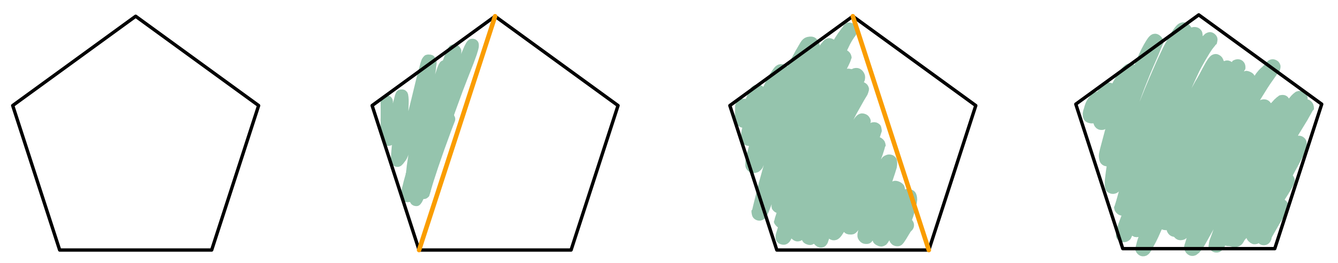
A New Twist on Time:

Differential Equations for Cosmological Correlators



Hayden Lee

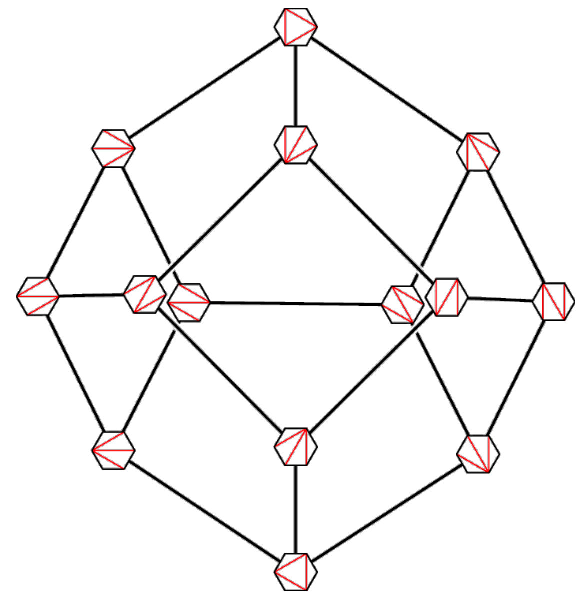
University of Chicago

w/ N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, G. Pimentel [to appear]

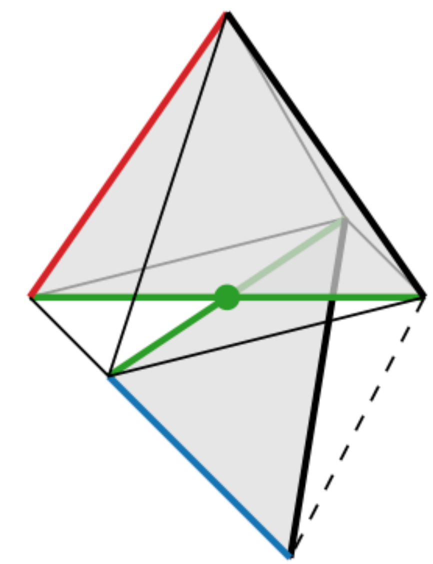
09/21/23 Cortona

Over the past decade, we have seen scattering amplitudes emerge from new mathematical structures in boundary kinematic space.

amplituhedron



associahedron



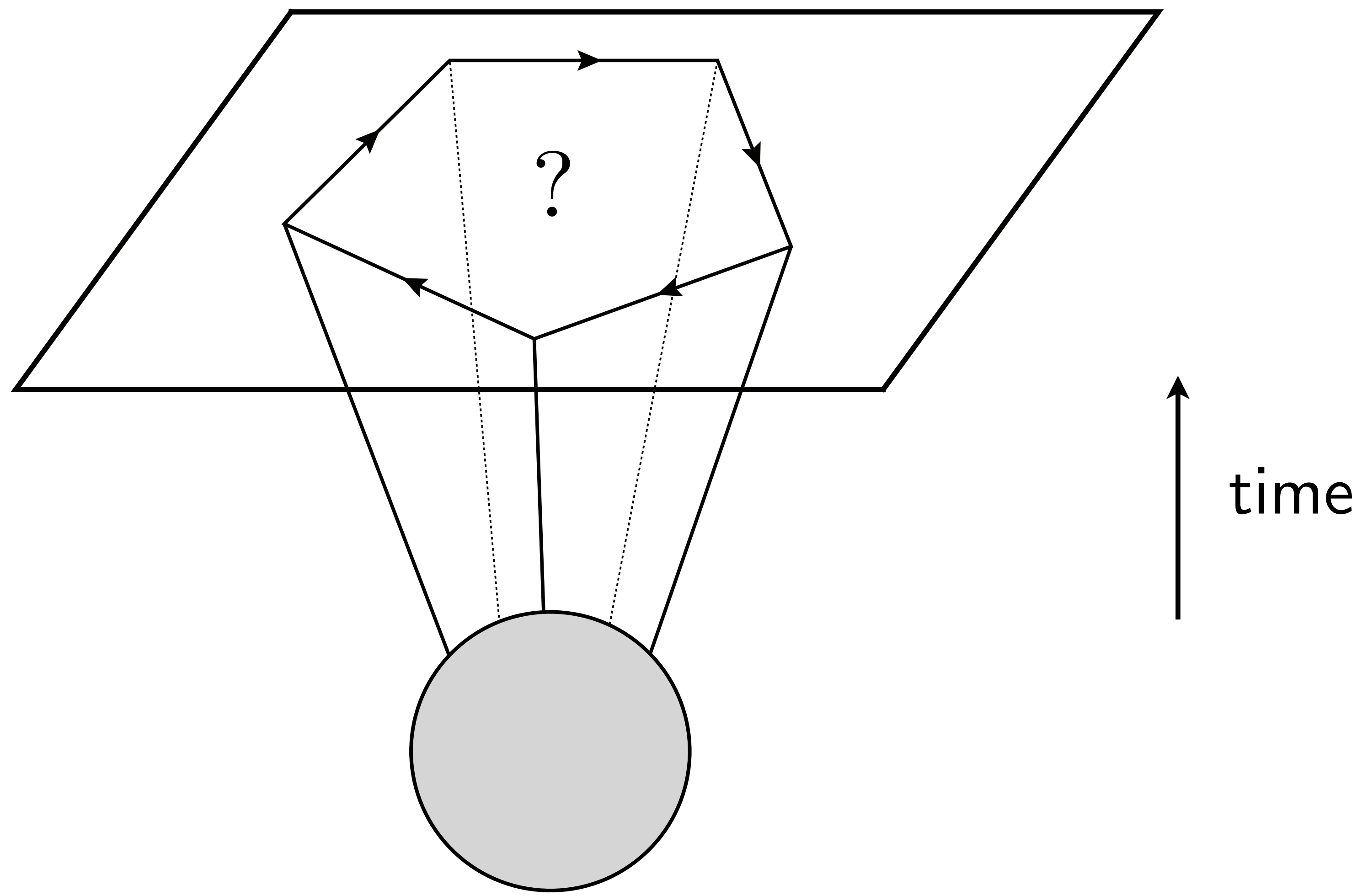
cosmological polytopes

conceptual advantage: focuses directly on boundary observables

practical advantage: simplifies calculations

We can't directly observe the pre-Big Bang evolution of the universe,

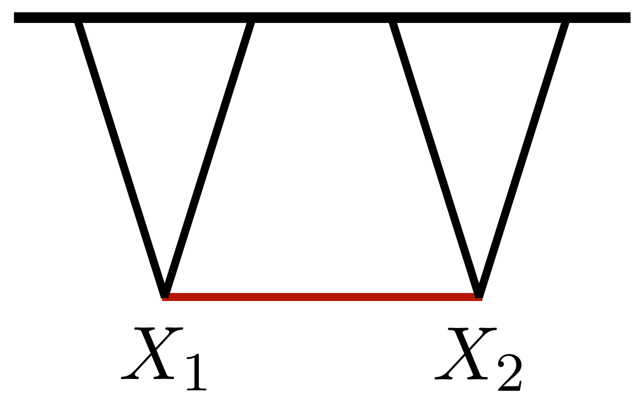
but instead must infer it from spatial correlations on the future boundary.



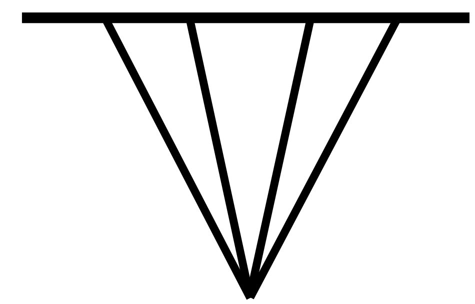
How can we see "time evolution" from boundary correlators?

In de Sitter space, boundary conformal correlators satisfy local differential equations that reflect bulk time evolution.

$(\Delta_{X_1} - \mu^2)$



$=$



$$\Delta_X = (X^2 - 1)\partial_X^2 + 2X\partial_X$$

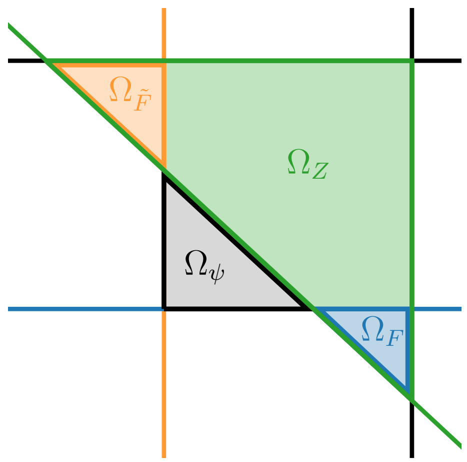
Arkani-Hamed, Maldacena [2015]
Arkani-Hamed, Baumann, HL, Pimentel [2018]

Is there a deeper reason for their existence beyond de Sitter?

Outline

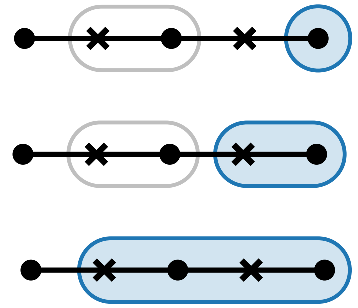
I.

Correlators as Twisted Integrals



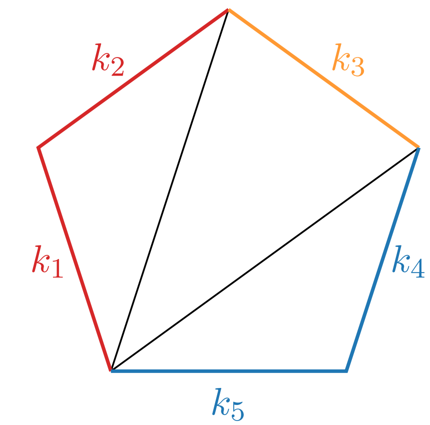
II.

