Massive inflation correlators at tree and loop levels

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Correlators in Cortona | September 22, 2023

w/ Zhehan Qin (秦哲涵)

1. Phase information in cosmological collider signals
JHEP 10 (2022) 192 [2205.01692]
2. Helical inflation correlators: partial Mellin-Barnes and bootstrap equations JHEP 04 (2023) 059 [2208.13790]
3. Closed-Form Formulae for Inflation Correlators
JHEP 07 (2023) 001 [2301.07047]
4. Inflation Correlators at the One-Loop Order: Nonanalyticity, Factorization, Cutting Rule, and OPE
JHEP 09 (2023) 116 [2304.13295]
5. Nonanalyticity and On-Shell Factorization of Inflation Correlators at All Loop Orders 2308.14802

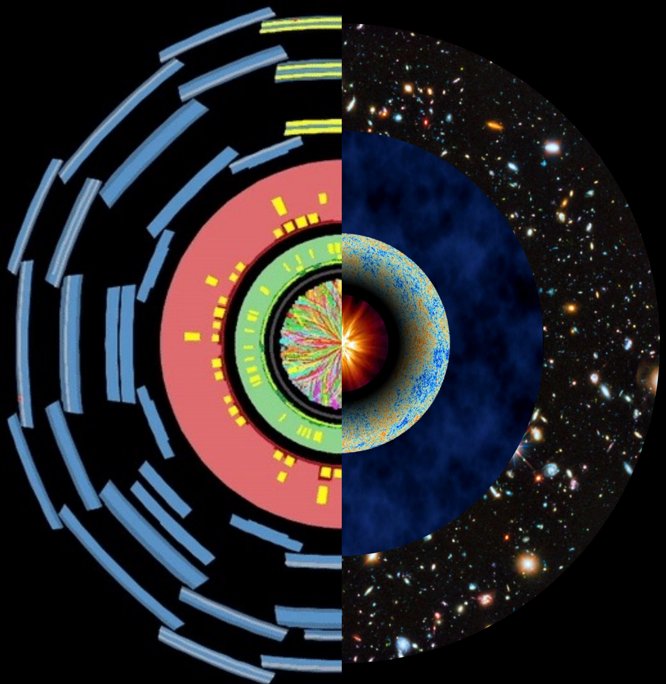
w/ Hongyu Zhang (张洪语)

6. Bootstrapping One-Loop Inflation Correlators with the Spectral Decomposition JHEP 04 (2023) 103 [2211.03810]

w/ Jiaju Zang (臧家驹)

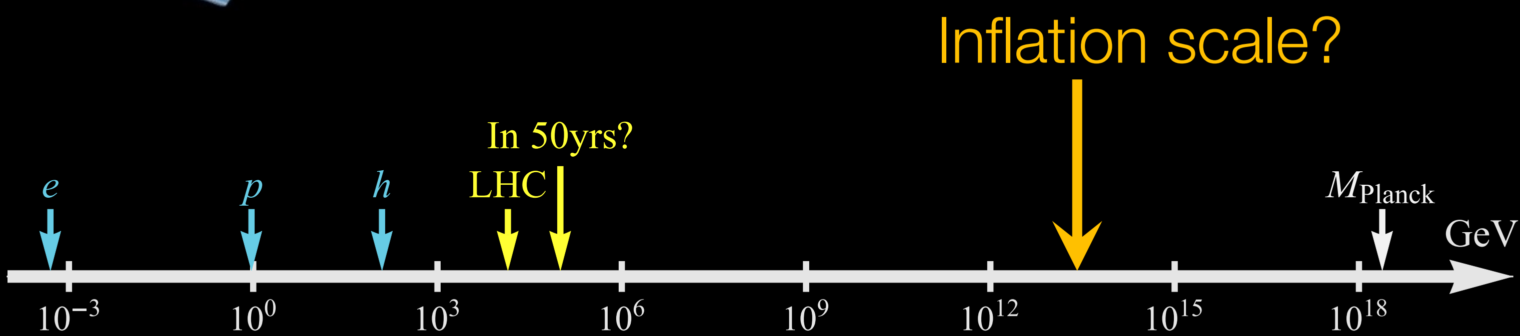
7. Inflation Correlators with Multiple Massive Exchanges
2309.10849

A cosmological collider program



The early universe was in a state of extremely high density / high temperature

Make use of the high energies of the early universe to study fundamental particle physics at high scale

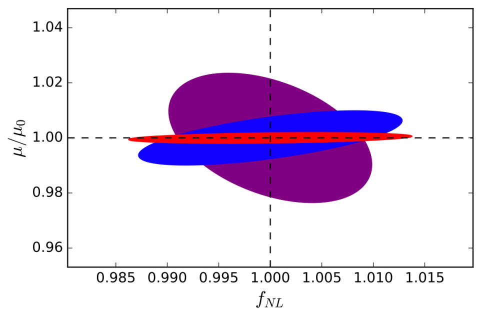


Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more

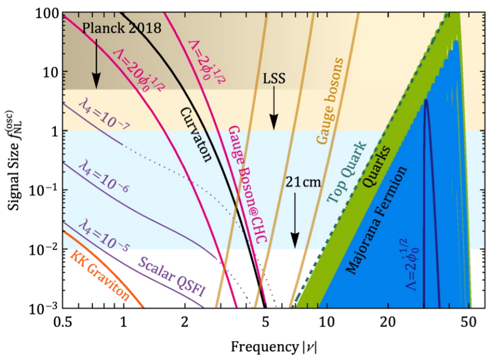
Progress in recent years

Very incomplete and a bit outdated, subject to my own limited knowledge

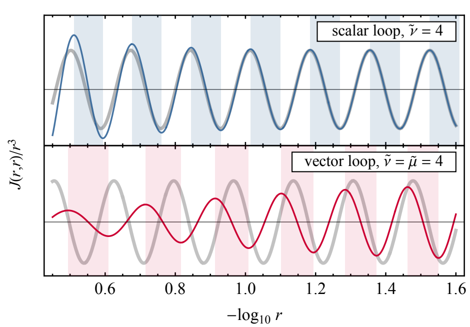
Meerburg et al. 1610.06559



Wang, ZX, 1910.12876



Qin, ZX, 2205.01692



Observation: methods,forecasts, results

21cm: Meerburg, Münchmeyer, Muñoz, Chen: 1610.06559
LSS: Dizgah, Lee, Muñoz, Dvorkin: 1801.07265
Galaxy imaging: Kogai, Akitsu, Schmidt, Urakawa, 2009.05517
P-violating trispectrum: Philcox: 2206.04227; Cabass, Ivanov, Philcox, 2210.16320

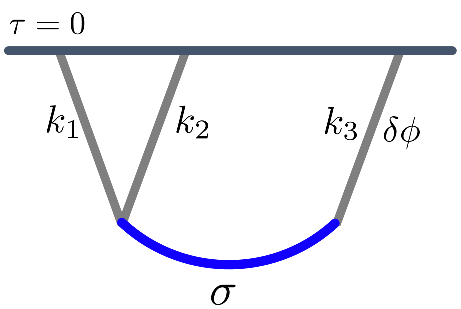
Particle physics and phenomenology

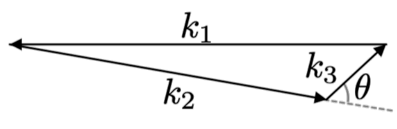
SM: Chen, Wang, ZX: 1604.07841, 1610.06597, 1612.08122; Hook, Huang, Racco, 1907.10624, 1908.00019; New scalars: An et al: 1706.09971; Wang, 1911.04459; Bodas, Kumar, Sundrum 2010.04727; Lu, Reece, ZX, 2108.11385; Higgs: Wu, 1812.10654; Kumar, Sundrum, 1711.03988, Lu, Wang, ZX, 1907.07390; CP violation: Liu et al, 1909.01819; Cui, ZX, 2112.10793; Spin: Lee, Baumann, Pimentel, 1607.03735; Chen, Wang, ZX, 1805.02656; Alexander et al, 1907.05829; Wang, ZX, 1910.12876, 2004.02887; Tong, ZX, 2203.06349, Niu et al., 2211.14324, 2211.14331; Beyond slow-roll: Tong, Wang, Zhou, 1801.05688; Kumar and Sundrum, 1908.11378; Chen, Ebadi, Kumar, 2205.01107; DM: Li et al: 1903.08842, 2002.01131; Lu, 2103.05958, and many more

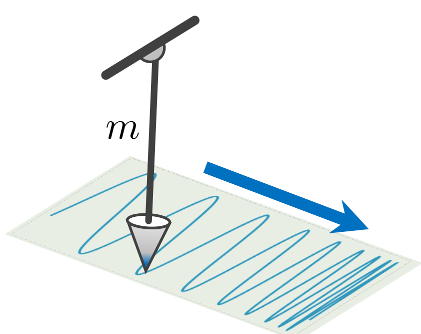
Cosmic correlators: structures and explicit results

Bootstrap: Arkani-Hamed, Baumann, Lee, Pimentel, et al: 1811.00024, 1910.14051, 2005.04234; Pajer et al: 2007.00027, 2010.12818, 10189; Pimentel, Wang, 2205.00013; Jazayeri, Renaux-Petel, 2205.10340, Wang, Pimentel, Achúcarro 2212.14035; Qin, ZX, 2301.07047 Mellin-Barnes: Sleight, Taronna, 1906.12302, 1907.01143, 2007.09993, 2109.02725; Qin, ZX, 2205.01692, 2208.13790; Spectral: ZX, Zhang, 2211.03810; Numerical: Wang, ZX, Zhong, 2109.14635 Analytical properties: Pajer et al: 2009.02898, 2103.09832, 2104.06587; Di Pietro, Gorbenko, S. Komatsu, 2108.01695; Tong, Wang, Zhu, 2112.03448, and many more

Cosmological collider signal







Where are the signals from?

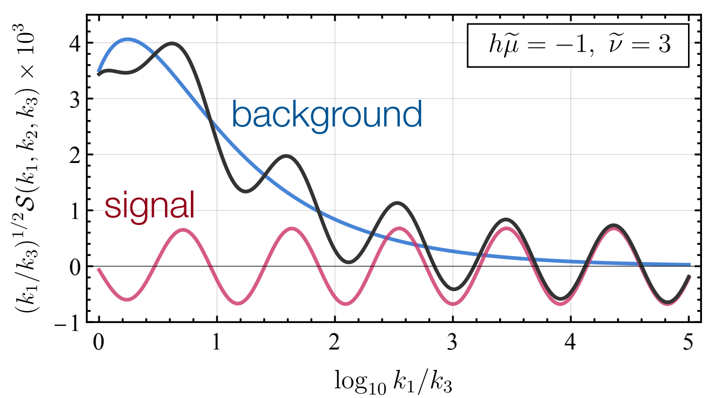
1. Particle production (~ Schwinger effect)
2. Superhorizon interference (shown in the squeezed limit)

signal size — scaling exponent — phase

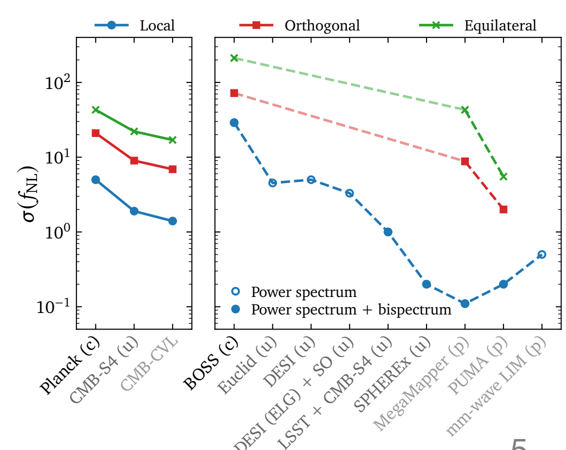
$$\mathcal{S}_{\rm signal}(k_1,k_2,k_3) \sim B\left(\frac{k_1}{k_3}\right)^L \sin\left[\omega \log\left(\frac{k_1}{k_3}\right)+\delta\right]$$

frequency

Zhehan Qin, ZZX, 2208.13790



Snowmass 2021: 2203.08128

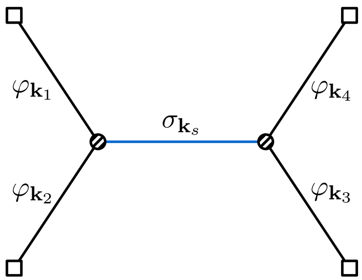


See also Arkani-Hamed, Maldacena, 1503.08043, Chen, Namjoo, Wang, 1509.03930, Tong, Wang, Zhu, 2112.03448

Theoretical challenges

Analyatical computation of inflation correlators with massive exchanges is challenging.

