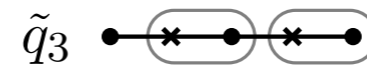
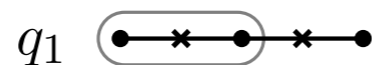
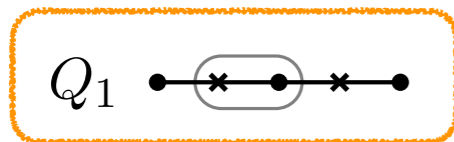


The equation can be predicted using simple graphical rules.

# Graphical Rules

Taking the differential of one of the source functions gives

$$dQ_1 = \underbrace{Q_1 \bullet \times \bullet \times \bullet}_{\text{activation}} + (Q_1 - q_1) \bullet \times \bullet \times \bullet + (Q_1 - \tilde{q}_3) \bullet \times \bullet \times \bullet + q_1 \bullet \times \bullet \times \bullet + (\tilde{q}_3 + \tilde{q}_2) \bullet \times \bullet \times \bullet - \tilde{q}_2 \bullet \times \bullet \times \bullet$$

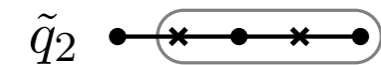
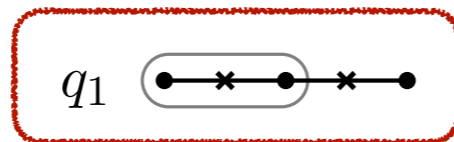
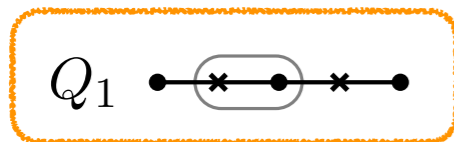


The equation can be predicted using simple graphical rules.

# Graphical Rules

Taking the differential of one of the source functions gives

$$dQ_1 = \underbrace{Q_1 \text{ (activation)}}_{\text{activation}} + \underbrace{\left( (Q_1 - q_1) \text{ (merger)} + q_1 \text{ (merger)} \right)}_{\text{merger}} + (Q_1 - \tilde{q}_3) \text{ (blue)} + (\tilde{q}_3 + \tilde{q}_2) \text{ (blue)} - \tilde{q}_2 \text{ (blue)}$$

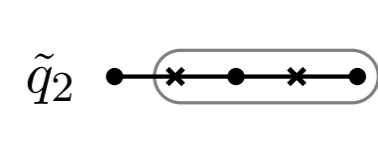
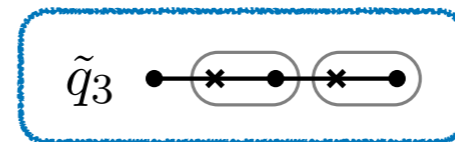
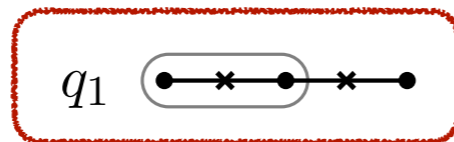
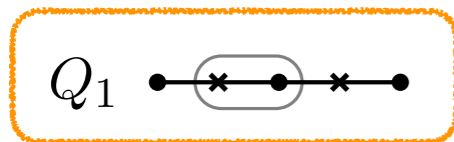


The equation can be predicted using simple graphical rules.

# Graphical Rules

Taking the differential of one of the source functions gives

$$dQ_1 = \underbrace{Q_1 \text{ (activation)}} + \underbrace{\left( (Q_1 - q_1) \text{ (merger)} + q_1 \text{ (merger)} \right)} + \underbrace{\left( (Q_1 - \tilde{q}_3) \text{ (nucleation)} + (\tilde{q}_3 + \tilde{q}_2) \text{ (nucleation)} - \tilde{q}_2 \text{ (nucleation)} \right)}$$