Time Evolution as Kinematic Flow



III.

Beyond Single Graphs

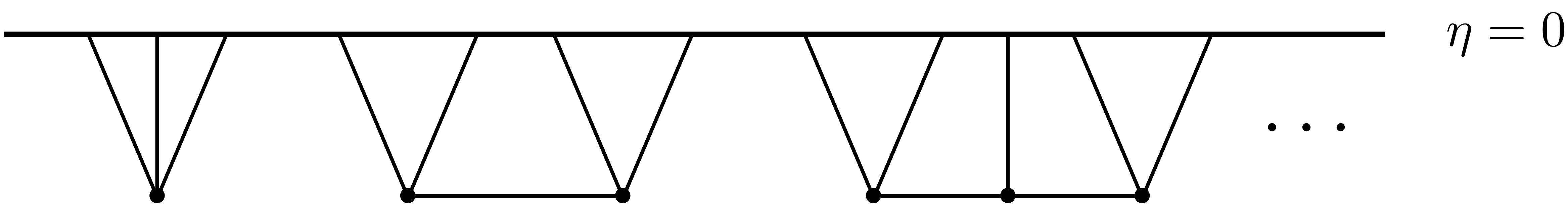


Today, I'll present a new mathematical perspective on cosmological time evolution.

I. Correlators as Twisted Integrals

Cosmological Wavefunction

We'll be interested in the wavefunction (coefficients) in FRW cosmology.



Specifically, we'll consider conformally-coupled scalars with (non-conformal) polynomial interactions and a power-law scale factor.

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \frac{\lambda}{3!}\phi^3 \right] \qquad a(\eta) \propto \frac{1}{\eta^{1+\varepsilon}}$$

$$[\varepsilon = 0 : \text{ dS}\,, \quad \varepsilon = -1 : \text{ flat}\,, \quad \varepsilon = -2 : \text{ radiation}\,, \quad \varepsilon = -3 : \text{ matter}]$$

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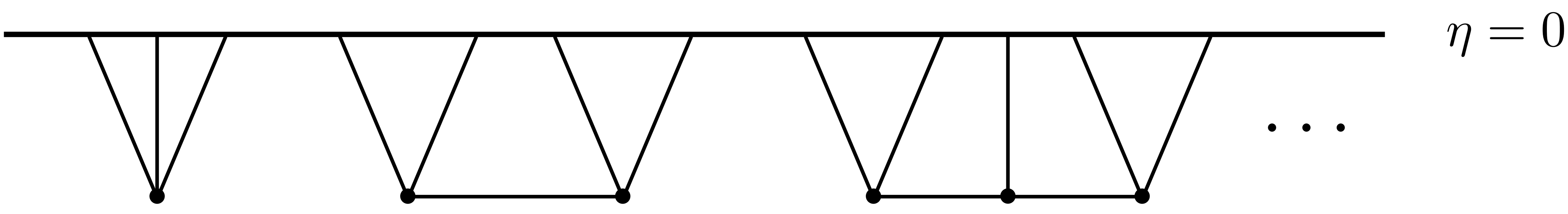


This can be mapped to the action of massless scalars with time-dependent couplings.

$$S = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{\tilde{\lambda}(\eta)}{3!}\phi^3 \right] \qquad \tilde{\lambda}(\eta) \propto \frac{1}{\eta^{1+\varepsilon}} \propto \int_0^\infty d\omega\, \omega^{\varepsilon} e^{i\omega\eta}$$

$$[\varepsilon = 0 : \text{ dS}\,, \quad \varepsilon = -1 : \text{ flat}\,, \quad \varepsilon = -2 : \text{ radiation}\,, \quad \varepsilon = -3 : \text{ matter}]$$

Cosmological Wavefunction

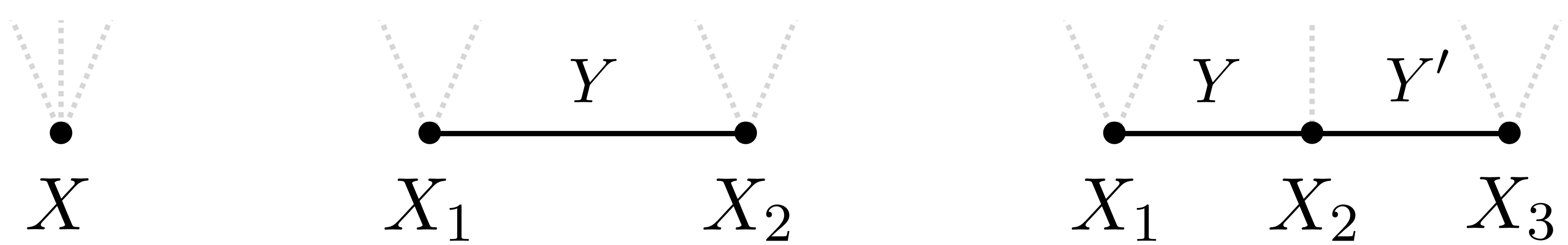
The wavefunction in FRW is related to the flat-space one as

$$\psi^{(n)}_{\mathrm{FRW}}(X_i) = \int_0^\infty dx_1 \cdots dx_n \, (x_1 \cdots x_n)^\varepsilon \, \psi^{(n)}_{\mathrm{flat}}(X_i \to X_i + x_i)$$

"twist"

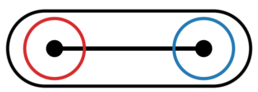
rational function (tree level)

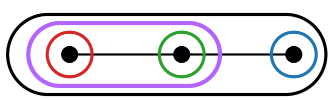
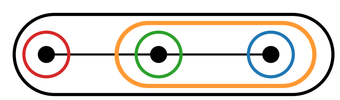
It's convenient to consider graphs with all external propagators truncated.



Wavefunction in Flat Space

Flat-space wavefunction is described by rational functions with simple poles.

$\psi^{(2)}_{\text{flat}} =$  $= \dfrac{1}{(X_1+X_2)(X_1+Y)(X_2+Y)}$

$\psi^{(3)}_{\text{flat}} =$  $+$ 

$$= \frac{1}{(X_1+X_2+X_3)(X_1+Y)(X_2+Y+Y')(X_3+Y')}\left(\frac{1}{X_1+X_2+Y'}+\frac{1}{X_2+X_3+Y}\right)$$

There is a simple combinatorial prescription for constructing the flat-space wavefunction.

Arkani-Hamed, Benincasa, Postnikov [2017]

Case Study: Two-Site Chain

The simplest nontrivial example in FRW is the two-site chain graph:

$$\psi^{(2)}_{\mathrm{FRW}}(X_1, X_2) = \int_0^\infty dx_1 dx_2 \, \frac{(x_1 x_2)^\varepsilon}{(x_1 + x_2 + X_1 + X_2)(x_1 + X_1 + 1)(x_2 + X_2 + 1)}$$

Modern approaches to compute integrals of this type include

- the method of differential equations
- twisted cohomology

Similar techniques can be applied to our cosmology problem.

Family of Integrals

Consider a family of integrals with the same singularities.

$$I_{\vec{n}} = \int (x_1 x_2)^{\varepsilon}\, \Omega_{\vec{n}}\,, \quad \Omega_{\vec{n}} = \frac{dx_1 \wedge dx_2}{T_1^{m_1} T_2^{m_2} B_1^{n_1} B_2^{n_2} B_3^{n_3}}$$

$$T_1 = x_1\,, \quad B_1 = x_1 + X_1 + 1$$

$$T_2 = x_2\,, \quad B_2 = x_2 + X_2 + 1\,, \quad B_3 = x_1 + x_2 + X_1 + X_2$$

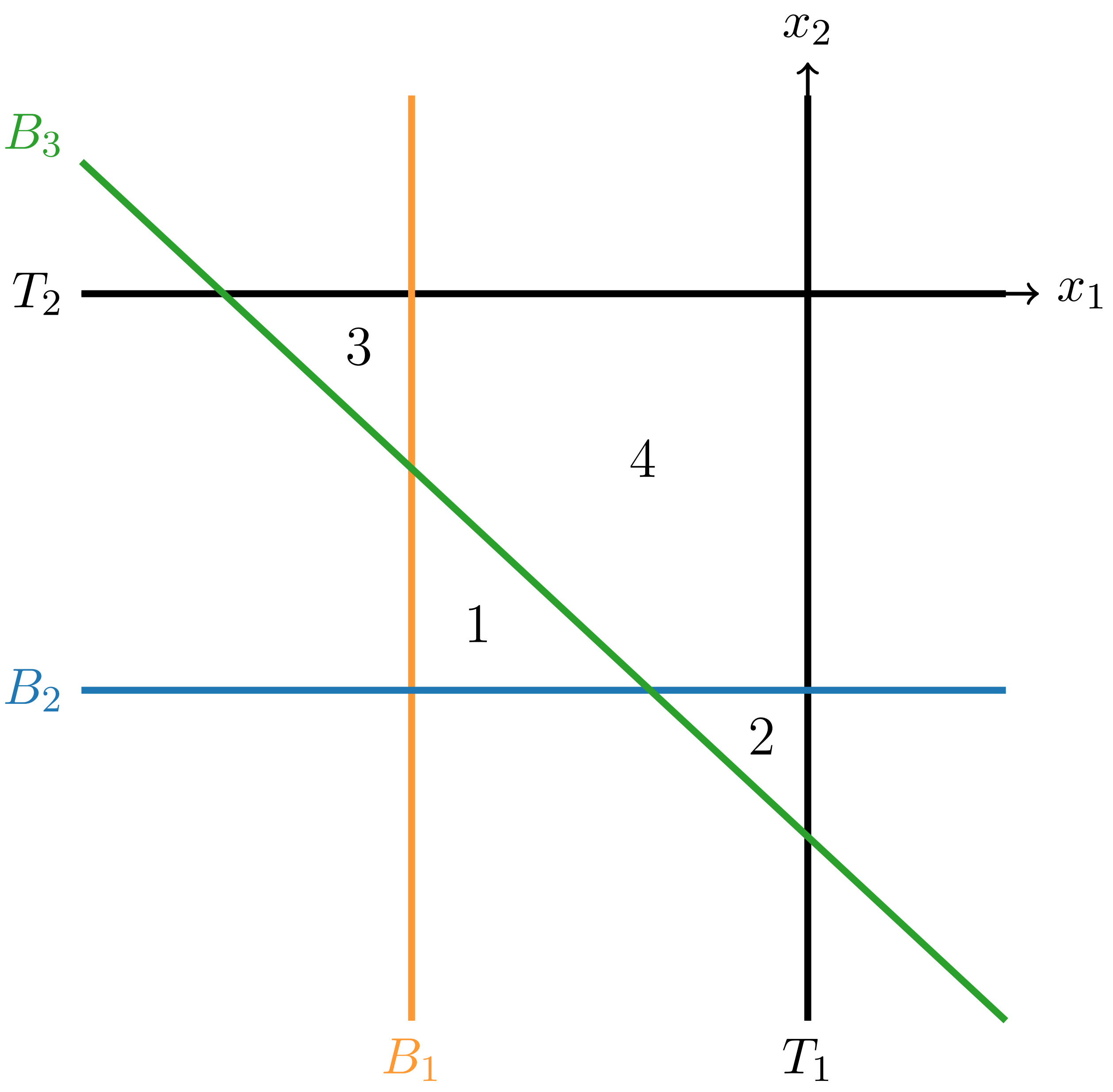
These integrals form a finite-dimensional vector space.