1. Space and time treated separately
2. Reduced symmetry (boost breaking)
3. Propagators in terms of special functions
4. Time-ordered integral



$$
\begin{aligned}
\mathcal{T}_\varphi &\equiv \langle \varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4}\rangle_s' \\
&= -\lambda^2 \sum_{\mathsf{a},\mathsf{b}=\pm} \mathsf{ab} \int_{-\infty}^0 \frac{\mathrm{d}\tau_1}{(-\tau_1)^2}\frac{\mathrm{d}\tau_2}{(-\tau_2)^2} G_\mathsf{a}'(k_1,\tau_1)G_\mathsf{a}'(k_2,\tau_1)G_\mathsf{b}'(k_3,\tau_2)G_\mathsf{b}'(k_4,\tau_2)D_{\mathsf{ab}}(k_s;\tau_1,\tau_2)
\end{aligned}
$$

$$
\begin{aligned}
D_{\pm\pm}(k;\tau_1,\tau_2) &= D_{\gtrless}(k;\tau_1,\tau_2)\theta(\tau_1-\tau_2) + D_{\lessgtr}(k;\tau_1,\tau_2)\theta(\tau_2-\tau_1), \\
D_{\pm\mp}(k;\tau_1,\tau_2) &= D_{\lessgtr}(k;\tau_1,\tau_2)
\end{aligned}
$$

$$
\begin{aligned}
D_>(k;\tau_1,\tau_2) &= \frac{\pi}{4}e^{-\pi\widetilde{\nu}}H^2(\tau_1\tau_2)^{3/2}\mathrm{H}_{\mathrm{i}\widetilde{\nu}}^{(1)}(-k\tau_1)\mathrm{H}_{-\mathrm{i}\widetilde{\nu}}^{(2)}(-k\tau_2) \\
G_\mathsf{a}(k,\tau) &= \frac{1}{2k^3}(1-\mathrm{i}\mathsf{a}k\tau)e^{\mathrm{i}\mathsf{a}k\tau}
\end{aligned}
$$

Why bother?

If difficult to compute, and if the final answer must be in monstrous special functions, why bother to do it?

Practically: Analytical expressions indispensable for understanding parameter dependences: particle model building

Much simpler and faster than brute-force numerical computations parameter scanning / template design / …

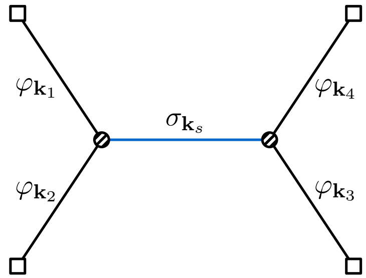
For a direct numerical approach, see L.-T. Wang, ZX, Y.-M. Zhong, 2109.14635

Theoretically: Analytical properties encode rich physics.

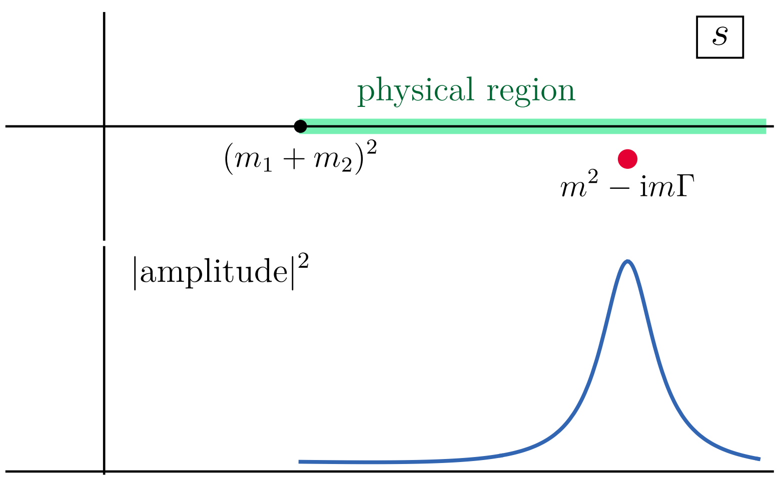
Inflation correlators are dS counterparts of scattering amplitudes in flat space and boundary correlators in AdS, but are least understood among the three.

We live in a universe presumably asymptotic to dS when $t \to \pm\infty$, and it's our responsibility to better understand amplitudes in dS

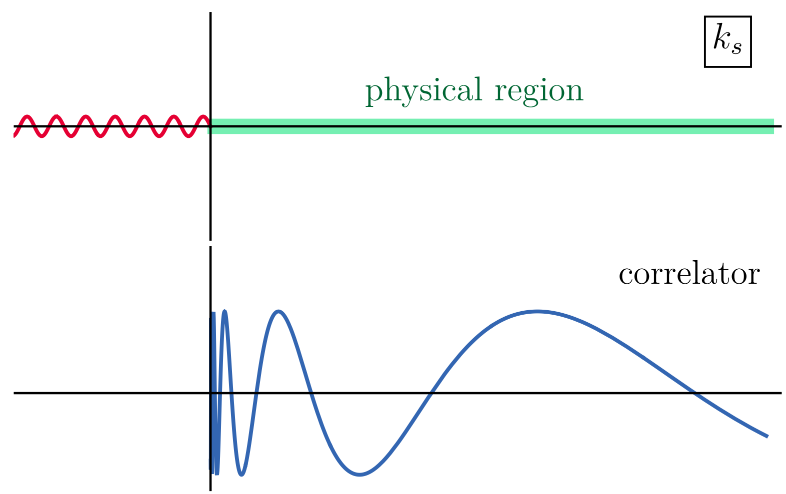
Analytical property ~ phenomenology



Flat-space scattering amplitude



dS boundary correlator



Rest of the talk:

Analytical computation of arbitrary tree graphs

Partial Mellin-Barnes representation | family-tree decomposition

Full analytical results at 1-loop level

spectral decomposition

Nonlocal signals at the 1-loop order

Factorization theorem | proof | cutting rule

Applications: nonlocal sigals in all 1-loop 4-point graphs

Nonlocal signals in multiloop graphs

Nonlocal cut | signal detection algorithm | factorization theorem

Partial Mellin-Barnes (PMB) representation

Zhehan Qin, ZX, 2205.01692, 2208.13790

Mellin transformation & Mellin-Barnes representation

$$F(s)=\int_0^\infty \mathrm{d}x\, x^{s-1}f(x) \qquad\qquad f(x)=\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\,x^{-s}F(s)$$

Expanding in eigenmode of dilatation operator
dS counterpart of Fourier transform in flat space

The MB rep of the Hankel functions

$$\mathrm{H}_\nu^{(j)}(az)=\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{(az/2)^{-2s}}{\pi}e^{(-1)^{j+1}(2s-\nu-1)\pi\mathrm{i}/2}\Gamma\left[s-\frac{\nu}{2},s+\frac{\nu}{2}\right]$$

The Euler Gamma function has poles at all nonpositive integers
Picking up residues at these poles amounts to an IR expansion of Hankel when the expansion is valid

$$\Gamma\left[z_1,\cdots,z_m\right]\equiv\Gamma(z_1)\cdots\Gamma(z_m)$$

$$\Gamma\begin{bmatrix}z_1,\cdots,z_m\\ w_1,\cdots,w_n\end{bmatrix}\equiv\frac{\Gamma(z_1)\cdots\Gamma(z_m)}{\Gamma(w_1)\cdots\Gamma(w_n)}$$

See also C. Sleight, 1906.12302,
C. Sleight, M. Taronna, 1907.01143, 2007.09993, 2109.02725

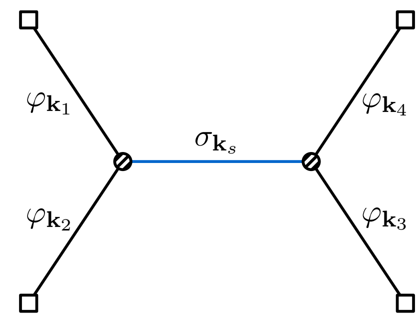
Partial Mellin-Barnes (PMB) representation

Zhehan Qin, ZX, 2205.01692，2208.13790

Why “partial”?

1. We eventually want expressions of equal-time correlators in “time-momentum” rep
=> better to keep time variables of external modes

2. External modes are typically massless scalar, tensor, conformal scalar, etc
=> Simple mode function; exp and powers

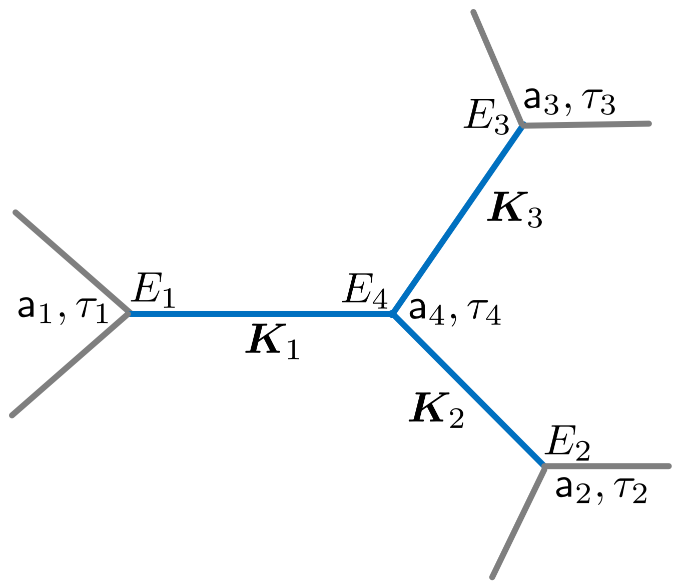


We only use MB for all bulk modes (Partial MB)

$$D^{(\widetilde{\nu})}_{-+}(k;\tau_1,\tau_2)=\frac{\pi}{4}e^{-\pi\widetilde{\nu}}(\tau_1\tau_2)^{3/2}\mathrm{H}^{(1)}_{\mathrm{i}\widetilde{\nu}}(-k\tau_1)\mathrm{H}^{(2)}_{-\mathrm{i}\widetilde{\nu}}(-k\tau_2)$$

$$D^{(\widetilde{\nu})}_{\pm\mp}(k;\tau_1,\tau_2)=\frac{1}{4\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{\mathrm{d}\bar{s}}{2\pi\mathrm{i}}\,e^{\mp\mathrm{i}\pi(s-\bar{s})}\left(\frac{k}{2}\right)^{-2(s+\bar{s})}(-\tau_1)^{-2s+3/2}(-\tau_2)^{-2\bar{s}+3/2}$$
$$\times\Gamma\left[s-\frac{\mathrm{i}\widetilde{\nu}}{2},s+\frac{\mathrm{i}\widetilde{\nu}}{2},\bar{s}-\frac{\mathrm{i}\widetilde{\nu}}{2},\bar{s}+\frac{\mathrm{i}\widetilde{\nu}}{2}\right]$$

Arbitrary tree graphs with PMB



External legs:
massless scalar, tensor, conformal scalar, etc

Internal legs:
massive scalars in the principal series

$$\widetilde{\nu}_i = \sqrt{m_i^2 - 9/4} > 0$$

Ignore tensor/flavor structures

total energy of external modes

3-momentum

$$\mathcal{I} = \sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V=\pm} \int \prod_{\ell=1}^{V} \left[\mathrm{d}\tau_\ell \, \mathrm{i}\mathsf{a}_\ell (-\tau_\ell)^{p_\ell} e^{\mathrm{i}\mathsf{a}_\ell E_\ell \tau_\ell} \right] \prod_{i=1}^{I} D_{\mathsf{a}_{i1}\mathsf{a}_{i2}}(K_i, \tau_{i1}, \tau_{i2})$$

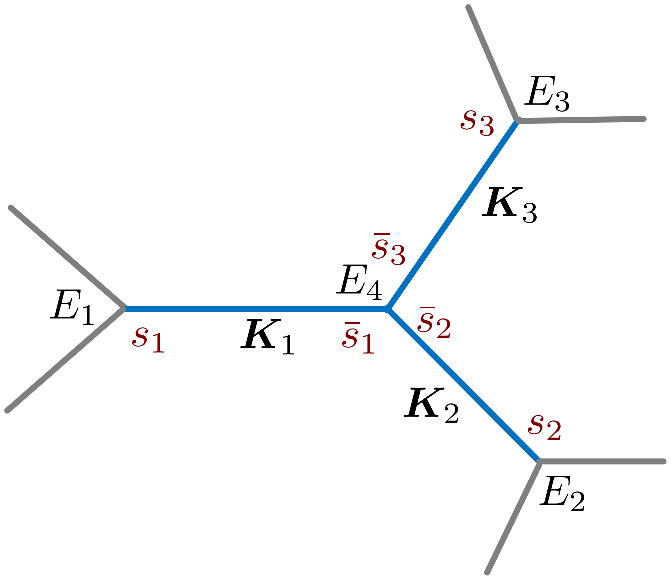
SK indices

Couplings

External modes

bulk lines

Arbitrary tree graphs with PMB



Taking MB for all bulk modes:

$$
\begin{aligned}
&D^{(\widetilde{\nu})}_{\pm\mp}(k;\tau_1,\tau_2)\\
&= \frac{1}{4\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{\mathrm{d}\bar{s}}{2\pi\mathrm{i}}\,e^{\mp\mathrm{i}\pi(s-\bar{s})}\left(\frac{k}{2}\right)^{-2(s+\bar{s})}\\
&\quad\times(-\tau_1)^{-2s+3/2}(-\tau_2)^{-2\bar{s}+3/2}\\
&\quad\times\Gamma\left[s-\frac{\mathrm{i}\widetilde{\nu}}{2},s+\frac{\mathrm{i}\widetilde{\nu}}{2},\bar{s}-\frac{\mathrm{i}\widetilde{\nu}}{2},\bar{s}+\frac{\mathrm{i}\widetilde{\nu}}{2}\right]
\end{aligned}
$$