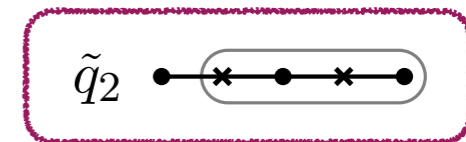
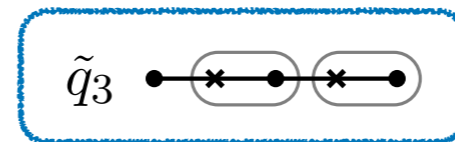
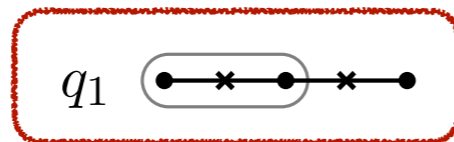
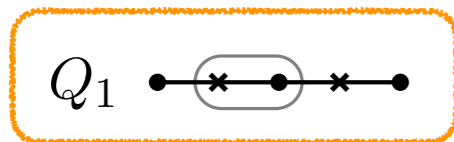


The equation can be predicted using simple graphical rules.

# Graphical Rules

Taking the differential of one of the source functions gives

$$dQ_1 = \underbrace{Q_1 \text{ (activation)}} + \underbrace{\begin{aligned} &(Q_1 - q_1) \text{ (merger)} \\ &+ q_1 \text{ (merger)} \end{aligned}} + \underbrace{\begin{aligned} &(Q_1 - \tilde{q}_3) \text{ (nucleation)} \\ &+ (\tilde{q}_3 + \tilde{q}_2) \text{ (absorption)} \\ &- \tilde{q}_2 \text{ (absorption)} \end{aligned}}$$



The equation can be predicted using simple graphical rules.



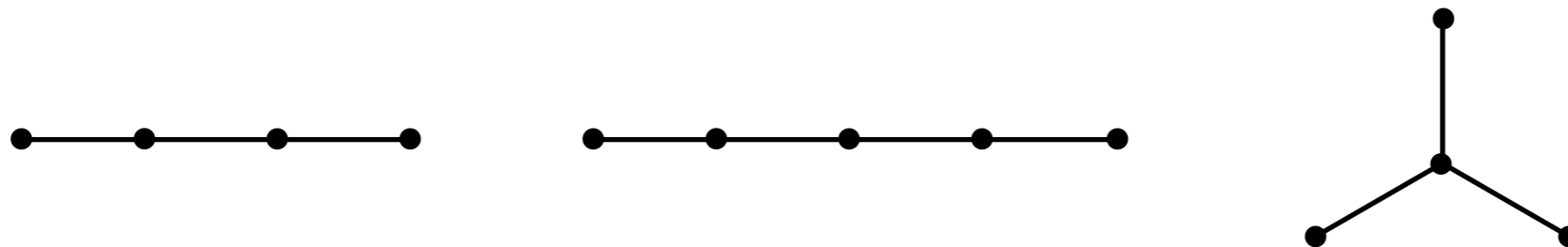
# Graphical Rules

Taking the differential of one of the source functions gives

$$dQ_1 = \underbrace{Q_1 \cdot \text{[activation diagram]}}_{\text{activation}} + \underbrace{\left( (Q_1 - q_1) \cdot \text{[merger diagram 1]} + q_1 \cdot \text{[merger diagram 2]} \right)}_{\text{merger}} + \underbrace{\left( (Q_1 - \tilde{q}_3) \cdot \text{[nucleation diagram 1]} + (\tilde{q}_3 + \tilde{q}_2) \cdot \text{[nucleation diagram 2]} - \tilde{q}_2 \cdot \text{[absorption diagram]} \right)}_{\text{absorption}}$$