Twisted cohomology provides a geometric way to determine the size of this vector space.

Aomoto, Kita [2011]
Mastrolia, Mizera [2018]

Master Integrals

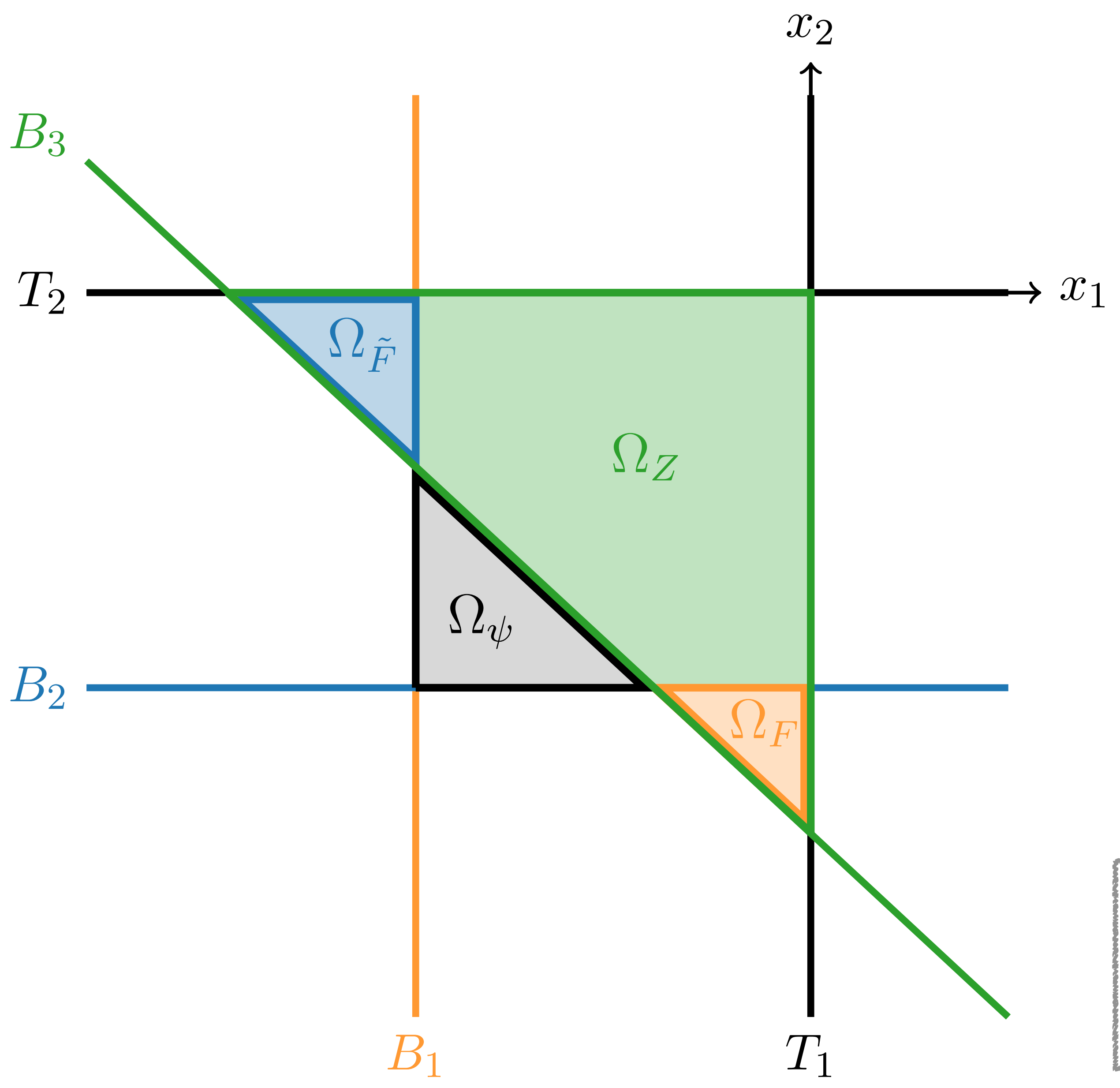
The number of master integrals equals the number of bounded regions defined by the singular factors of the integrand.



$$T_1 = x_1$$
$$T_2 = x_2$$
$$B_1 = x_1 + X_1 + 1$$
$$B_2 = x_2 + X_2 + 1$$
$$B_3 = x_1 + x_2 + X_1 + X_2$$

Master Integrals

A good choice for the basis of integrals is given by the canonical forms of bounded regions.



$$\vec{I} = \begin{bmatrix} \psi \\ F \\ \tilde{F} \\ Z \end{bmatrix} = \int (x_1 x_2)^\varepsilon \begin{bmatrix} \Omega_\psi \\ \Omega_F \\ \Omega_{\tilde{F}} \\ \Omega_Z \end{bmatrix}$$

$$\Omega_{\text{can}}[\triangle_{L_1 L_2 L_3}] = d\log\left(\frac{L_1}{L_3}\right) \wedge d\log\left(\frac{L_2}{L_3}\right)$$

Differential Equations

The basis of master integrals satisfy differential equations.

For instance, taking an X1 derivative of the wavefunction gives

$$\partial_{X_1}\psi = \int (x_1 x_2)^{\varepsilon}\, \partial_{x_1}\Omega_{\psi} \qquad (\partial_{X_1}\Omega_{\psi} = \partial_{x_1}\Omega_{\psi})$$

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$$= \varepsilon \int (x_1 x_2)^{\varepsilon} \left[-\frac{1}{T_1}\Omega_{\psi}\right] \qquad \text{(IBP)}$$

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$$= \varepsilon \int (x_1x_2)^{\varepsilon} \left[-\frac{1}{T_1}\Omega_{\psi}\right] \qquad \text{(IBP)}$$

$$= \varepsilon \int (x_1x_2)^{\varepsilon} \left[\frac{1}{X_1+1}(\Omega_{\psi} - \Omega_F) + \frac{1}{X_1-1}\Omega_F\right]$$

$$= \varepsilon \left[\frac{1}{X_1+1}(\psi - F) + \frac{1}{X_1-1}F\right]$$

Differential Equations

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For instance, taking an X1 derivative of the wavefunction gives

$$\partial_{X_1}\psi = \varepsilon\left[\frac{1}{X_1+1}(\psi - F) + \frac{1}{X_1-1}F\right]$$

This can be expressed in terms of the total differential as

$$\begin{aligned}\mathrm{d}\psi = \varepsilon\Big[&\mathrm{d}\log(X_1+1)(\psi - F) + \mathrm{d}\log(X_1-1)F\\ &+ \mathrm{d}\log(X_3+1)(\psi - \tilde{F}) + \mathrm{d}\log(X_3-1)\tilde{F}\Big]\end{aligned}$$

Differential Equations

Taking the total differential of the basis vector gives

$$\mathrm{d}\vec{I} = \varepsilon A \vec{I}$$

$$\mathrm{d} = \sum_i \mathrm{d}X_i \frac{\partial}{\partial X_i}$$

$$\mathrm{d}^2 = 0 \quad \Rightarrow$$
$$\mathrm{d}A = A \wedge A = 0$$

with

letters

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathrm{d}\log(X_1 + X_2) + \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 + 1) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 - 1)$$
$$+ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 + 1) + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 - 1)$$

Locality and Time Evolution

The equations can also be combined into a local second-order equation.

$$\Big[(X_1^2-1)\partial_{X_1}^2+2(1-\varepsilon)X_1\partial_{X_1}-\varepsilon(1-\varepsilon)\Big]\psi=\frac{1}{(X_1+X_2)^{1-2\varepsilon}}$$

depends only on X_1

The solution after imposing appropriate boundary conditions is

$$\begin{aligned}\psi&=c_A(\varepsilon)(1+X_1)^{\varepsilon}(1+X_2)^{\varepsilon}\\&+c_B(\varepsilon)(X_1+X_2)^{2\varepsilon}\left(1-{}_2F_1\left[\begin{matrix}1,\varepsilon\\1-\varepsilon\end{matrix}\,\middle|\,\frac{1-X_2}{1+X_1}\right]-{}_2F_1\left[\begin{matrix}1,\varepsilon\\1-\varepsilon\end{matrix}\,\middle|\,\frac{1-X_1}{1+X_2}\right]\right)\end{aligned}$$

General Tree Graphs

There are several challenges for the conventional approach:

(!) Hard to visualize higher-dimensional planes.

(!!) Finding an optimal basis is a bit of an art.

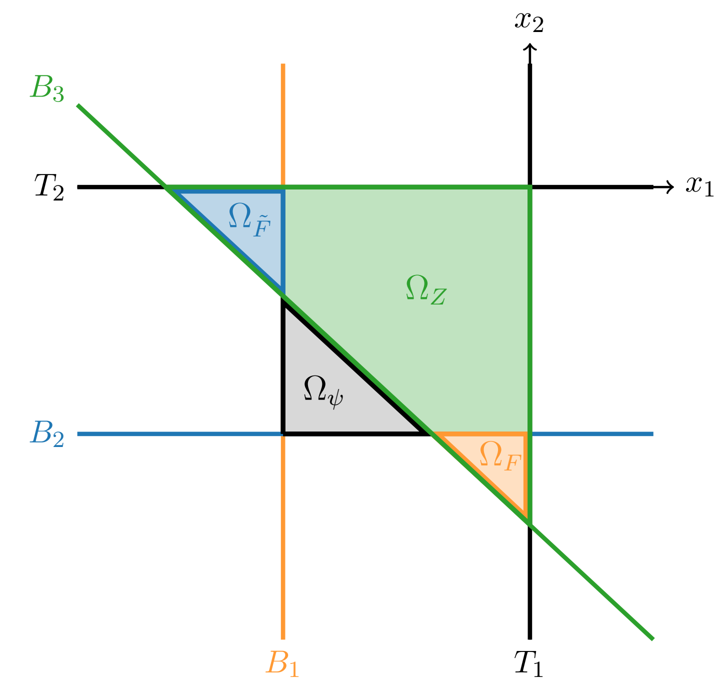
(!!!) The derivatives are not automatically expressible in terms of the original basis.

Remarkably, there are hidden combinatorial and geometric structures underlying these differential equations that allow us to bypass these challenges.