Mellin integrals over 2*I* variables

$$
\begin{aligned}
\mathcal{I} = &\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\prod_{i=1}^{I}\left\{\frac{1}{4\pi}\frac{\mathrm{d}s_i}{2\pi\mathrm{i}}\frac{\mathrm{d}\bar{s}_i}{2\pi\mathrm{i}}\left(\frac{K_i}{2}\right)^{-2s_{i\bar{i}}}\Gamma\left[s_i-\frac{\mathrm{i}\widetilde{\nu}}{2},s_i+\frac{\mathrm{i}\widetilde{\nu}}{2},\bar{s}_i-\frac{\mathrm{i}\widetilde{\nu}}{2},\bar{s}_i+\frac{\mathrm{i}\widetilde{\nu}}{2}\right]\right\}\\
&\times\left\{\sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V=\pm}\int_{-\infty}^{0}\prod_{\ell=1}^{V}\left[\mathrm{d}\tau_\ell\,\mathrm{i}\mathsf{a}_\ell(-\tau_\ell)^{p_\ell-2\sum_\ell s}e^{\mathrm{i}\mathsf{a}_\ell E_\ell\tau_\ell}\right]\mathcal{N}_{\mathsf{a}_1\cdots\mathsf{a}_V}\Big(\tau_1,\cdots,\tau_V;\{s,\bar{s}\}\Big)\right\}
\end{aligned}
$$

time integrals

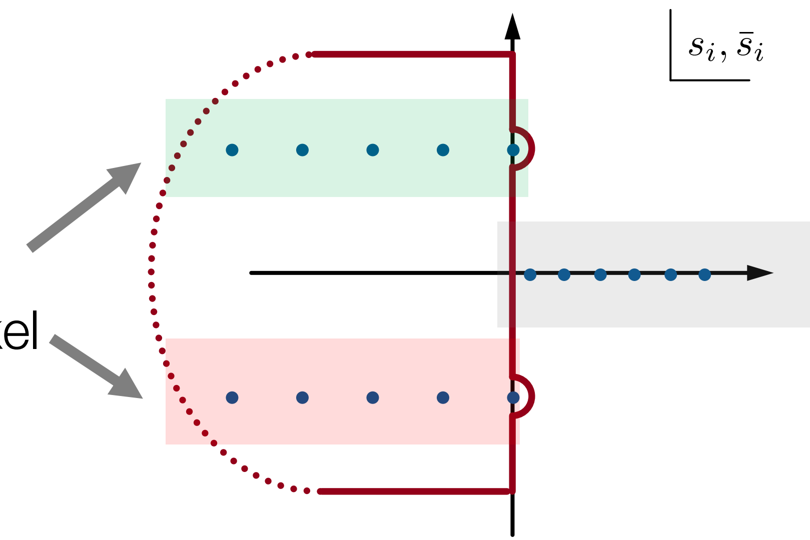
arbitrary time-orderings

MB integrals: Poles structures

$$\mathcal{I} = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \prod_{i=1}^{I} \left\{ \frac{1}{4\pi} \frac{\mathrm{d}s_i}{2\pi\mathrm{i}} \frac{\mathrm{d}\bar{s}_i}{2\pi\mathrm{i}} \left(\frac{K_i}{2}\right)^{-2s_{i\bar{i}}} \Gamma\left[s_i - \frac{\mathrm{i}\widetilde{\nu}}{2}, s_i + \frac{\mathrm{i}\widetilde{\nu}}{2}, \bar{s}_i - \frac{\mathrm{i}\widetilde{\nu}}{2}, \bar{s}_i + \frac{\mathrm{i}\widetilde{\nu}}{2} \right] \right\}$$

$$\times \left\{ \sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V=\pm} \int_{-\infty}^{0} \prod_{\ell=1}^{V} \left[\mathrm{d}\tau_\ell \, \mathrm{i}\mathsf{a}_\ell (-\tau_\ell)^{p_\ell - 2\sum_\ell s} e^{\mathrm{i}\mathsf{a}_\ell E_\ell \tau_\ell} \right] \mathcal{N}_{\mathsf{a}_1\cdots\mathsf{a}_V} \Big(\tau_1, \cdots, \tau_V; \{s, \bar{s}\} \Big) \right\}$$

All left poles are from MB reps of Hankel



Time integrals only generate right poles
[proved: 2304.13295]

For soft-bulk-line configurations: we pick up all left poles.

[All Mellin variables are balanced in the Euler Gamma factors]

Arbitrary nested time integrals: Family-tree decomposition

ZX, Jiaju Zang, 2309.10849

The most general nested time integral in PMB reads:

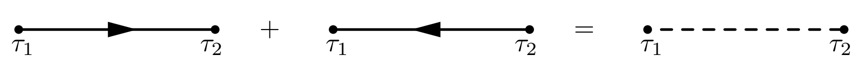
$$\mathbb{T}_{q_1\cdots q_V}(E_1,\cdots,E_V)=\int\prod_{\ell=1}^{V}\left[\mathrm{d}\tau_\ell(-\tau_\ell)^{q_\ell-1}e^{\mathrm{i}E_\ell\tau_\ell}\right]\prod_{i,j}\theta(\tau_i-\tau_j)$$

Answers are in hypergeometric series not yet named
The best we can do:
1) get the series expansion; 2) know how to do analytical continuation

=>We have to work with a special configuration at a time

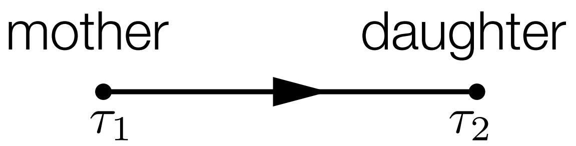
Family-tree decomposition: Pick up a maximal energy and switch some bulk lines so that: a) all graphs are partially ordered; b) maximal energy sits at the earliest site.

$$\theta(\tau_1-\tau_2)+\theta(\tau_2-\tau_1)=1$$



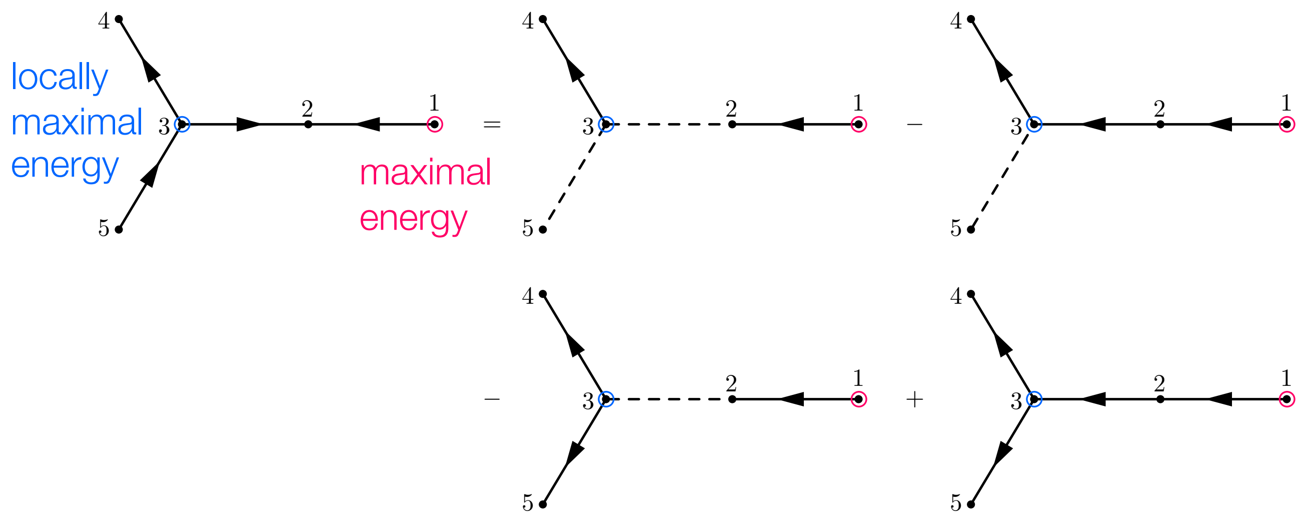
Partial-ordered graph:

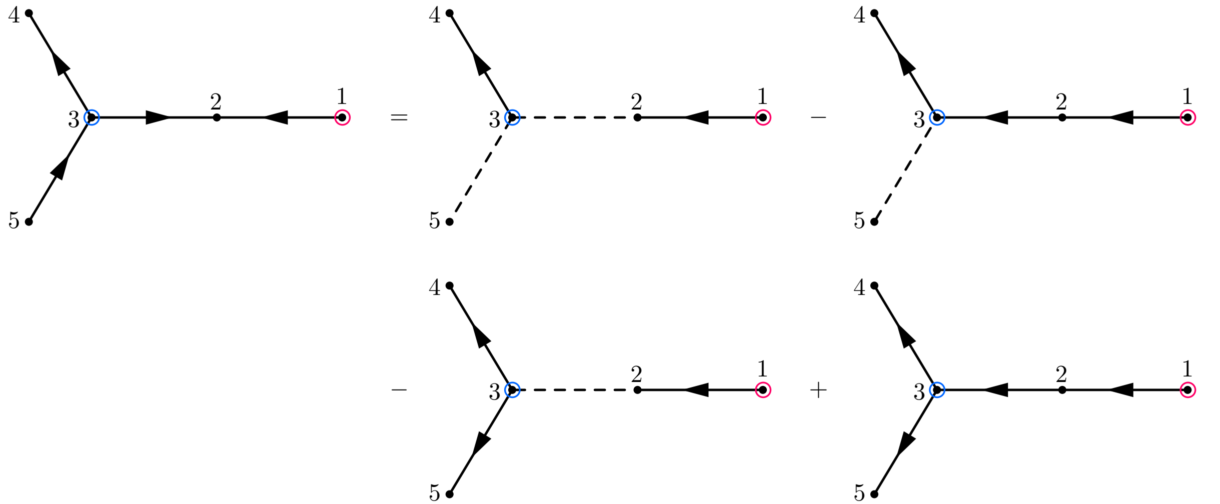
1) Every site has only one mother, except the earliest-time site, who is motherless;
2) A site can have many daughters.



A partially ordered graph is naturally a (maternal) family tree

An example: $$\mathbb{T}_{q_1\cdots q_5}(E_1,\cdots,E_5) \equiv \int_{-\infty}^0 \prod_{\ell=1}^5 \left[\mathrm{d}\tau_\ell(-\tau_\ell)^{q_\ell-1}e^{\mathrm{i}E_\ell\tau_\ell}\right]$$
$$\times\,\theta(\tau_2-\tau_1)\theta(\tau_2-\tau_3)\theta(\tau_4-\tau_3)\theta(\tau_3-\tau_5)$$





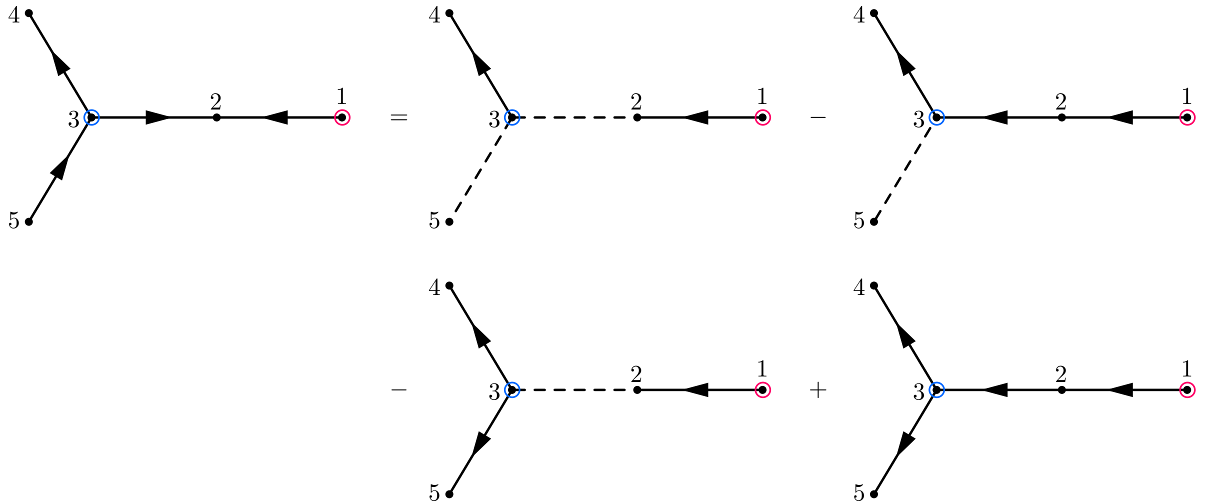
$$\mathbb{T}_{q_1\cdots q_5}(\widehat{E}_1,\cdots,E_5) = \mathcal{C}_{q_1q_2}(\widehat{E}_1,E_2)\mathcal{C}_{q_3q_4}(\widehat{E}_3,E_4)\mathcal{C}_{q_5}(E_5) - \mathcal{C}_{q_1q_2q_3q_4}(\widehat{E}_1,E_2,E_3,E_4)\mathcal{C}_{q_5}(E_5) - \mathcal{C}_{q_1q_2}(\widehat{E}_1,E_2)\mathcal{C}_{q_4q_3q_5}(E_4,\widehat{E}_3,E_5) + \mathcal{C}^{(\text{iso})}_{q_1q_2q_3q_4q_5}(\widehat{E}_1,E_2,E_3,E_4,E_5)$$

For each family-tree integral, the answer is a one-line formula:

$$\mathcal{C}_{q_1\cdots q_N}(\widehat{E}_1,E_2,\cdots,E_N) = \frac{1}{(\mathrm{i}E_1)^{q_{1\cdots N}}}\sum_{n_2,\cdots,n_N=0}^{\infty}\Gamma(q_{1\cdots N}+n_{2\cdots N})\prod_{j=2}^{N}\frac{(-E_j/E_1)^{n_j}}{(\widetilde{q}_j+\widetilde{n}_j)n_j!}$$

$\widetilde{q}_j$: the sum of all q's of Site j and her descendents.