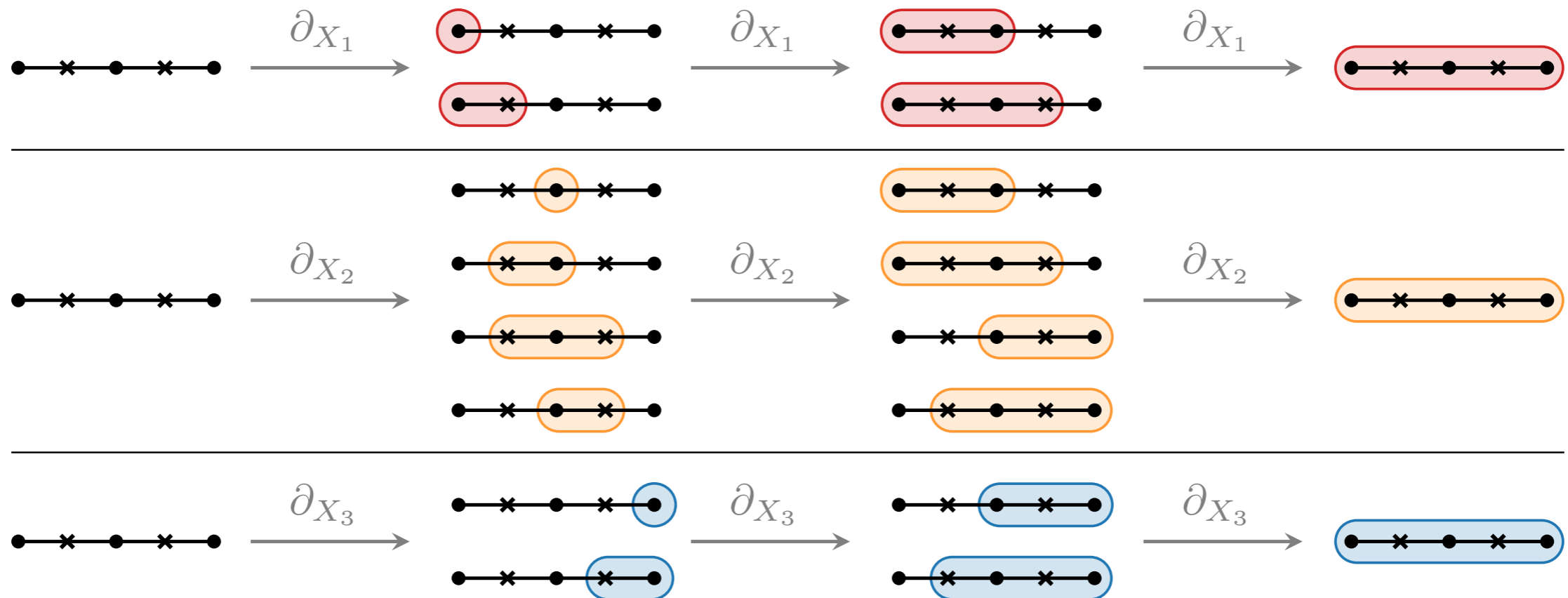
The diagram shows the differential equation for  $dQ_1$  as a sum of three terms, each represented by a graphical rule. The first term, labeled "activation", is  $Q_1$  multiplied by a diagram of a horizontal line with four nodes, where the second node from the left is enclosed in an orange oval. The second term, labeled "merger", consists of two sub-terms:  $(Q_1 - q_1)$  multiplied by a diagram where the first node is in a red circle and the second node is in a grey oval, and  $q_1$  multiplied by a diagram where the second node is in a red oval. The third term, labeled "absorption", consists of three sub-terms:  $(Q_1 - \tilde{q}_3)$  multiplied by a diagram where the third node is in a blue circle and the second node is in a grey oval;  $(\tilde{q}_3 + \tilde{q}_2)$  multiplied by a diagram where the second and third nodes are in blue ovals; and  $-\tilde{q}_2$  multiplied by a diagram where the second node is in a blue oval. The labels "nucleation" and "absorption" are placed above and below their respective sub-terms.

The graphical rules are local and can be used to predict the differential equations for arbitrary tree graphs with different topologies.