II. Time Evolution as Kinematic Flow

Two-Site Chain Revisited

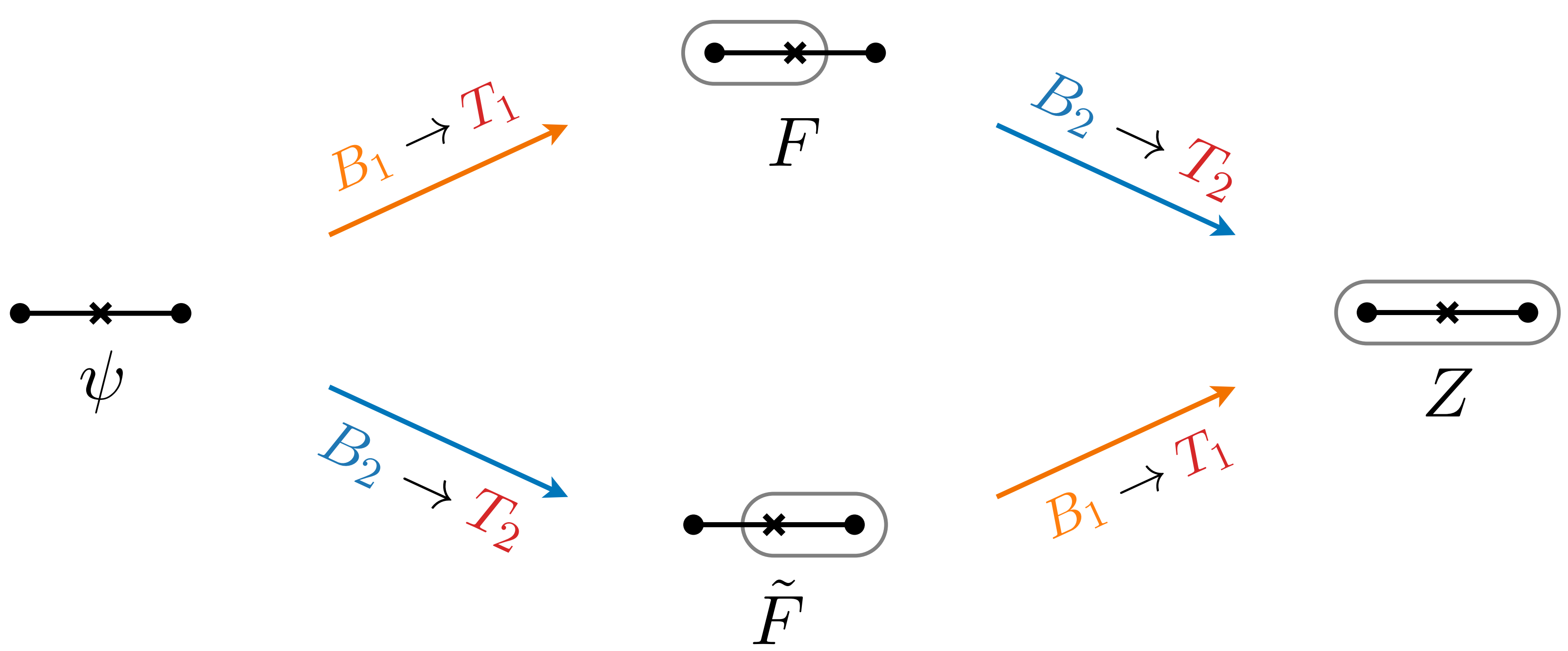
For the two-site chain, we had



$$\Omega_\psi = \Omega_{\text{can}}[\Delta_{B_1 B_2 B_3}] \qquad \Omega_F = \Omega_{\text{can}}[\Delta_{B_2 B_3 T_1}]$$

$$\Omega_Z = \Omega_{\text{can}}[\Delta_{B_3 T_1 T_2}] \qquad \Omega_{\tilde{F}} = \Omega_{\text{can}}[\Delta_{B_3 B_1 T_2}]$$

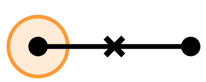
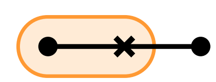
Notice that the source functions are obtained by substituting twisted planes.

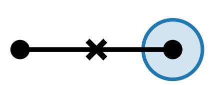
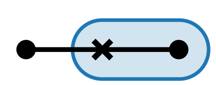


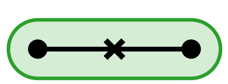
Graphical Representation

It's natural to represent the basis integrals by (disconnected) tubings that enclose at least one cross

and the letters by connected tubings as

 $\equiv \; \varepsilon \, \mathrm{d}\log(X_1 + 1)\,,$  $\equiv \; \varepsilon \, \mathrm{d}\log(X_1 - 1)\,,$

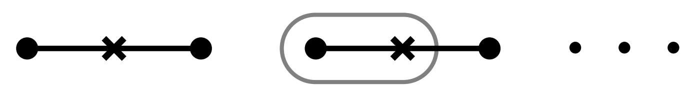
 $\equiv \; \varepsilon \, \mathrm{d}\log(X_2 + 1)\,,$  $\equiv \; \varepsilon \, \mathrm{d}\log(X_2 - 1)\,,$

 $\equiv \; \varepsilon \, \mathrm{d}\log(X_1 + X_2)\,.$

Under the action of "d", these tubings grow according to simple graphical rules.

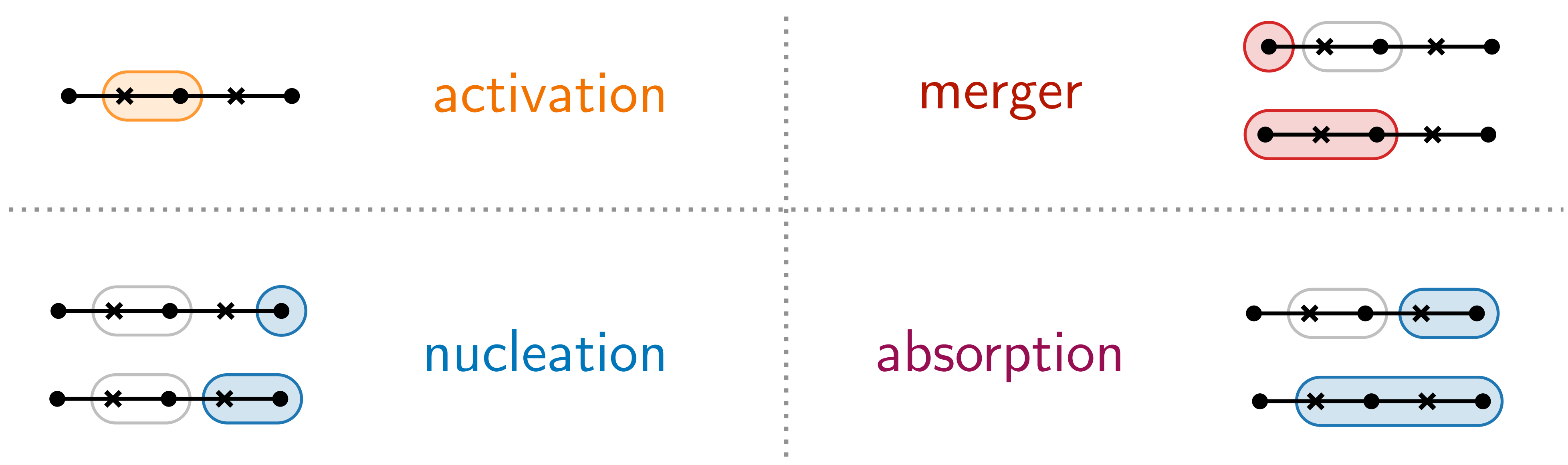
Graphical Rules for All Trees

- Enumerate all possible tubings for the basis functions.



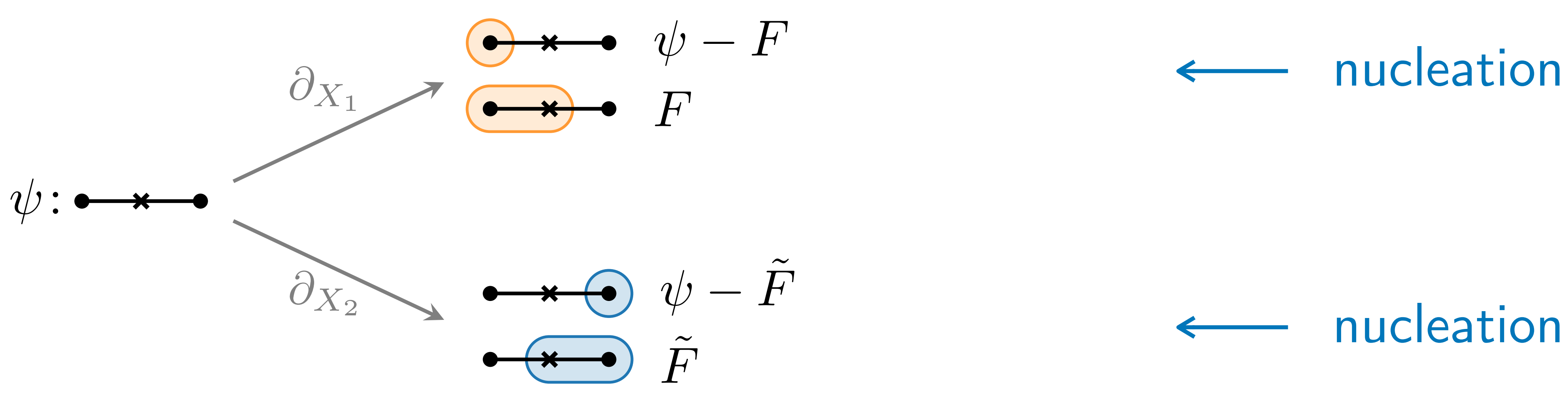
$$N_{\text{basis}} = 4^{n-1}$$

- Taking the differential, these tubings grow and merge according to four graphical rules:



Growth of Tubings

Let's see these rules in action for the two-site chain:



Taking derivatives nucleates a pair of tubings around each vertex, which correspond to letters with $\pm$ sign.

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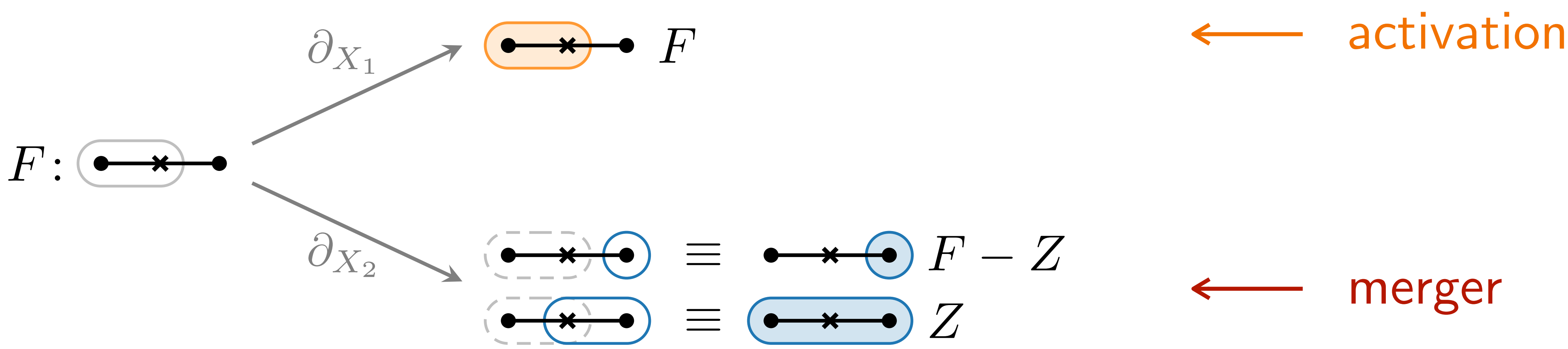

When a vertex is inside the parent tubing, then the tubing gets activated.