[See 2309.10849 for the proof]



$$\mathbb{T}_{q_1\cdots q_5}(\widehat{E}_1,\cdots,E_5) = \mathcal{C}_{q_1q_2}(\widehat{E}_1,E_2)\mathcal{C}_{q_3q_4}(\widehat{E}_3,E_4)\mathcal{C}_{q_5}(E_5) - \mathcal{C}_{q_1q_2q_3q_4}(\widehat{E}_1,E_2,E_3,E_4)\mathcal{C}_{q_5}(E_5)$$
$$- \mathcal{C}_{q_1q_2}(\widehat{E}_1,E_2)\mathcal{C}_{q_4q_3q_5}(E_4,\widehat{E}_3,E_5) + \mathcal{C}^{(\mathrm{iso})}_{q_1q_2q_3q_4q_5}(\widehat{E}_1,E_2,E_3,E_4,E_5)$$

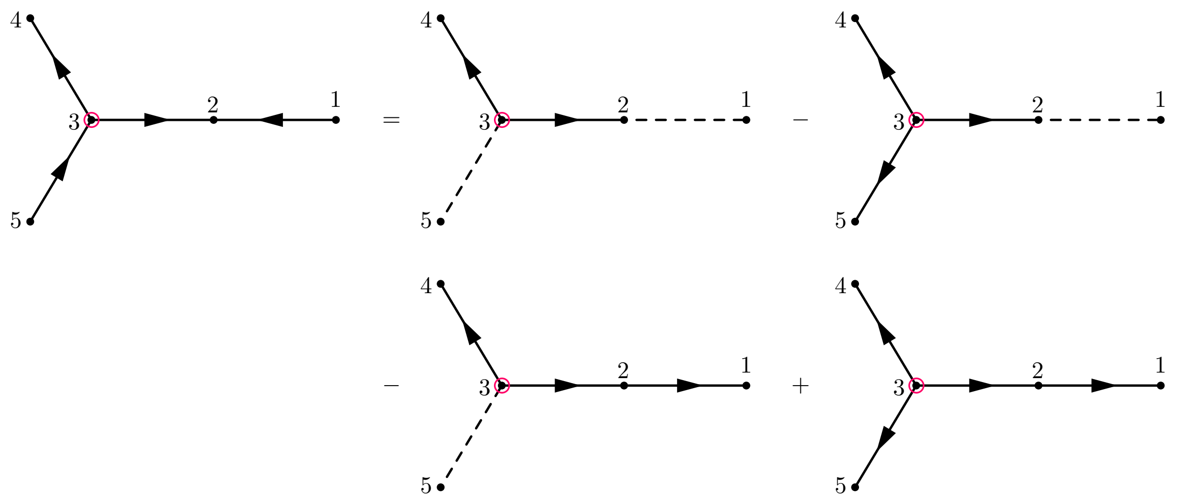
For our five-site example, we have, say:

$$\widetilde{\mathcal{C}}_{q_1q_2q_3q_4}(E_1,E_2,\widehat{E}_3,E_4) = \sum_{n_1,n_2,n_4=0}^{\infty} \frac{(-1)^{n_{124}}\Gamma(q_{1234}+n_{124})}{(q_1+n_1)(q_{12}+n_{12})(q_4+n_4)} \frac{\varrho_{13}^{n_1}}{n_1!}\frac{\varrho_{23}^{n_2}}{n_2!}\frac{\varrho_{43}^{n_4}}{n_4!},$$

$$\widetilde{\mathcal{C}}^{(\mathrm{iso})}_{q_1q_2q_3q_4q_4}(E_1,E_2,\widehat{E}_3,E_4,E_5)$$
$$= \sum_{n_1,n_2,n_4,n_5=0}^{\infty} \frac{(-1)^{n_{1245}}\Gamma(q_{12345}+n_{1245})}{(q_1+n_1)(q_{12}+n_{12})(q_4+n_4)(q_5+n_5)} \frac{\varrho_{13}^{n_1}}{n_1!}\frac{\varrho_{23}^{n_2}}{n_2!}\frac{\varrho_{43}^{n_4}}{n_4!}\frac{\varrho_{53}^{n_5}}{n_5!}$$

Analytical continuation

Suppose we choose E_3 as maximal (instead of E_1):



$$\mathbb{T}_{q_1\cdots q_5}(\widehat{E}_1,\cdots,E_5) = \mathcal{C}_{q_1}(E_1)\mathcal{C}_{q_2q_3q_4}(E_2,\widehat{E}_3,E_4)\mathcal{C}_{q_5}(E_5) - \mathcal{C}_{q_1}(E_1)\mathcal{C}^{(\mathrm{iso})}_{q_2q_4q_5q_3}(E_2,E_4,E_5,\widehat{E}_3)$$
$$- \mathcal{C}_{q_1q_2q_3q_4}(E_1,E_2,\widehat{E}_3,E_4)\mathcal{C}_{q_5}(E_5) + \mathcal{C}^{(\mathrm{iso})}_{q_1q_2q_3q_4q_5}(E_1,E_2,\widehat{E}_3,E_4,E_5)$$

The result expanded in terms of $1/E_3$

Also possible to expand in terms of $1/E_{\mathrm{Total}}$

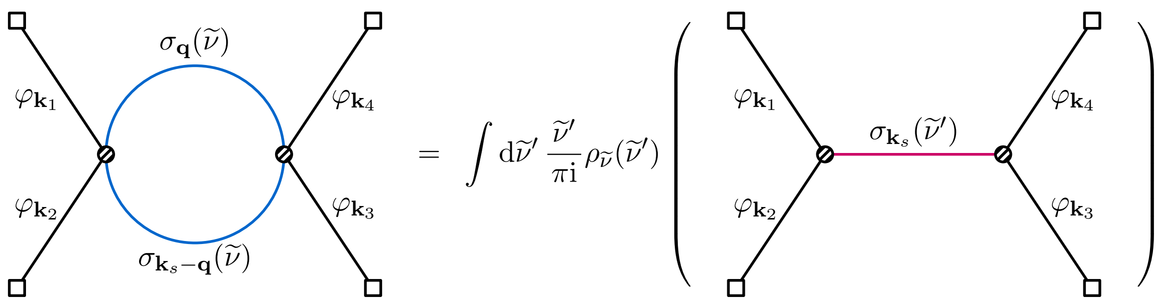
The computation of massive tree graphs largely trivialized

[See 2309.10849 for applications]

Difficulties in loop diagrams

Complete analytical calculation is challenging even with PMB rep.

The first and hitherto only known complete result for massive 1-loop:
(w/ Hongyu Zhang, 2211.03810)



$$\rho^{\mathrm{dS}}_{\widetilde{\nu}}(\widetilde{\nu}') = \frac{1}{(4\pi)^{(d+1)/2}}\frac{\cos[\pi(\frac{d}{2}-\mathrm{i}\widetilde{\nu})]}{\sin(-\pi\mathrm{i}\widetilde{\nu})}\Gamma\left[\begin{matrix}\frac{3-d}{2}, \frac{d}{2}-\mathrm{i}\widetilde{\nu}\\ \frac{2-d}{2}-\mathrm{i}\widetilde{\nu}\end{matrix}\right]$$
$$\times\, {}_7\mathcal{F}_6\left[\begin{matrix}\frac{2-d}{2}+\mathrm{i}\widetilde{\nu}'-\mathrm{i}\widetilde{\nu}, \frac{3-d/2+\mathrm{i}\widetilde{\nu}'-\mathrm{i}\widetilde{\nu}}{2}, \frac{2-d}{2}, \frac{2-d}{2}-\mathrm{i}\widetilde{\nu}, \frac{2-d}{2}+\mathrm{i}\widetilde{\nu}', \frac{\mathrm{i}\widetilde{\nu}'-2\mathrm{i}\widetilde{\nu}+d/2}{2}, \frac{\mathrm{i}\widetilde{\nu}'+d/2}{2}\\ \frac{1-d/2+\mathrm{i}\widetilde{\nu}'-\mathrm{i}\widetilde{\nu}}{2}, 1+\mathrm{i}\widetilde{\nu}'-\mathrm{i}\widetilde{\nu}, 1+\mathrm{i}\widetilde{\nu}', 1-\mathrm{i}\widetilde{\nu}, \frac{4+\mathrm{i}\widetilde{\nu}'-3d/2}{2}, \frac{4+\mathrm{i}\widetilde{\nu}'-2\mathrm{i}\widetilde{\nu}-3d/2}{2}\end{matrix}\middle| 1\right]$$
$$+\,(\widetilde{\nu}\to-\widetilde{\nu}).$$

Marolf, Morrison, 1006.0035

1-loop bubble from spectral decomposition

ZX, Hongyu Zhang, 2211.03810

An example of 4-point function with 4 external massless scalars

$$\mathcal{L}_{\varphi,\widetilde{\nu}} = \frac{1}{16k_1k_2k_3k_4(k_{12}k_{34})^{5/2}}\Big[\widehat{\mathcal{J}}_{\rm NS}(r_1,r_2) + \widehat{\mathcal{J}}_{\rm LS}(r_1,r_2) + \widehat{\mathcal{J}}_{\rm BG}(r_1,r_2)\Big].$$

$$\widehat{\mathcal{J}}_{\rm NS} = \frac{2(r_1r_2)^{3/2+2\mathrm{i}\widetilde{\nu}}}{\pi^2\cos(2\pi\mathrm{i}\widetilde{\nu})}\sum_{n=0}^{\infty}\frac{(1+n)_{\frac12}\big[(1+\mathrm{i}\widetilde{\nu}+n)_{\frac12}\big]^2(1+2\mathrm{i}\widetilde{\nu}+n)_{\frac12}}{(1+2\mathrm{i}\widetilde{\nu}+2n)_2}(\tfrac32+2\mathrm{i}\widetilde{\nu}+2n)$$
$$\times\,{}_2\mathcal{F}_1\left[\begin{matrix}2+\mathrm{i}\widetilde{\nu}+n,\frac52+\mathrm{i}\widetilde{\nu}+n\\ \frac52+2\mathrm{i}\widetilde{\nu}+2n\end{matrix}\middle|r_1^2\right]{}_2\mathcal{F}_1\left[\begin{matrix}2+\mathrm{i}\widetilde{\nu}+n,\frac52+\mathrm{i}\widetilde{\nu}+n\\ \frac52+2\mathrm{i}\widetilde{\nu}+2n\end{matrix}\middle|r_2^2\right](r_1r_2)^{2n}+\text{c.c.}.$$