# Time Evolution as Kinematic Flow

Taking successive partial derivatives, the tubings grow as

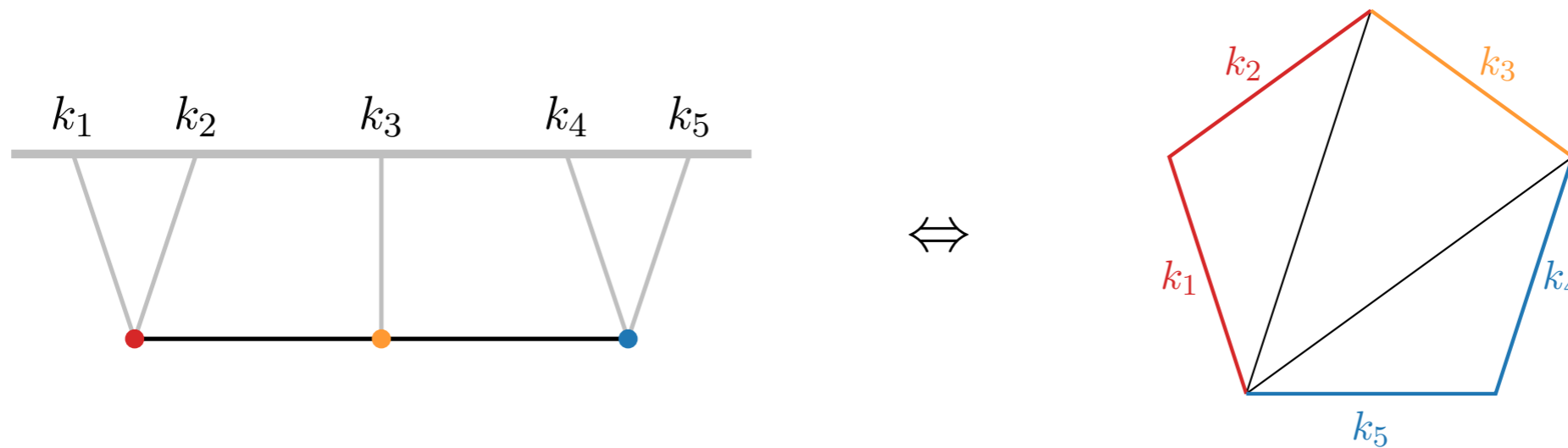


The growth ends, and the system closes, when all vertices are enclosed.

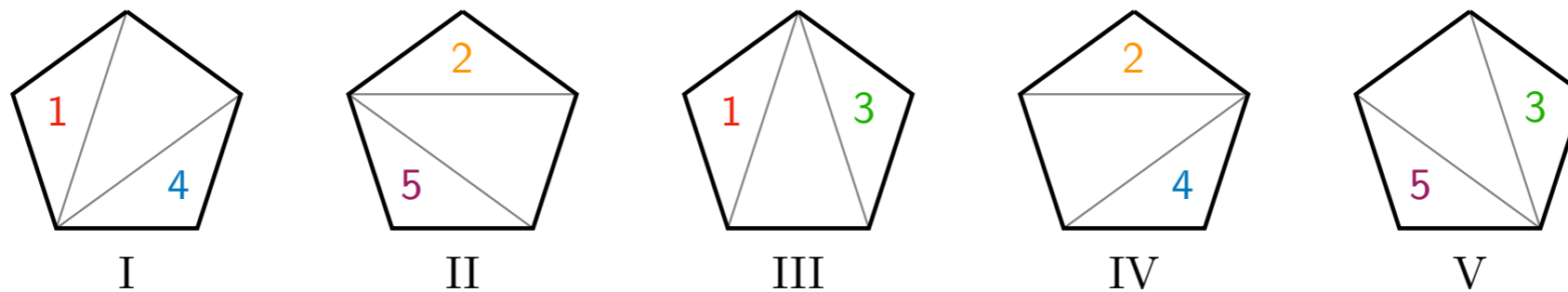
## **III. Beyond Single Graphs**

# Beyond Single Graphs

A single graph corresponds to a specific triangulation of a polygon.