If two tubings overlap, then they merge to form a bigger tubing.

Growth of Tubings

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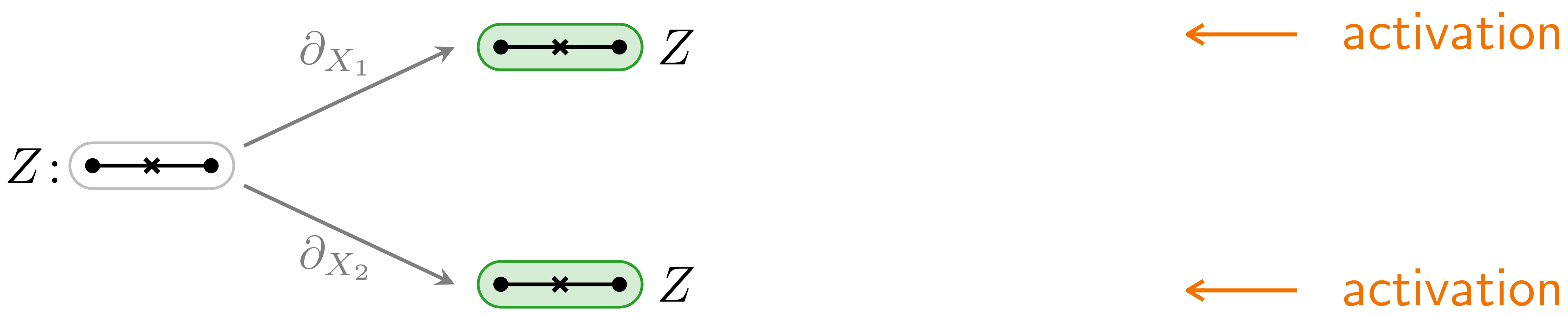
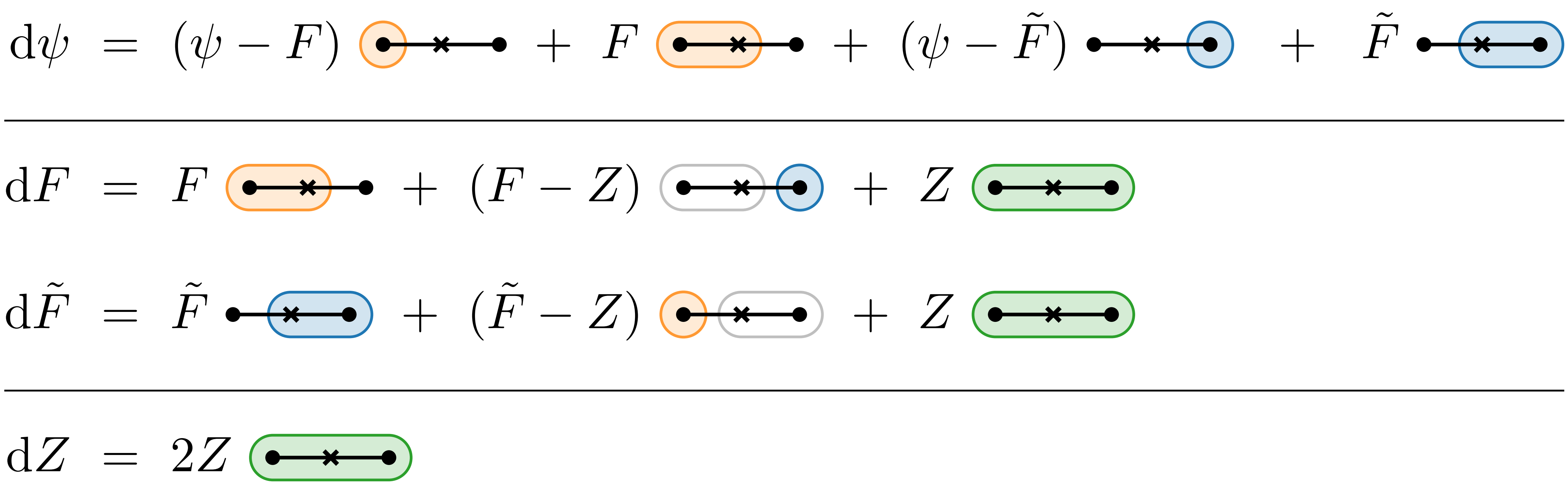


The growth ends when all vertices are enclosed inside a tubing.

No new source functions appear, and the system closes.

Differential Equations

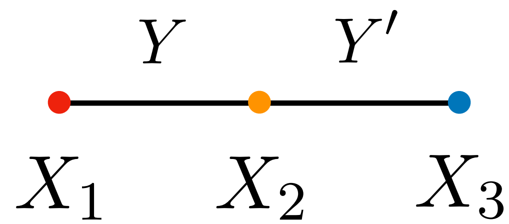
The differential equations for the two-site chain can be represented as



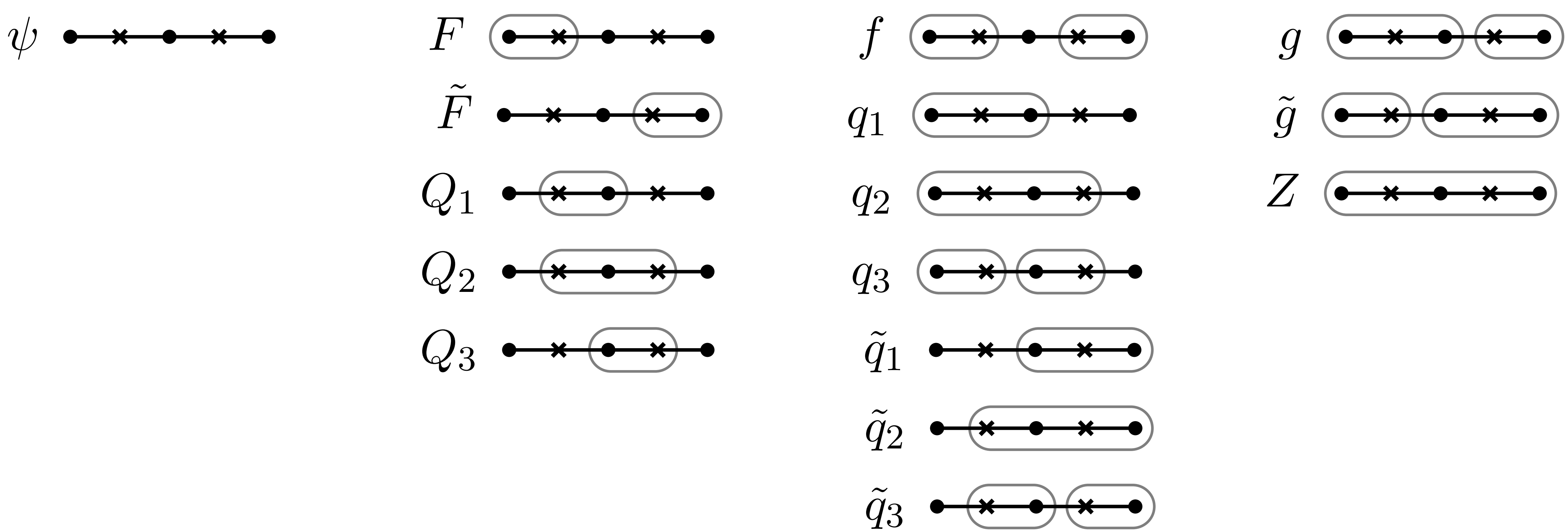
The rules are local and completely general – no more artistic choices of basis integrals and their IBP reduction are needed.

Three-Site Chain

A similar pattern holds for a three-site graph:



There are 16 basis functions, less than the 25 we get from twisted cohomology.



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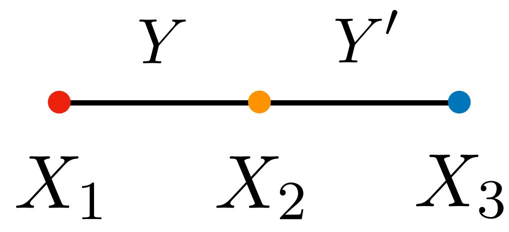
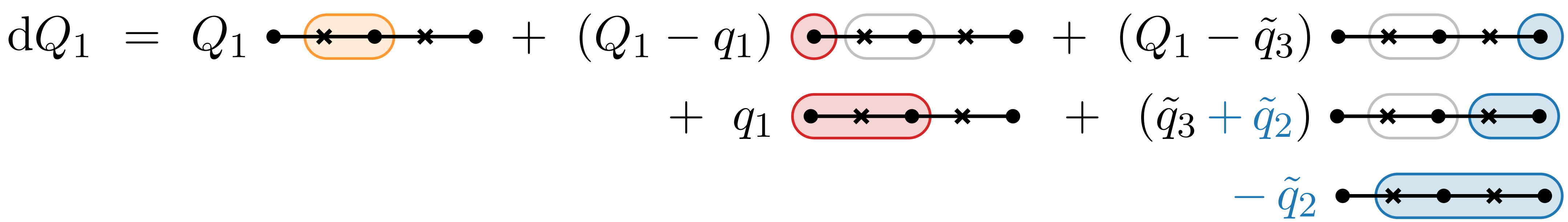


There are 13 letters, less than the naive counting of 19.

$$\equiv \varepsilon\,\mathrm{d}\log(X_1+Y)$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_3+Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_2+Y+Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_1-Y)$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_3-Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_2-Y+Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_2-Y-Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_2+Y-Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_1+X_2+Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_2+X_3+Y)$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_1+X_2+X_3)$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_1+X_2-Y')$$
$$\equiv \varepsilon\,\mathrm{d}\log(X_2+X_3-Y)$$

Graphical Rules

Taking the differential of one of the source functions gives



The equation can be predicted using simple graphical rules.