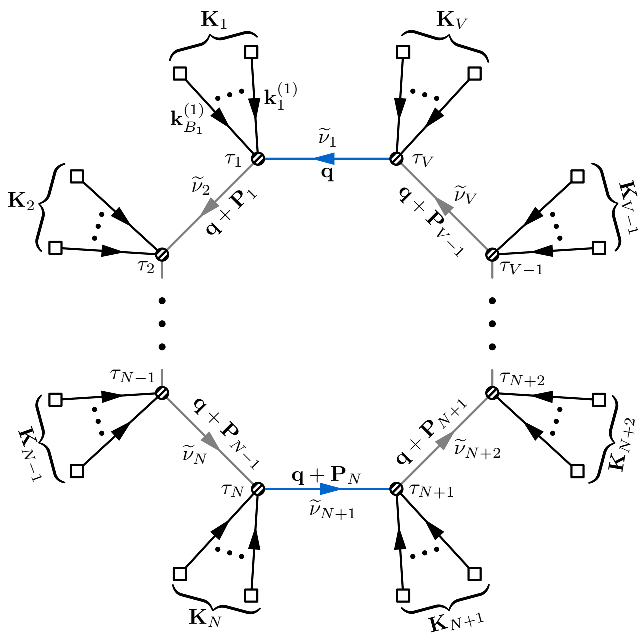
$$\widehat{\mathcal{J}}_{\rm LS} = -\frac{2(r_1/r_2)^{3/2+2\mathrm{i}\widetilde{\nu}}}{\pi^2\cos(2\pi\mathrm{i}\widetilde{\nu})}\sum_{n=0}^{\infty}\frac{(1+n)_{\frac12}\big[(1+\mathrm{i}\widetilde{\nu}+n)_{\frac12}\big]^2(1+2\mathrm{i}\widetilde{\nu}+n)_{\frac12}}{(1+2\mathrm{i}\widetilde{\nu}+2n)_2}(\tfrac32+2\mathrm{i}\widetilde{\nu}+2n)$$
$$\times\,{}_2\mathcal{F}_1\left[\begin{matrix}2+\mathrm{i}\widetilde{\nu}+n,\frac52+\mathrm{i}\widetilde{\nu}+n\\ \frac52+2\mathrm{i}\widetilde{\nu}+2n\end{matrix}\middle|r_1^2\right]{}_2\mathcal{F}_1\left[\begin{matrix}\frac12-\mathrm{i}\widetilde{\nu}-n,1-\mathrm{i}\widetilde{\nu}-n\\ -\frac12-2\mathrm{i}\widetilde{\nu}-2n\end{matrix}\middle|r_2^2\right]\left(\frac{r_1}{r_2}\right)^{2n}+\text{c.c.}.$$

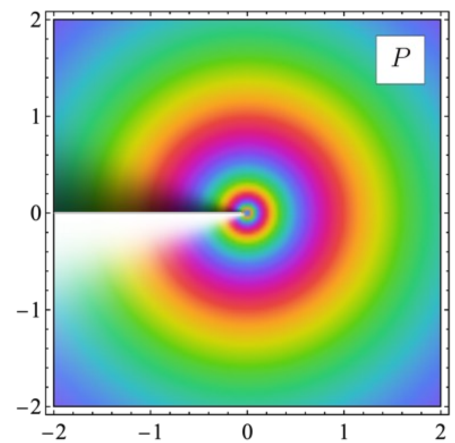
$$\widehat{\mathcal{J}}_{\rm BG} = \sum_{\ell,m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^{\ell+n+1}(\ell+1)_{2m+4}(\frac52+\ell+2n)}{2^{2m}n!(m-n)!(\frac52+\ell+n)_{m+1}}$$
$$\times\left[\widehat{\rho}^{\rm dS}_{\widetilde{\nu}}(-\tfrac{\mathrm{i}5}{2}-\mathrm{i}\ell-2\mathrm{i}n)-\frac{1}{(4\pi)^2}\log\mu_R^2\right]r_1^{2m}\left(\frac{r_1}{r_2}\right)^{5/2+\ell}.$$

Most general 1PI 1-loop correlators

Complete result for arbitrary 1-loop certainly beyond our reach for now

Question: Can we at least identify all nonlocal signals in a given graph?
Answer: Yes, to all loop orders.



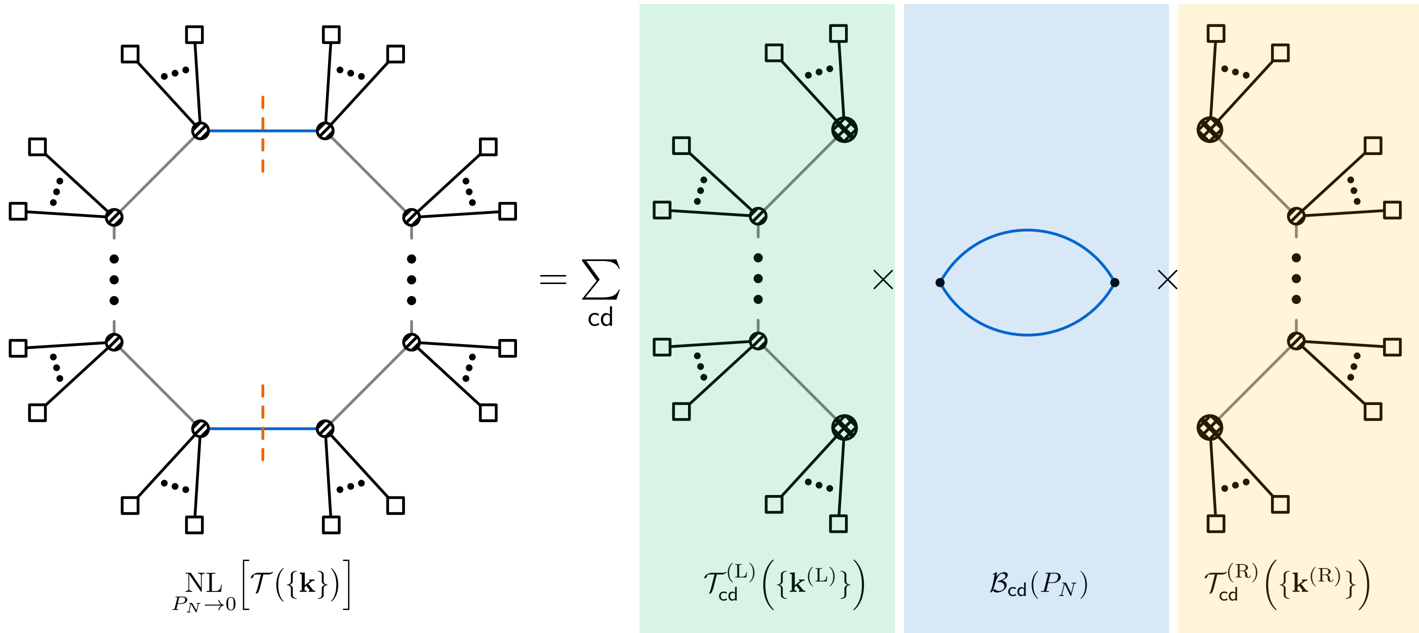


$P^{i\omega}$

nonlocal signal ~ branch point

Theorem: factorization of LO nonlocal signal

Zehan Qin, ZX, 2304.13295



$$\lim_{P_N\to 0}\mathcal{T}(\{\mathbf{k}\}) = \sum_{\mathsf{c},\mathsf{d}=\pm}\mathcal{T}^{(\mathrm{L})}_{\mathsf{cd}}\left(\{\mathbf{k}^{(\mathrm{L})}\}\right)\mathcal{T}^{(\mathrm{R})}_{\mathsf{cd}}\left(\{\mathbf{k}^{(\mathrm{R})}\}\right)\mathcal{B}_{\mathsf{cd}}(P_N) + \text{analytic}$$

Left subgraph

Bubble signal

Right subgraph

Left subgraph

$$\mathcal{T}_{\mathsf{cd}}^{(\mathrm{L})}\left(\{\mathbf{k}^{(\mathrm{L})}\}\right)=\sum_{\mathsf{a}_1,\cdots,\mathsf{a}_N=\pm}\int_{-\infty}^{0}\prod_{\ell=1}^{N}\left[\mathrm{d}\tau_\ell\,(\mathrm{i}\mathsf{a}_\ell)(-\tau_\ell)^{p_\ell}\prod_{i=1}^{B_\ell}C_{\mathsf{a}_\ell}\left(k_i^{(\ell)};\tau_\ell\right)\right]$$

$$\times\prod_{n=1}^{N-1}\left[D_{\mathsf{a}_n\mathsf{a}_{n+1}}^{(\widetilde{\nu}_{n+1})}(P_n;\tau_n,\tau_{n+1})\right](-\tau_1)^{3/2+\mathsf{c}\mathrm{i}\widetilde{\nu}_1}(-\tau_N)^{3/2+\mathsf{d}\mathrm{i}\widetilde{\nu}_{N+1}}$$

Right subgraph

$$\mathcal{T}_{\mathsf{cd}}^{(\mathrm{R})}\left(\{\mathbf{k}^{(\mathrm{R})}\}\right)=\sum_{\mathsf{a}_{N+1},\cdots,\mathsf{a}_V=\pm}\int_{-\infty}^{0}\prod_{\ell=N+1}^{V}\left[\mathrm{d}\tau_\ell\,(\mathrm{i}\mathsf{a}_\ell)(-\tau_\ell)^{p_\ell}\prod_{i=1}^{B_\ell}C_{\mathsf{a}_\ell}\left(k_i^{(\ell)};\tau_\ell\right)\right]$$

$$\times\prod_{n=N+1}^{V-1}\left[D_{\mathsf{a}_n\mathsf{a}_{n+1}}^{(\widetilde{\nu}_{n+1})}(P_n;\tau_n,\tau_{n+1})\right](-\tau_{N+1})^{3/2+\mathsf{d}\mathrm{i}\widetilde{\nu}_{N+1}}(-\tau_V)^{3/2+\mathsf{c}\mathrm{i}\widetilde{\nu}_V}$$

Bubble signal

$$\mathcal{B}_{\mathsf{cd}}(P_N)=\Gamma\left[\begin{matrix}-\frac{3}{2}-\mathsf{c}\mathrm{i}\widetilde{\nu}_1-\mathsf{d}\mathrm{i}\widetilde{\nu}_{N+1},\frac{3}{2}+\mathsf{c}\mathrm{i}\widetilde{\nu}_1,\frac{3}{2}+\mathsf{d}\mathrm{i}\widetilde{\nu}_{N+1},-\mathsf{c}\mathrm{i}\widetilde{\nu}_1,-\mathsf{d}\mathrm{i}\widetilde{\nu}_{N+1}\\3+\mathsf{c}\mathrm{i}\widetilde{\nu}_1+\mathsf{d}\mathrm{i}\widetilde{\nu}_{N+1}\end{matrix}\right]$$

$$\times\frac{P_N^3}{(4\pi)^{7/2}}\left(\frac{P_N}{2}\right)^{2\mathrm{i}(\mathsf{c}\widetilde{\nu}_1+\mathsf{d}\widetilde{\nu}_{N+1})}$$

Outline of the proof

1. Partial Mellin-Barnes representation
All momentum dependences go into the loop integral

2. Loop integral
Isolate the nonanalytic term in the P_N -> 0 limit
The nonanalytic contributions must be from left poles

3. Time integrals only generate right poles

4. Pole structure
UV poles regular; IR poles contain signals (Signal poles)

5. Cutting rule: At all signal poles, D = Re D

6. Completing the Mellin integral

Proof

A general L-loop graph:

Vertices — External lines

$$\mathcal{T}(\{\mathbf{k}\}) = \sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V=\pm} \int_{-\infty}^{0} \prod_{i=1}^{V} \Big[\mathrm{i}\mathsf{a}_i \mathrm{d}\tau_i (-\tau_i)^{p_i} \Big] \prod_{j=1}^{B} \Big[C_{\mathsf{a}_j}(k_j;\tau_j) \Big]$$

$$\times \int \prod_{k=1}^{L} \left[\frac{\mathrm{d}^3 \mathbf{q}_k}{(2\pi)^3} \right] \prod_{\ell=1}^{I} \Big[D_{\mathsf{a}_{\ell 1}\mathsf{a}_{\ell 2}}(p_\ell;\tau_{\ell 1},\tau_{\ell 2}) \Big]$$

Bulk lines

Loop int

Partial Mellin-Barnes representation

$$D_{\pm\mp}^{(\widetilde{\nu})}(k;\tau_1,\tau_2) = \frac{1}{4\pi} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\mathrm{d}\bar{s}}{2\pi\mathrm{i}} e^{\mp \mathrm{i}\pi(s-\bar{s})} \left(\frac{k}{2} \right)^{-2(s+\bar{s})} (-\tau_1)^{-2s+3/2} (-\tau_2)^{-2\bar{s}+3/2}$$

$$\times \Gamma \left[s - \frac{\mathrm{i}\widetilde{\nu}}{2}, s + \frac{\mathrm{i}\widetilde{\nu}}{2}, \bar{s} - \frac{\mathrm{i}\widetilde{\nu}}{2}, \bar{s} + \frac{\mathrm{i}\widetilde{\nu}}{2} \right]$$

$$\mathcal{T}(\{\mathbf{k}\}) = \sum_{\mathsf{a}_1,\cdots,\mathsf{a}_V=\pm} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \prod_{i=1}^{I} \left[\frac{\mathrm{d}s_i}{2\pi\mathrm{i}} \frac{\mathrm{d}\bar{s}_i}{2\pi\mathrm{i}} \mathbb{D}(s_i,\bar{s}_i) \right] \mathbb{T}\Big(\{k\};\{s,\bar{s}\} \Big) \mathbb{L}\Big(\{\mathbf{k}\};\{s,\bar{s}\} \Big)$$