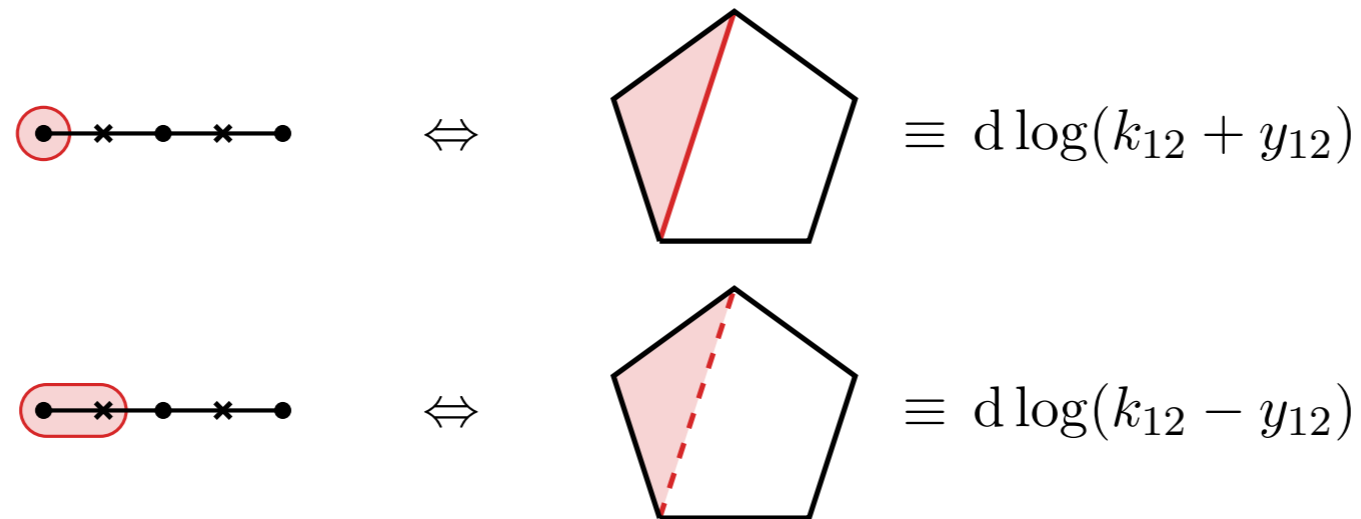


Distinct triangulations a pentagon correspond to different permutations.

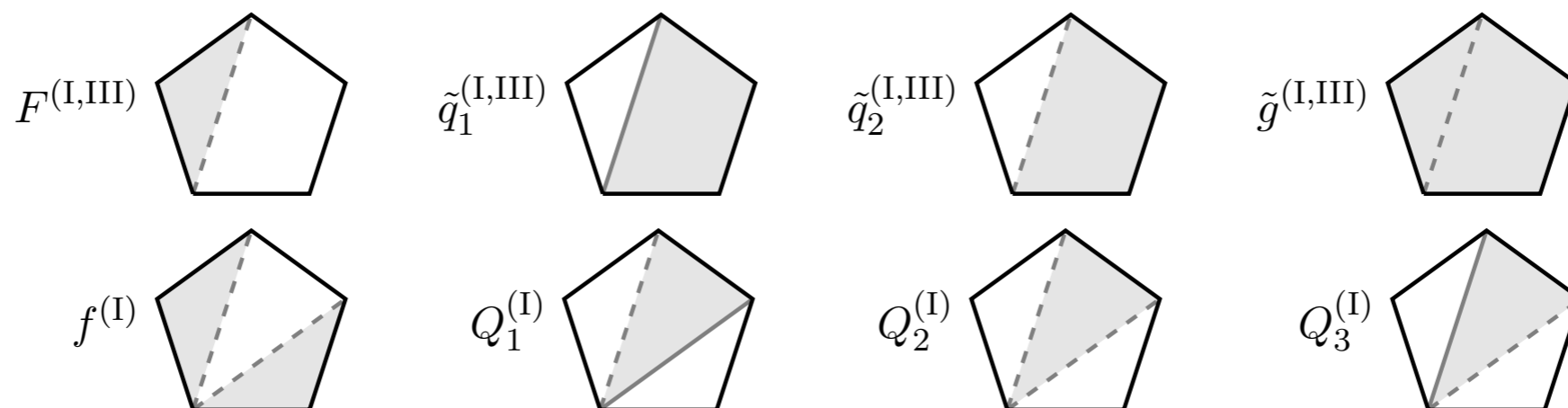


# Kinematic Polygons

Letters are now represented by subpolygons with (dashed) internal edges.