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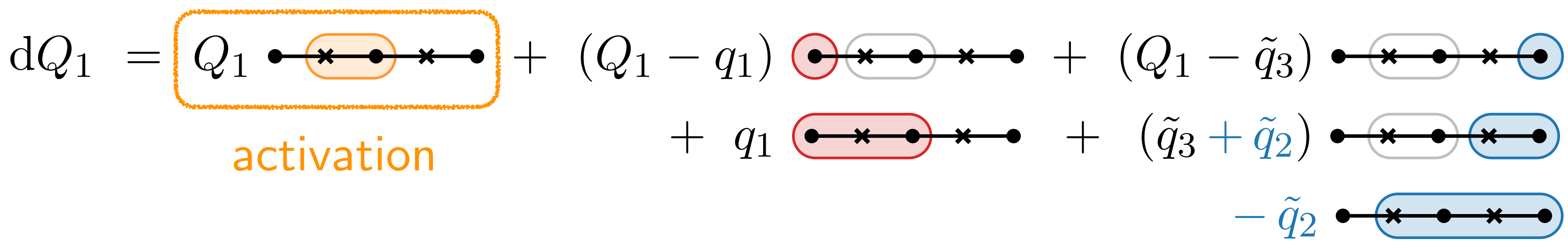

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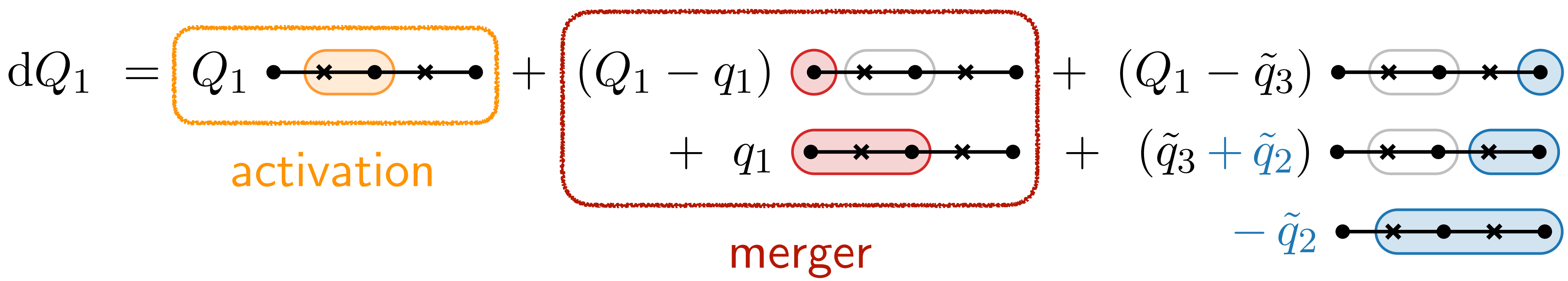

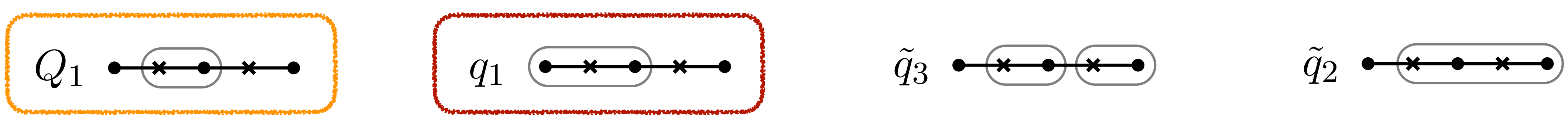

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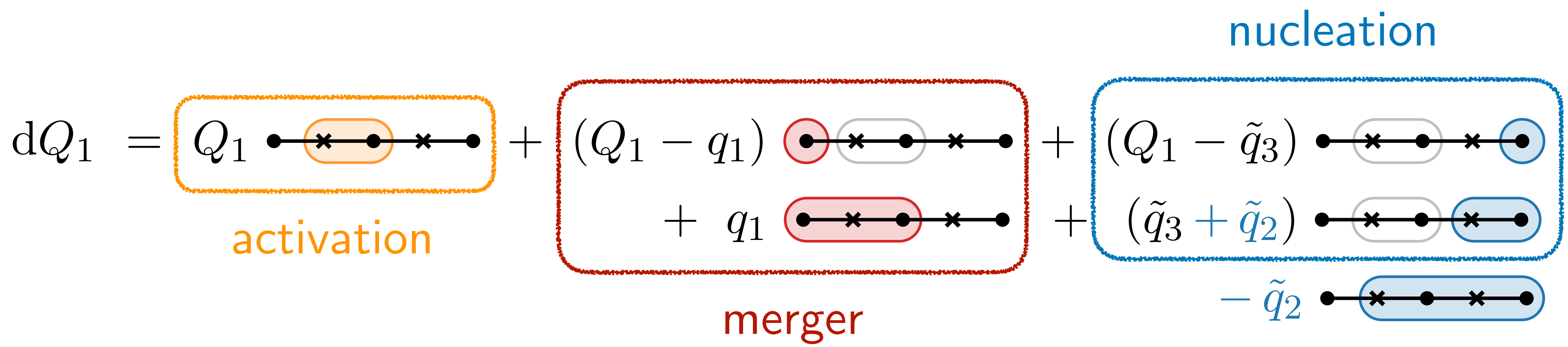
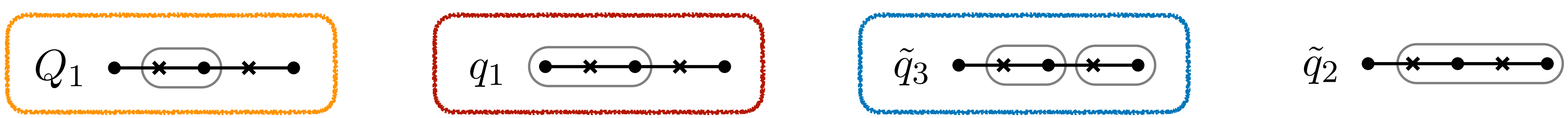
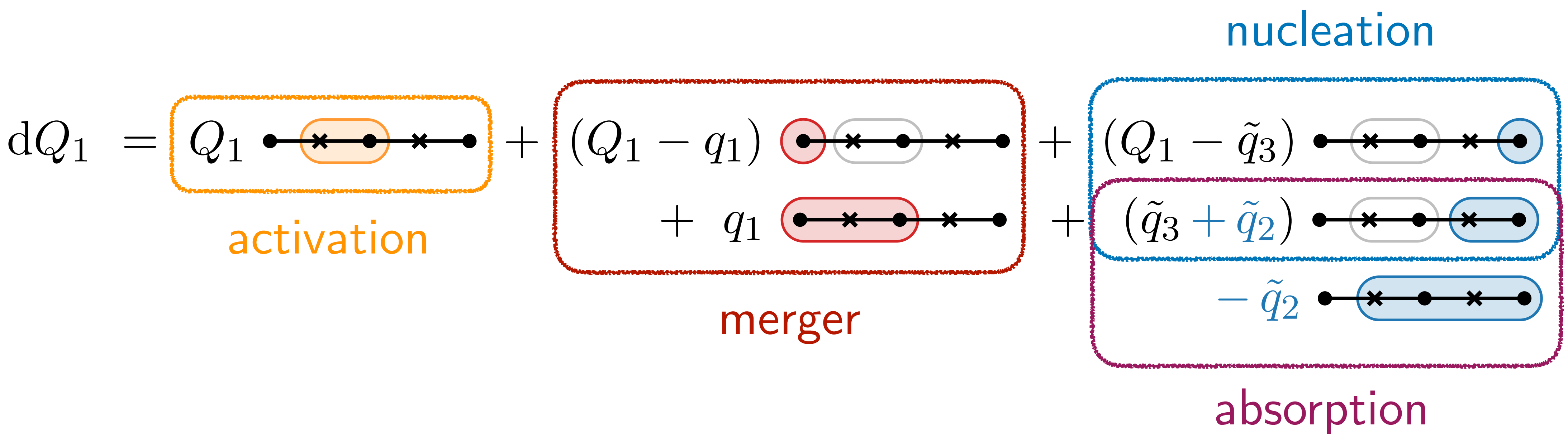


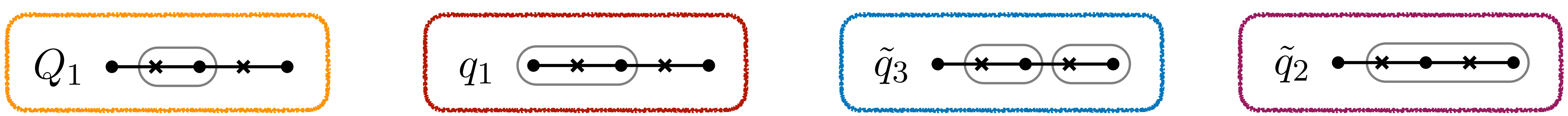


The equation can be predicted using simple graphical rules.

Graphical Rules

Taking the differential of one of the source functions gives





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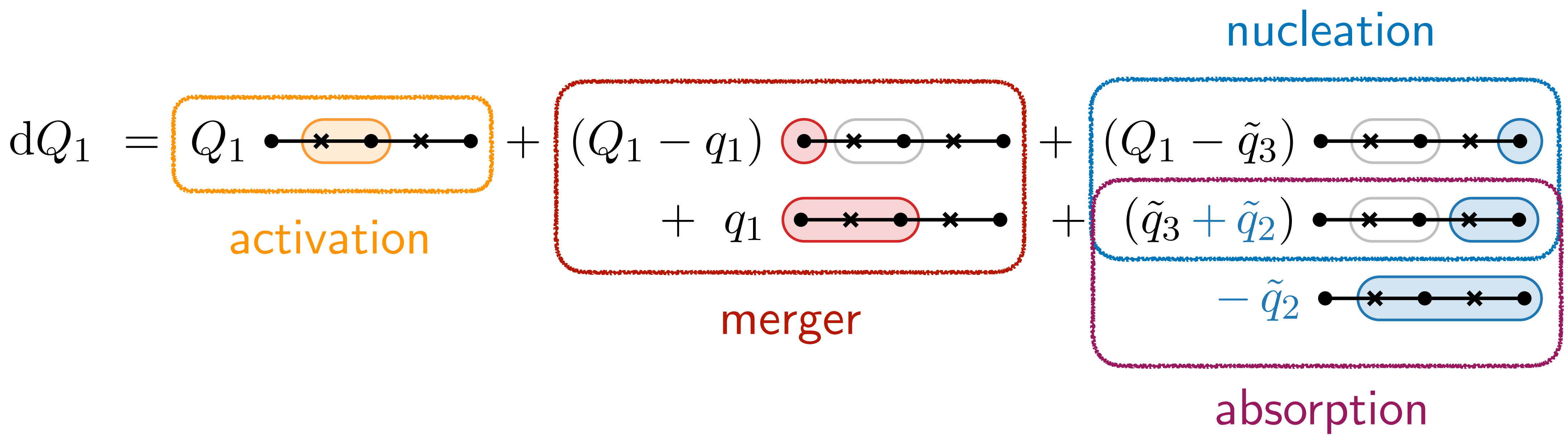
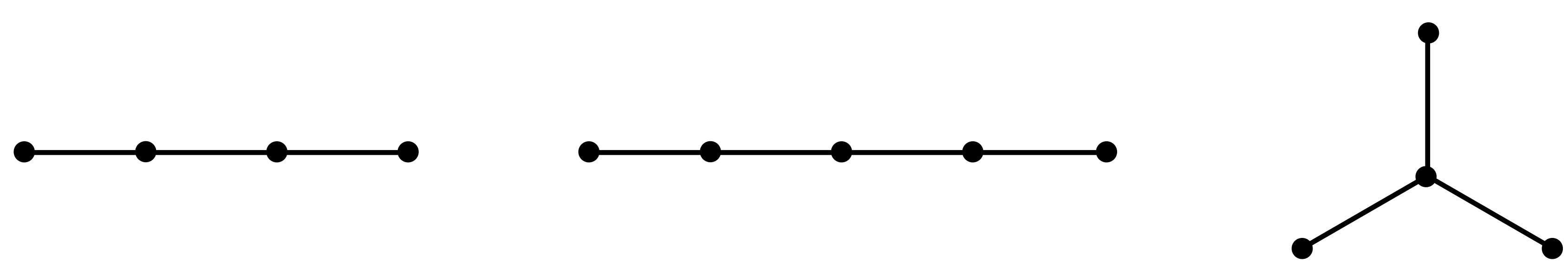
Graphical Rules