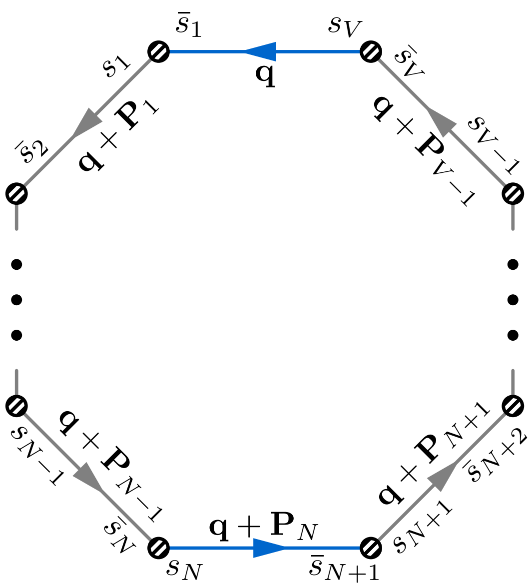
Bulk lines — Time int — Loop int

Loop momentum integral

$$\mathbb{L} = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\mathcal{P}$$

$$\mathcal{P} \equiv |\mathbf{q}|^{-2s_{V\bar{1}}}|\mathbf{q}+\mathbf{P}_1|^{-2s_{1\bar{2}}}|\mathbf{q}+\mathbf{P}_2|^{-2s_{2\bar{3}}}\cdots|\mathbf{q}+\mathbf{P}_{V-1}|^{-2s_{(V-1)\bar{V}}}$$

$P_N \to 0$ $\qquad$ $P_N < Q < \min\{P_1, \cdots, P_{N-1}, P_{N+1}, \cdots, P_{V-1}\}$



$$\mathbb{L} = \mathbb{L}_{\mathrm{S}} + \mathbb{L}_{\mathrm{H}}$$

$$\mathbb{L}_{\mathrm{S}} = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\mathcal{P}\,\theta(Q - |\mathbf{q}|)$$

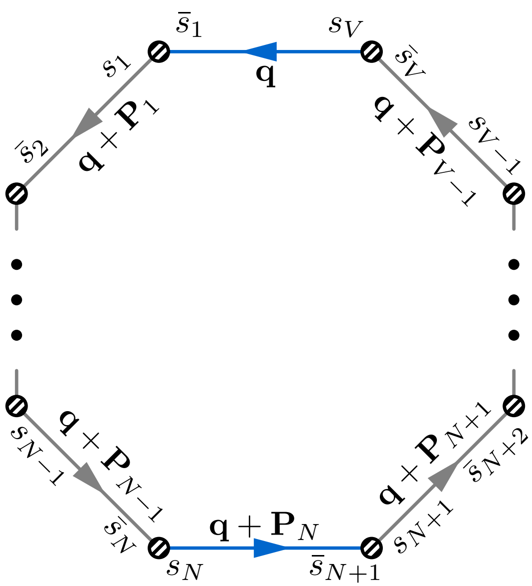
$$\mathbb{L}_{\mathrm{H}} = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\mathcal{P}\,\theta(|\mathbf{q}| - Q)$$

The hard part is analytic in P_N as P_N -> 0

$$\mathbb{L}_{\mathrm{H}} = \int_Q \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} |\mathbf{q}|^{-2s_{V\bar{1}}} |\mathbf{q}+\mathbf{P}_1|^{-2s_{1\bar{2}}} \cdots |\mathbf{q}+\mathbf{P}_{V-1}|^{-2s_{(V-1)\bar{V}}}$$

We can Taylor-expand all but the two soft lines around $P_N = 0$

$$\mathbb{L}_{\mathrm{S}} = \int^Q \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} |\mathbf{q}|^{-2s_{V\bar{1}}} |\mathbf{q}+\mathbf{P}_1|^{-2s_{1\bar{2}}} \cdots |\mathbf{q}+\mathbf{P}_{V-1}|^{-2s_{(V-1)\bar{V}}}$$



$$\begin{aligned}\mathbb{L}_{\mathrm{S}} = &\left(P_1^{-2s_{1\bar{2}}} \cdots P_{N-1}^{-2s_{(N-1)\bar{N}}}\right)\\ &\times \left(P_{N+1}^{-2s_{(N+1)(\overline{N+2})}} \cdots P_{V-1}^{-2s_{(V-1)\bar{V}}}\right)\\ &\times \int^Q \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} |\mathbf{q}|^{-2s_{V\bar{1}}} |\mathbf{q}+\mathbf{P}_N|^{-2s_{N(\overline{N+1})}}\\ &\times \left[1+\mathcal{O}(P_N)\right]\end{aligned}$$

The cutoff Q can be removed, generating a term analytic in P_N

$$\mathbb{L} = \left(P_1^{-2s_{1\bar{2}}} \cdots P_{N-1}^{-2s_{(N-1)\overline{N}}}\right)\left(P_{N+1}^{-2s_{(N+1)(\overline{N+2})}} \cdots P_{V-1}^{-2s_{(V-1)\bar{V}}}\right)$$
$$\times \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} |\mathbf{q}|^{-2s_{V\bar{1}}} |\mathbf{q} + \mathbf{P}_N|^{-2s_{N(\overline{N+1})}} \left[1 + \mathcal{O}(P_N)\right] + \text{terms analytic in } P_N$$

The remaining loop (bubble) integral can be done, and we get:

$$\mathbb{L} = \left(P_1^{-2s_{1\bar{2}}} \cdots P_{N-1}^{-2s_{(N-1)\overline{N}}}\right)\left(P_{N+1}^{-2s_{(N+1)(\overline{N+2})}} \cdots P_{V-1}^{-2s_{(V-1)\bar{V}}}\right)$$
$$\times \frac{P_N^{3-2s_{V\bar{1}N(\overline{N+1})}}}{(4\pi)^{3/2}} \Gamma\left[\begin{matrix} s_{V\bar{1}N(\overline{N+1})} - \frac{3}{2}, \frac{3}{2} - s_{V\bar{1}}, \frac{3}{2} - s_{N(\overline{N+1})} \\ 3 - s_{V\bar{1}N(\overline{N+1})}, s_{V\bar{1}}, s_{N(\overline{N+1})} \end{matrix}\right]$$
$$\times \left[1 + \mathcal{O}(P_N)\right] + \text{terms analytic in } P_N$$

Lessons:

1. Nonanalytic terms come from left poles
2. There is a single set of left poles in the loop integral (UV poles), but they do not yield signals

Pole structure

The loop integral only has a set of UV poles (no signal)

$$\mathbb{L} \propto \frac{P_N^{3-2s_{V\bar{1}N(\overline{N+1})}}}{(4\pi)^{3/2}} \Gamma\begin{bmatrix} s_{V\bar{1}N(\overline{N+1})} - \frac{3}{2}, \frac{3}{2} - s_{V\bar{1}}, \frac{3}{2} - s_{N(\overline{N+1})} \\ 3 - s_{V\bar{1}N(\overline{N+1})}, s_{V\bar{1}}, s_{N(\overline{N+1})} \end{bmatrix} + \cdots$$

The time integral only has right poles (irrelevant)

The IR poles from MB rep of bulk propagators:

$$\Gamma\left[s_V - \frac{\mathrm{i}\widetilde{\nu}_1}{2}, s_V + \frac{\mathrm{i}\widetilde{\nu}_1}{2}, \bar{s}_1 - \frac{\mathrm{i}\widetilde{\nu}_1}{2}, \bar{s}_1 + \frac{\mathrm{i}\widetilde{\nu}_1}{2}\right]$$

$$\times\, \Gamma\left[s_N - \frac{\mathrm{i}\widetilde{\nu}_{N+1}}{2}, s_N + \frac{\mathrm{i}\widetilde{\nu}_{N+1}}{2}, \bar{s}_{N+1} - \frac{\mathrm{i}\widetilde{\nu}_{N+1}}{2}, \bar{s}_{N+1} + \frac{\mathrm{i}\widetilde{\nu}_{N+1}}{2}\right]$$

16 combinations in total; only 4 survive due to zeros of loop integral:

$$s_V = -n_V - \mathsf{c}\frac{\mathrm{i}\widetilde{\nu}_1}{2},\ \ \bar{s}_1 = -\bar{n}_1 - \mathsf{c}\frac{\mathrm{i}\widetilde{\nu}_1}{2},$$

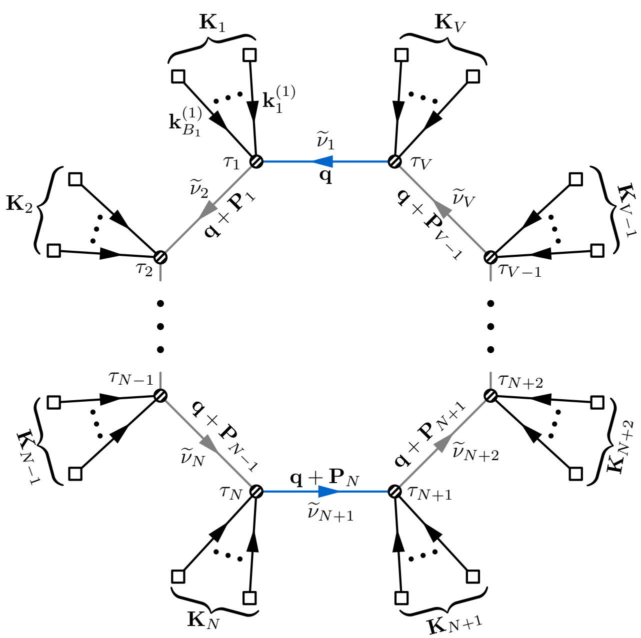
$$s_N = -n_N - \mathsf{d}\frac{\mathrm{i}\widetilde{\nu}_{N+1}}{2},\ \ \bar{s}_{N+1} = -\bar{n}_{N+1} - \mathsf{d}\frac{\mathrm{i}\widetilde{\nu}_{N+1}}{2}$$

Signal poles

$(\mathsf{c},\mathsf{d} = \pm 1)$

Corollary: nonlocal-signal cutting rule

A direct corollary of our proof of the factorization theorem



To compute nonlocal signals, the two soft lines can be cut:

Cutting rule: replace the soft propagator by its real part:

$$D_{\mathsf{ab}}\Big(q;\tau_1,\tau_2\Big) \Rightarrow \mathrm{Re}\,D\Big(q;\tau_1,\tau_2\Big)$$

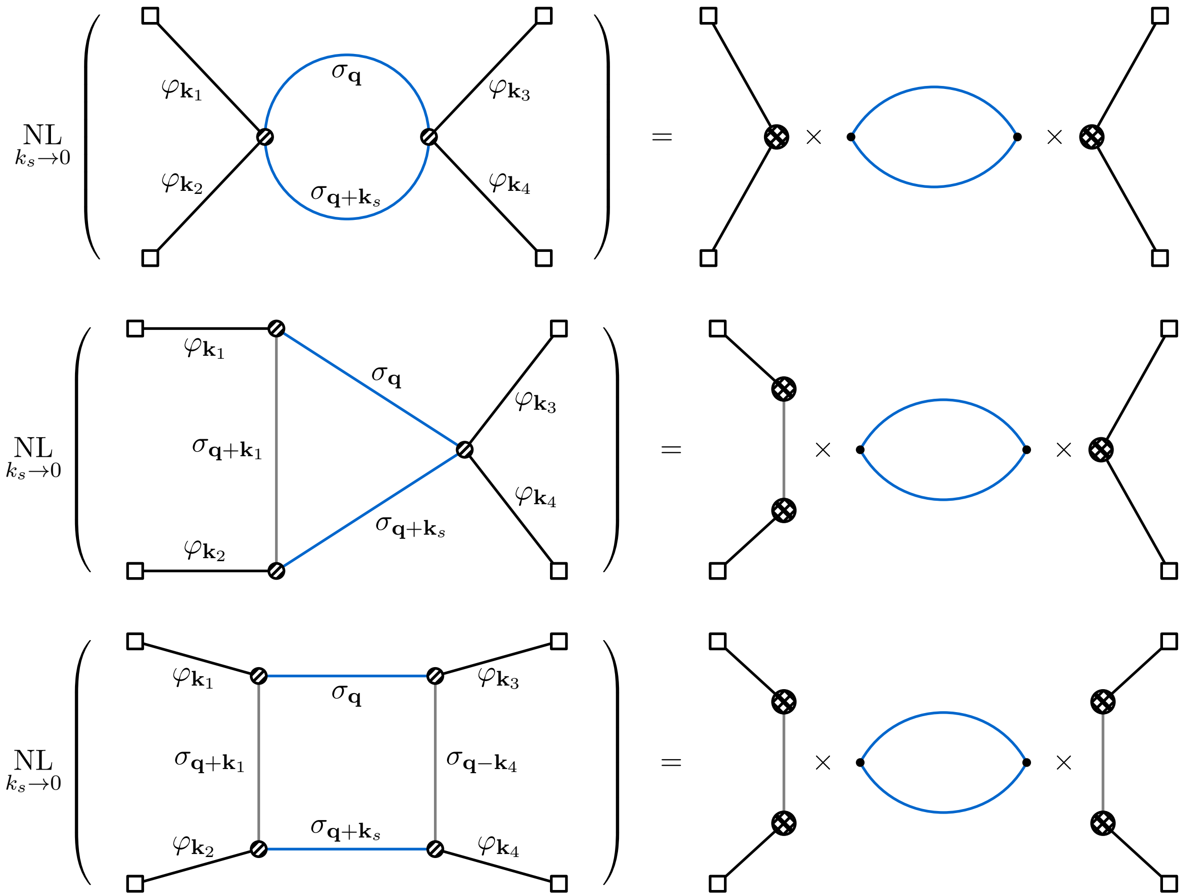
The real part automatically indep. of SK indices