Sources are given by subpolygons with at least one dashed internal edge.



# Growth of Kinematic Polygons

For example, taking the  $k_1$  derivative of the 5-point function gives

$$\begin{aligned}
 \partial_{k_1} \text{Pentagon} &= (\psi - F^{(I,III)}) \text{Pentagon}_{I,III} + (\psi - \sum Q_i^{(IV)}) \text{Pentagon}_{IV} \\
 &+ F^{(I,III)} \text{Pentagon}_{I,III}^{dashed} + Q_1^{(IV)} \text{Pentagon}_{IV}^{dashed} \\
 &(\psi - F^{(II,V)}) \text{Pentagon}_{II,V} + Q_2^{(IV)} \text{Pentagon}_{IV}^{dashed} \\
 &+ F^{(II,V)} \text{Pentagon}_{II,V}^{dashed} + Q_3^{(IV)} \text{Pentagon}_{IV}^{dashed}
 \end{aligned}$$

# Growth of Kinematic Polygons

The system of equations closes when the subpolygon is fully grown.

$$\begin{aligned}
 \partial_{k_1} \text{ (pentagon with dashed diagonal from top-left to bottom-right)} &= F^{(I,III)} \text{ (pentagon with dashed diagonal from bottom-left to top-right)} + q_1^{(I,IV)} \text{ (pentagon with solid diagonal from bottom-left to top-right)} + q_2^{(I,IV)} \text{ (pentagon with dashed diagonal from top-right to bottom-left)} \\
 &+ q_1^{(III,V)} \text{ (pentagon with solid diagonal from top-right to bottom-left)} + q_2^{(III,V)} \text{ (pentagon with dashed diagonal from top-right to bottom-left)} \\
 \partial_{k_1} \text{ (pentagon with dashed diagonal from top-right to bottom-left)} &= q_2^{(I,IV)} \text{ (pentagon with dashed diagonal from bottom-left to top-right)} + Z^{(I-V)} \text{ (solid green pentagon)} \\
 \partial_{k_1} \text{ (solid gray pentagon)} &= Z^{(I-V)} \text{ (solid green pentagon)}
 \end{aligned}$$

# Conclusions



# Conclusions

We have developed a systematic way of deriving the differential equations for the FRW wavefunction of conformally-coupled scalars at tree level.

$$d \left[ \begin{array}{c} \text{---} \times \text{---} \cdots \text{---} \\ \text{---} \times \text{---} \cdots \text{---} \\ \vdots \\ \text{---} \times \text{---} \cdots \text{---} \end{array} \right] = \varepsilon \left[ \begin{array}{c} \text{---} \times \text{---} \cdots \text{---} \\ \text{---} \times \text{---} \cdots \text{---} \\ \vdots \\ \text{---} \times \text{---} \cdots \text{---} \end{array} \right]$$

The differential equations can be predicted in terms of the dynamics of graphs.

The sum over graphs is captured by a kinematic polygon.

**Extra Slides**

# Simplex Forms

The previous example suggests that the canonical forms of simplices are natural objects to consider.

$$[L_1 \cdots L_n] = d \log L_1 \wedge \cdots \wedge d \log L_n$$

The differential of a (**projective**) simplex form obeys a nice formula:

$$d[L_1 \cdots L_n] = -\varepsilon \sum_a d \log \det(\hat{L}_1 \cdots \hat{L}_n \hat{T}_a) \times \partial[L_1 \cdots L_n T_a]$$

letters

$$L_i = \hat{L}_i \cdot X$$

$$X = (x_1, \cdots, x_n, 1)$$

boundary

# An Algorithm for All Trees

- ▶ Define all sources by substituting twisted planes in the wavefunction.

$$N_{\text{source}} = 4^{n-1}$$

- ▶ Take the differential of the sources using the formula

$$d[L_1 \cdots L_n] = -\varepsilon \sum_a d \log \det(\hat{L}_1 \cdots \hat{L}_n \hat{T}_a) \times \partial[L_1 \cdots L_n T_a]$$

- ▶ Express the result back in terms of the sources.