Taking the differential of one of the source functions gives

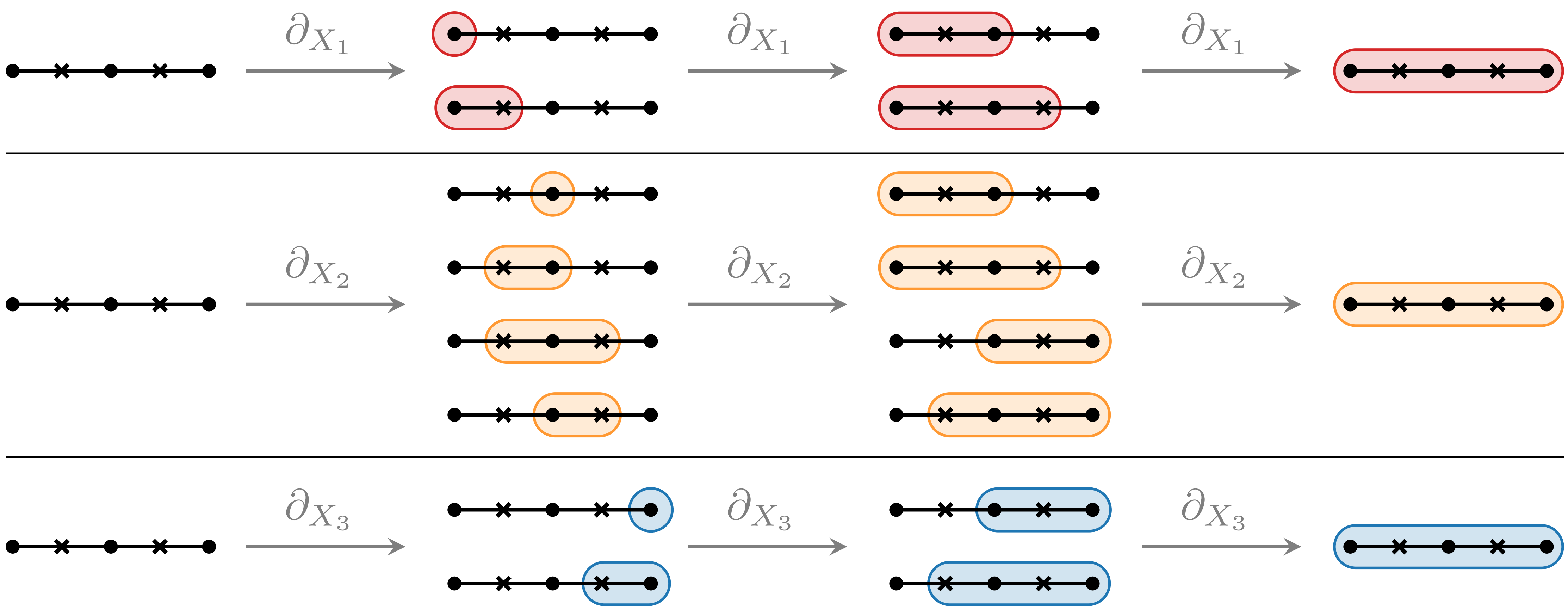


The graphical rules are local and can be used to predict the differential equations for arbitrary tree graphs with different topologies.



Time Evolution as Kinematic Flow

Taking successive partial derivatives, the tubings grow as

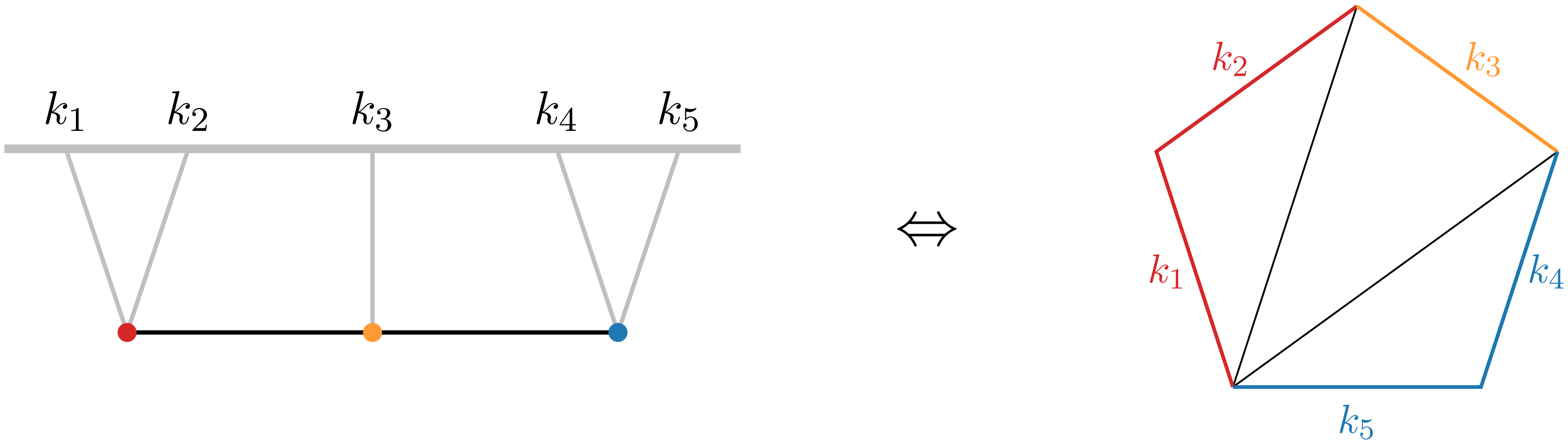


The growth ends, and the system closes, when all vertices are enclosed.

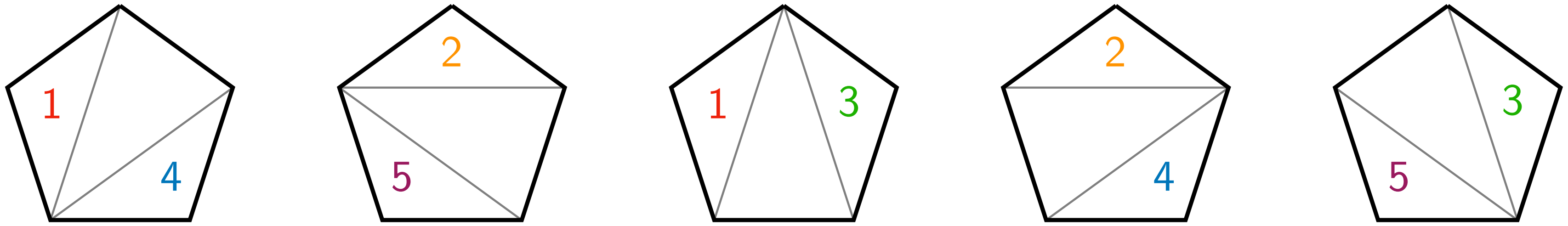
III. Beyond Single Graphs

Beyond Single Graphs

A single graph corresponds to a specific triangulation of a polygon.



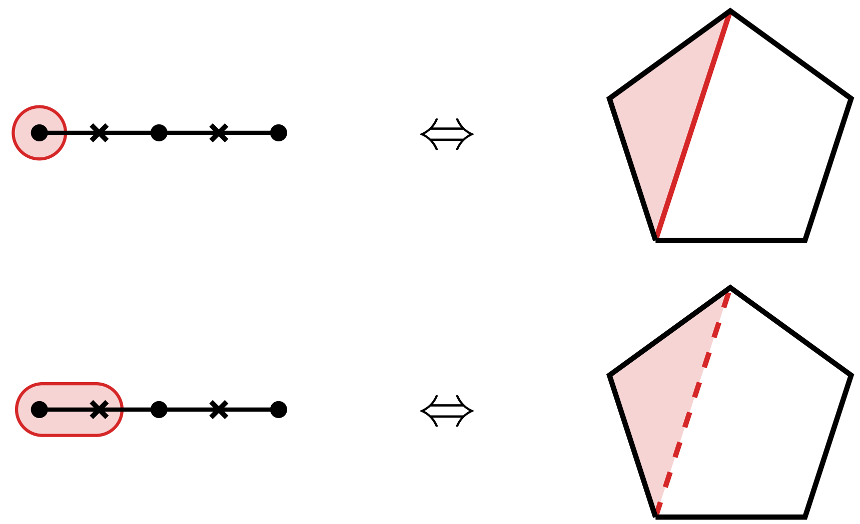
Distinct triangulations a pentagon correspond to different permutations.



I II III IV V

Kinematic Polygons

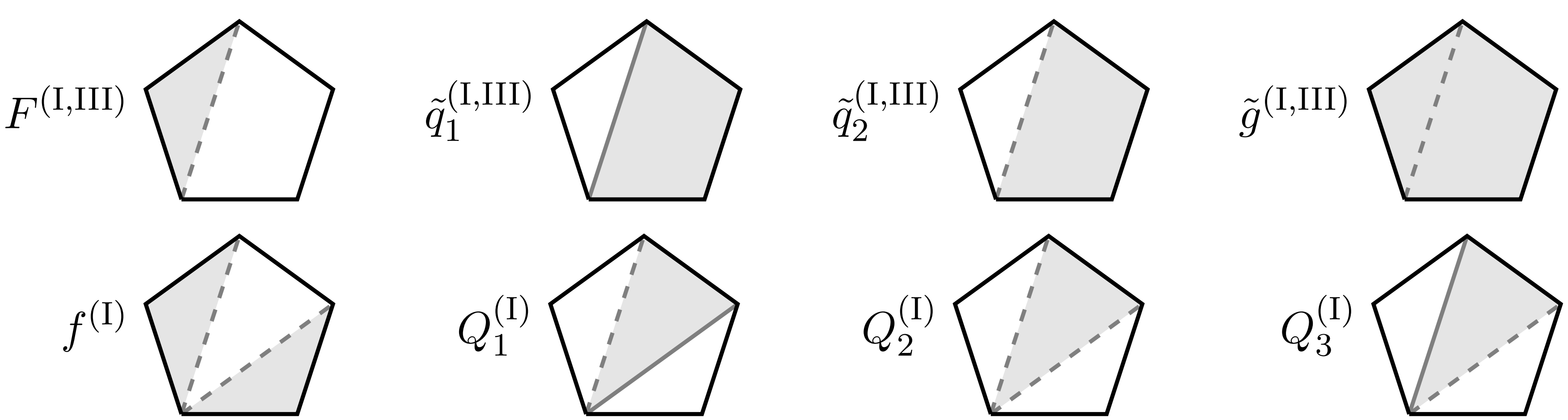
Letters are now represented by subpolygons with (dashed) internal edges.