➡ No nested time integrals

Applications

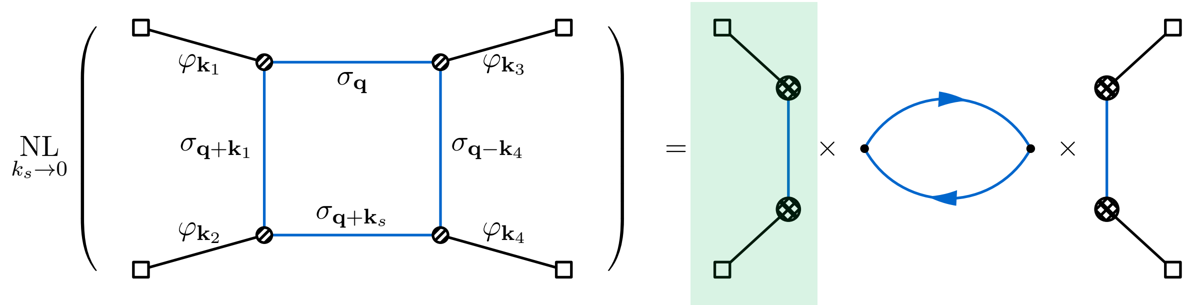
Zhehan Qin, ZX, 2304.13295

All nontrivial 1PI 1-loop 4-point correlators:



1-loop box graph:

$$\Delta\mathscr{L} = \frac{1}{2} a^3 \varphi' \sigma^2$$



$$\mathcal{T}_{\mathsf{c}}^{(\mathrm{L})}(k_1, k_2) = \frac{1}{4k_1k_2} \sum_{\mathsf{a}_1,\mathsf{a}_2=\pm} (-\mathsf{a}_1\mathsf{a}_2) \int_{-\infty}^{0} \frac{\mathrm{d}\tau_1}{\tau_1^2} \frac{\mathrm{d}\tau_2}{\tau_2^2} e^{\mathsf{a}_1 \mathrm{i} k_1 \tau_1 + \mathsf{a}_2 \mathrm{i} k_2 \tau_2} D^{(\widetilde{\nu})}_{\mathsf{a}_1\mathsf{a}_2}(k_1; \tau_1, \tau_2)$$
$$\times (-\tau_1)^{3/2+\mathrm{c}\mathrm{i}\widetilde{\nu}} (-\tau_2)^{3/2+\mathrm{c}\mathrm{i}\widetilde{\nu}}$$

Computed with an improved bootstrap method in 2301.07047

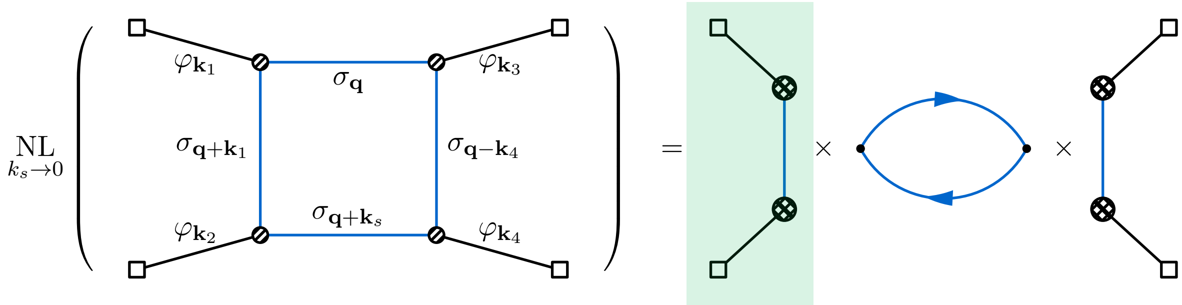
$$\mathcal{T}_{\mathsf{c}}^{(\mathrm{L})}(k_1, k_2) = -\frac{8(2+\mathrm{i}\mathrm{c}\widetilde{\nu})\Gamma(2+2\mathrm{i}\mathrm{c}\widetilde{\nu})\sin(\pi\mathrm{i}\mathrm{c}\widetilde{\nu})}{3+2\mathrm{i}\mathrm{c}\widetilde{\nu}} \frac{k_{12}^{-4-2\mathrm{c}\mathrm{i}\widetilde{\nu}}}{4k_1k_2}$$

The right subgraph is similar. Putting everything together, we get:

$$\lim_{k_s\to 0} \Big[\mathcal{T}_{\mathrm{box}}(\{\mathbf{k}\})\Big]_{\mathrm{NL}} = -\frac{k_s^3}{2(4\pi)^{7/2} k_1k_2k_3k_4k_{12}^4 k_{34}^4} \left(\frac{k_s^2}{4k_{12}k_{34}}\right)^{2\mathrm{i}\widetilde{\nu}} \frac{(2+\mathrm{i}\widetilde{\nu})^4}{(3+2\mathrm{i}\widetilde{\nu})^2} \sinh^2(\pi\widetilde{\nu})$$
$$\times \Gamma\left[3+2\mathrm{i}\widetilde{\nu}, -\frac{3}{2}-2\mathrm{i}\widetilde{\nu}\right] \Gamma^2\left[\frac{3}{2}+\mathrm{i}\widetilde{\nu}, -2-\mathrm{i}\widetilde{\nu}\right] + \mathrm{c.c.}.$$

1-loop box graph:

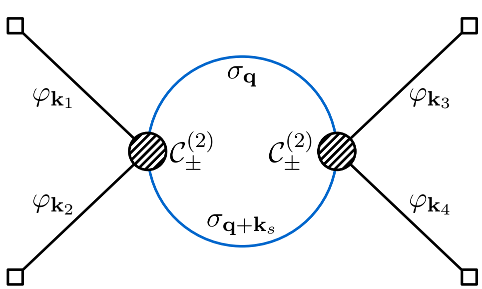
$$\Delta\mathscr{L} = \frac{1}{2}a^3\varphi'\sigma^2$$

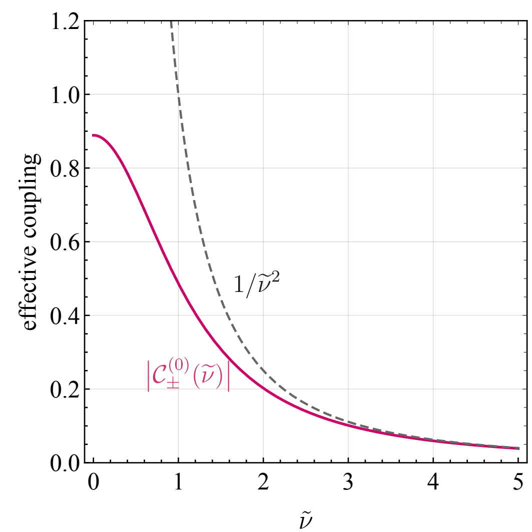


$$\lim_{k_s\to 0}\left[\mathcal{T}_{\rm box}(\{\mathbf{k}\})\right]_{\rm NL} = -\frac{k_s^3}{2(4\pi)^{7/2}k_1k_2k_3k_4k_{12}^4k_{34}^4}\left(\frac{k_s^2}{4k_{12}k_{34}}\right)^{2i\widetilde{\nu}}\frac{(2+i\widetilde{\nu})^4}{(3+2i\widetilde{\nu})^2}\sinh^2(\pi\widetilde{\nu})$$
$$\times\,\Gamma\left[3+2i\widetilde{\nu},-\frac{3}{2}-2i\widetilde{\nu}\right]\Gamma^2\left[\frac{3}{2}+i\widetilde{\nu},-2-i\widetilde{\nu}\right]+\text{c.c.}.$$

The nonlocal signal can be computed from a pinched graph with effective "pinched couplings"

→



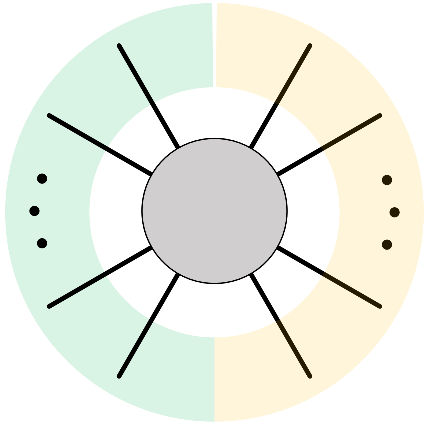


This computation can be pushsed to higher orders in k_s

Graphs with arbitrary number of loops

Zhehan Qin, ZX, 2308.14802

B-point L-loop graph specified by $\{\mathbf{k}\}$



Nonlocal bipartition

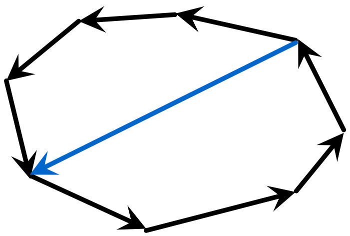
$$\{\mathbf{k}^{(\mathrm{L})}\} \cap \{\mathbf{k}^{(\mathrm{R})}\} = \varnothing$$

$$\{\mathbf{k}^{(\mathrm{L})}\} \cup \{\mathbf{k}^{(\mathrm{R})}\} = \{\mathbf{k}\}$$

$B_L \geq 2$, $B_R \geq 2$

Partial sums: $\widetilde{\mathbf{P}}_i = \sum_{j=1}^{B} \beta_{ij}\mathbf{k}_j$

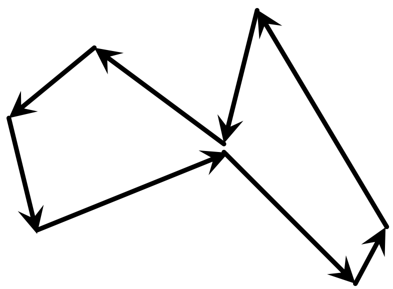
$\beta_{ij} \in \{0, 1\}$ at least one $\beta_{ij} = 0$



Nonlocal soft limit:

$$\mathbf{P} \equiv \sum \mathbf{k}^{(\mathrm{L})} \to \mathbf{0} \quad \left(\sum \mathbf{k}^{(\mathrm{R})} = -\mathbf{P} \to \mathbf{0}\right)$$

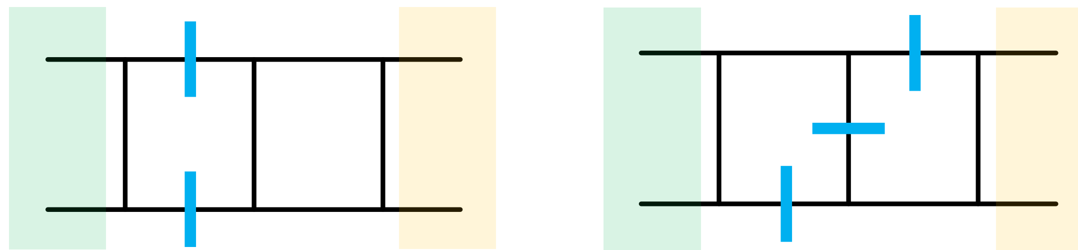
All $\mathbf{k}$ and. $\widetilde{\mathbf{P}}_j \neq \pm\mathbf{P}$ remain finite



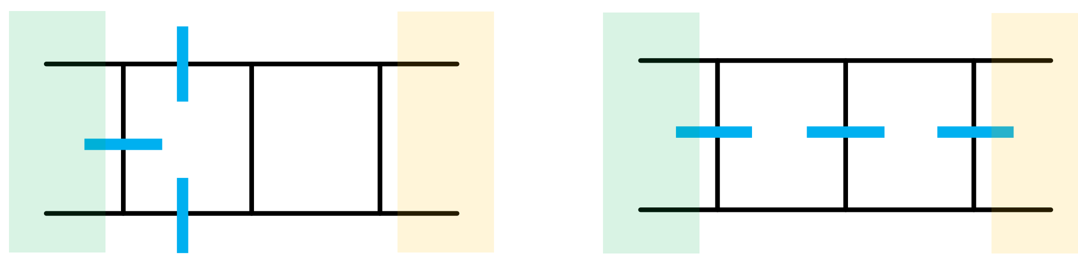
Given a nonlocal bipartition, a nonlocal cut means the removal of some bulk propagators, such that in the resultant graph:

1. The left boundary points are disconnected from the right
2. All left (right) boundary points are fully connected

Valid cuts:



Invalid cuts:



lines removed in a cut is called the degree of the cut
Given a bipartition, a cut of minimal degree is called a minimal cut

Signal-detection algorithm

Zhehan Qin, ZX, 2308.14802

For any B-point L-loop graph with a given nonlocal bipartition, we enumerate all possible nonlocal cuts with respect to this bipartition.