$$\equiv \mathrm{d}\log(k_{12}+y_{12})$$

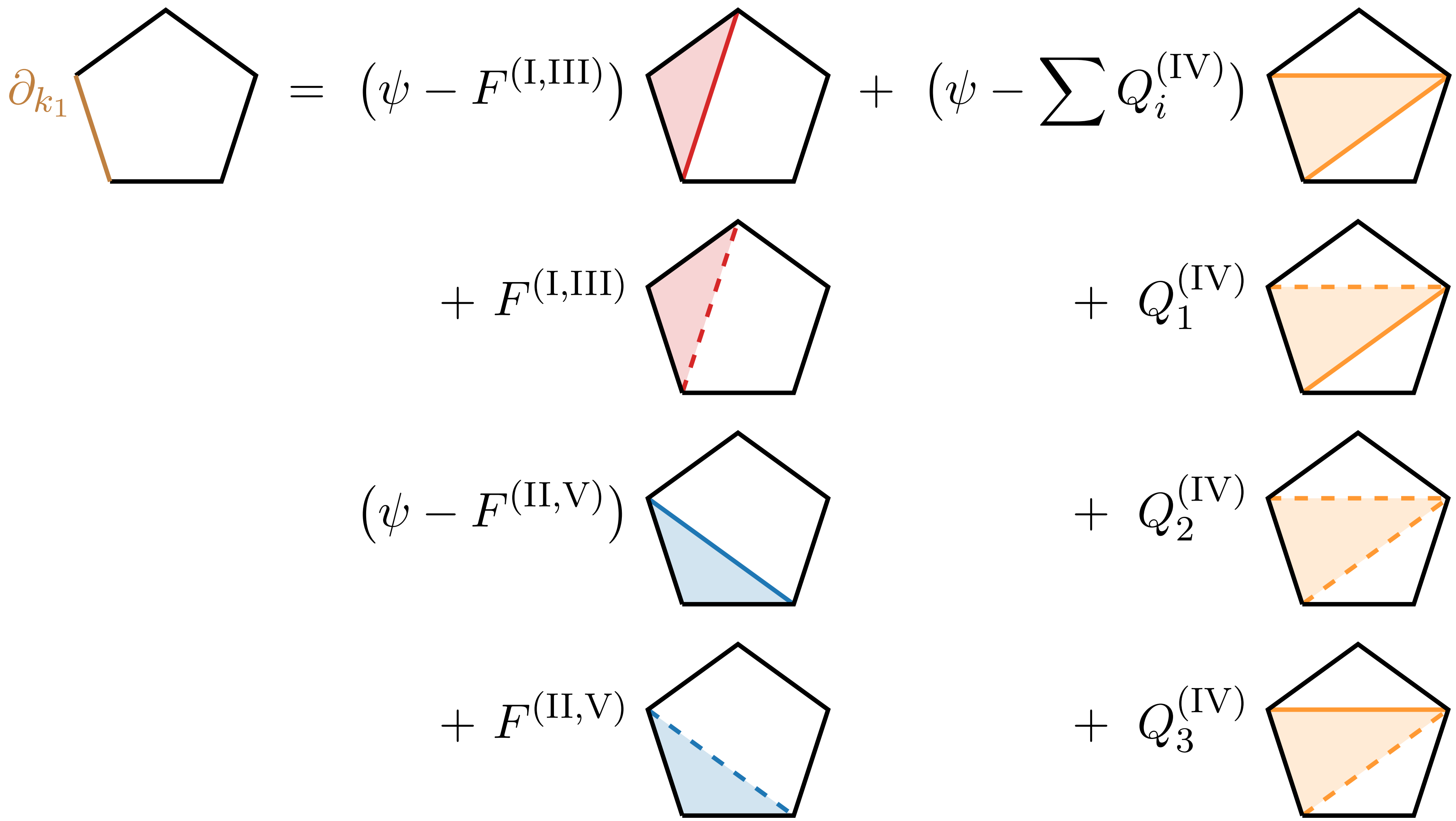
$$\equiv \mathrm{d}\log(k_{12}-y_{12})$$

Sources are given by subpolygons with at least one dashed internal edge.



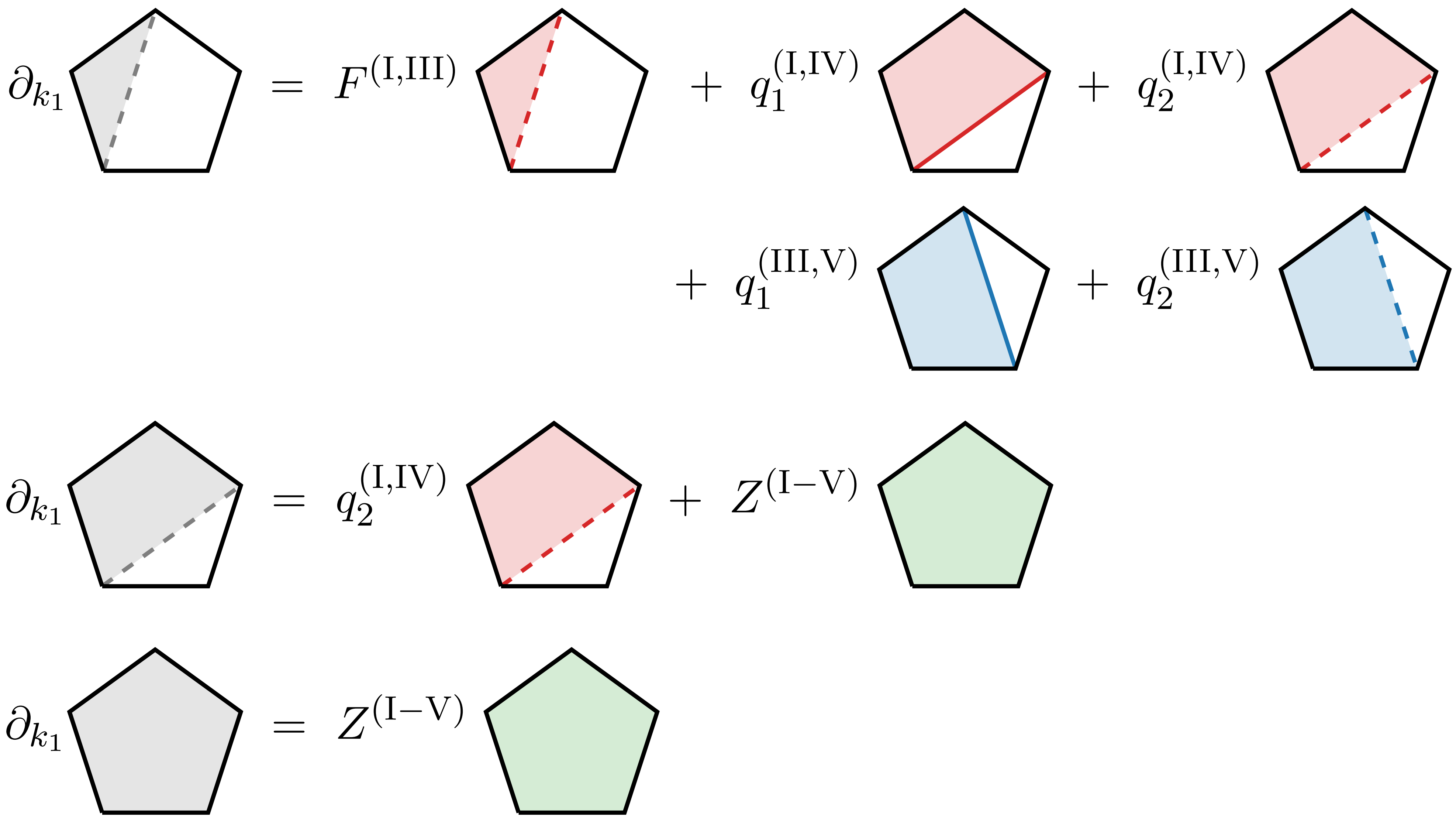
Growth of Kinematic Polygons

For example, taking the k_1 derivative of the 5-point function gives



Growth of Kinematic Polygons

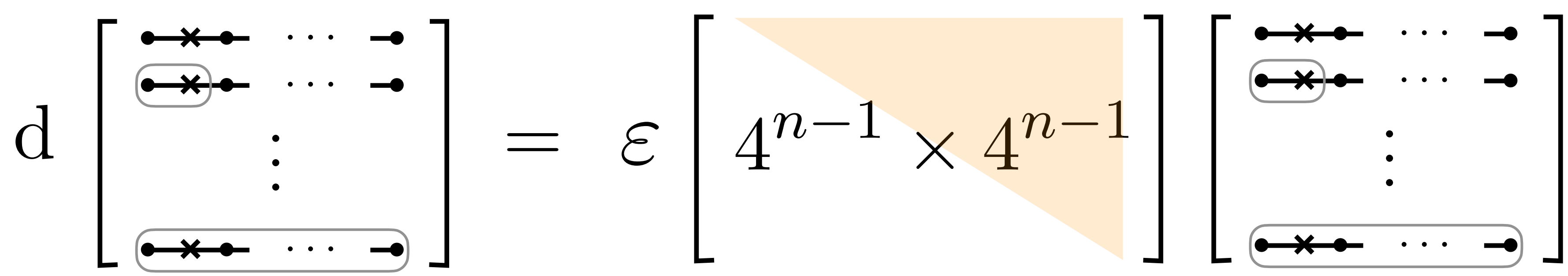
The system of equations closes when the subpolygon is fully grown.



Conclusions

Conclusions

We have developed a systematic way of deriving the differential equations for the FRW wavefunction of conformally-coupled scalars at tree level.



The differential equations can be predicted in terms of the dynamics of graphs.

The sum over graphs is captured by a kinematic polygon.

Extra Slides

Simplex Forms

The previous example suggests that the canonical forms of simplices are natural objects to consider.

$$[L_1 \cdots L_n] = \mathrm{d}\log L_1 \wedge \cdots \wedge \mathrm{d}\log L_n$$

The differential of a (projective) simplex form obeys a nice formula:

$$\mathrm{d}[L_1 \cdots L_n] = -\varepsilon \sum_a \mathrm{d}\log \det(\hat{L}_1 \cdots \hat{L}_n \hat{T}_a) \times \partial[L_1 \cdots L_n T_a]$$

letters

$$L_i = \hat{L}_i \cdot X$$

$$X = (x_1, \cdots, x_n, 1)$$

boundary

An Algorithm for All Trees

- Define all sources by substituting twisted planes in the wavefunction.

$$N_{\text{source}} = 4^{n-1}$$

- Take the differential of the sources using the formula

$$\mathrm{d}[L_1 \cdots L_n] = -\varepsilon \sum_a \mathrm{d}\log\det(\hat{L}_1 \cdots \hat{L}_n \hat{T}_a) \times \partial[L_1 \cdots L_n T_a]$$

- Express the result back in terms of the sources.