Then:

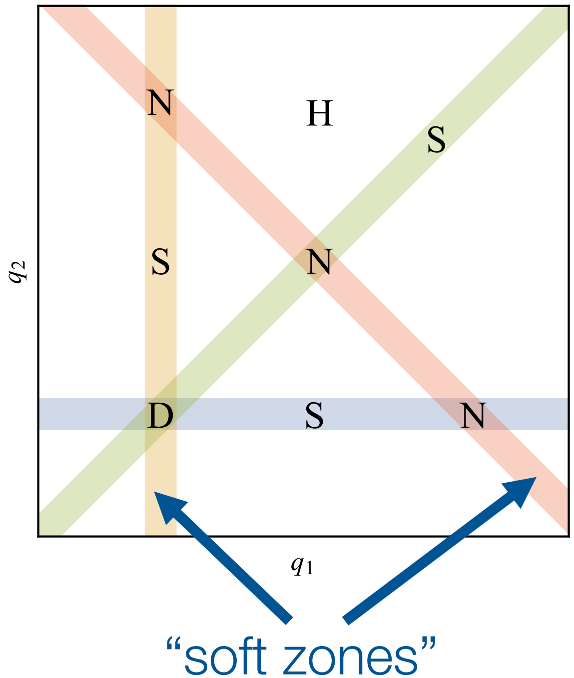
1. Each nonlocal cut gives a candidate of nonlocal signal in the nonlocal soft limit P -> 0

2. For dS covariant graphs with a unique minimal cut, this unique least cut gives the leading nonlocal signal in the limit P -> 0

3. For dS covariant graphs with more than one minimal cuts, the leading signal is the sum of nonlocal signals from all the least cuts

Nonlocal cut ⇔ degenerate singularity

Mellin-space loop integral: $$\mathbb{L}\Big(\{\mathbf{k}\};\{s,\bar{s}\}\Big)=\int\prod_{\ell=1}^{L}\left[\frac{\mathrm{d}^3\mathbf{q}_\ell}{(2\pi)^3}\right]\prod_{i=1}^{I}|\mathbf{p}_i|^{-2s_{i\bar{i}}}$$



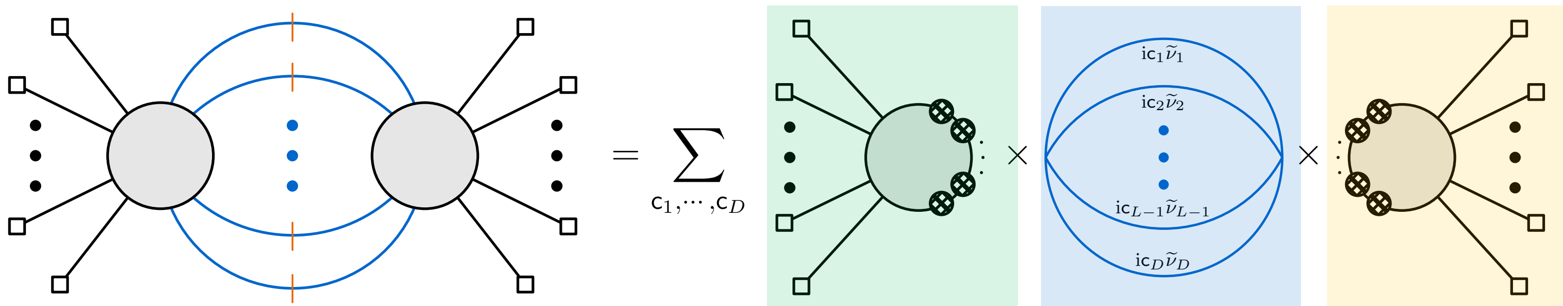
The (3L-dimensional) space of loop integral can be classified into 4 regions:

1. Hard region (H): all loop lines are hard
2. Singly soft region (S): only one line soft; all others hard
3. Nondegenerate region (N): more than one lines soft, but their momenta are linearly independent
4. Degenerate region (D): more than one lines soft, and their momenta are linearly dependent

1. Only the degenerate region could possibly yield nonlocal signals
2. Degenerate region ⇔ nonlocal cut

Factorization (with unique minimal cut) at all loop orders

There might be many cuts, but the minimal cut gives the leading signal:



$$\lim_{P\to 0} \mathcal{T}(\{\mathbf{k}\}) \supset \sum_{\substack{c_1,\cdots,c_D=\pm \\ \text{nonlocal}}} \mathcal{G}^{(\mathrm{L})}_{c_1\cdots c_D}\left(\{\mathbf{k}^{(\mathrm{L})}\}\right)\mathcal{G}^{(\mathrm{R})}_{c_1\cdots c_D}\left(\{\mathbf{k}^{(\mathrm{R})}\}\right)\mathfrak{M}_{c_1\cdots c_D}(P)$$

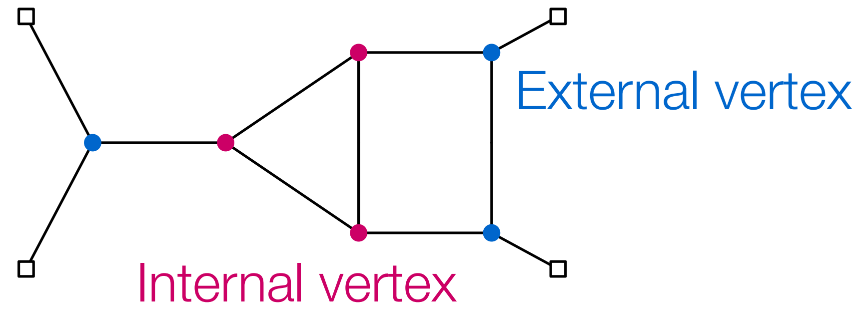
$$+ \text{(subleading nonlocal signals)} + \text{(terms analytic in } K)$$

The nonlocal signal is fully encoded in the following "melon signal":

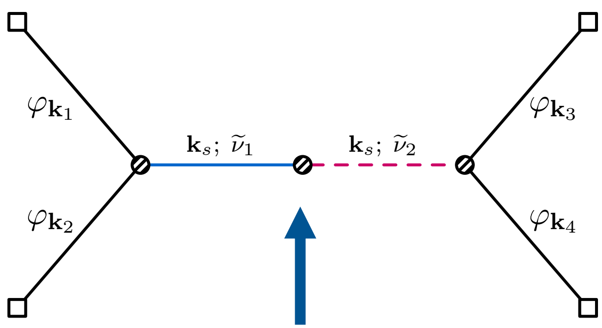
$$\mathfrak{M}_{c_1\cdots c_D}(P) \equiv \frac{P^{3(D-1)}}{(4\pi)^{(5D-3)/2}}\Gamma\begin{bmatrix} -\sum_{i=1}^{D} c_i\mathrm{i}\widetilde{\nu}_i - \frac{3}{2}(D-1) \\ \frac{3}{2}D + \sum_{i=1}^{D} c_i\mathrm{i}\widetilde{\nu}_i \end{bmatrix} \prod_{\ell=1}^{D}\left\{\Gamma\left[\tfrac{3}{2} + c_\ell\mathrm{i}\widetilde{\nu}_\ell, -c_\ell\mathrm{i}\widetilde{\nu}_\ell\right]\left(\frac{P}{2}\right)^{2\mathrm{i}c_\ell\widetilde{\nu}_\ell}\right\}$$

Subtleties with multiple least cuts

There are many subtleties with multiloop graphs with internal vertices



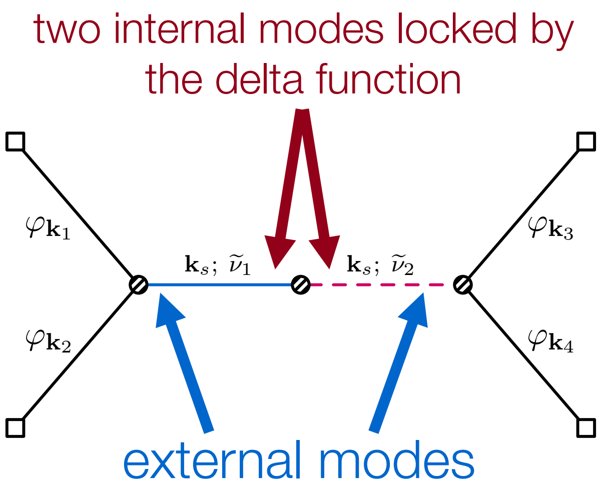
Take the two-point mixing tree-graph as an example:



$$\int \mathrm{d}\tau_3(-\tau_3)^{p_3+3-2s_{\bar{1}\bar{2}}}$$
$$= 2\pi\mathrm{i}\delta(p_3+4-2s_{\bar{1}\bar{2}})$$

Either the blue (solid) or the magenta (dashed) line can be cut. Either choice yields a valid nonlocal signal;

One can also cut both lines simultaneously, but it's impossible to get two copies of nonlocal signa

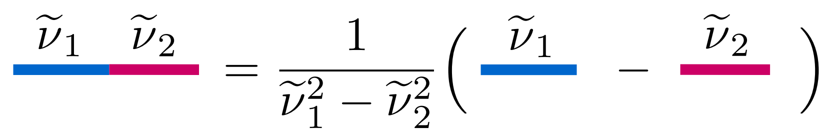


However, there is a peculiar possibility that the two "external" modes produce a signal together:

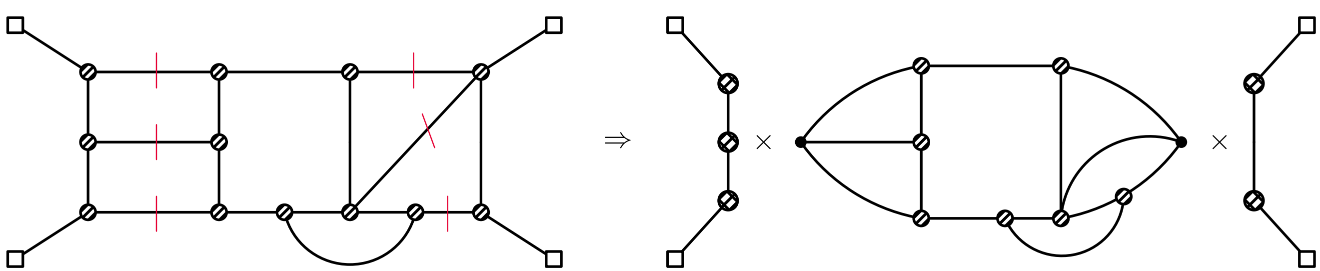
$$\mathcal{T} \supset k_s^{+\mathrm{i}\widetilde{\nu}_1+\mathrm{i}\widetilde{\nu}_2} + \text{c.c.}$$

In comparison, ordinary signals looks like $k_s^{+2\mathrm{i}\widetilde{\nu}_1} + \text{c.c.}$ or $k_s^{+2\mathrm{i}\widetilde{\nu}_2} + \text{c.c.}$

This odd possibility never happens for dS covariant mixing:

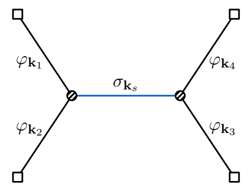


Can also be understood more generally from a CFT / OPE argument:

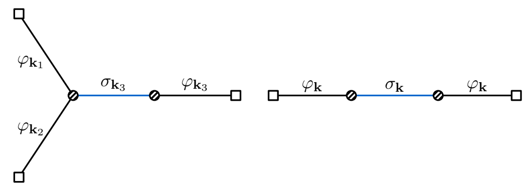


Summary: What we have learned since 2022

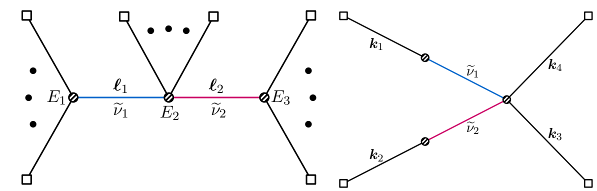
1. Complete analytical results:



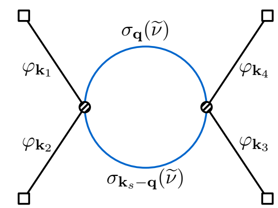
boost breaking graphs
PMB / bootstrap



Closed-form formula
improved bootstrap

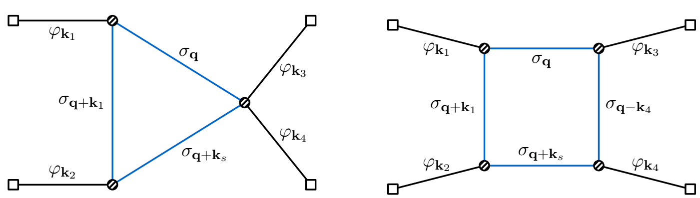


Multiple massive exchange
family-tree decomposition

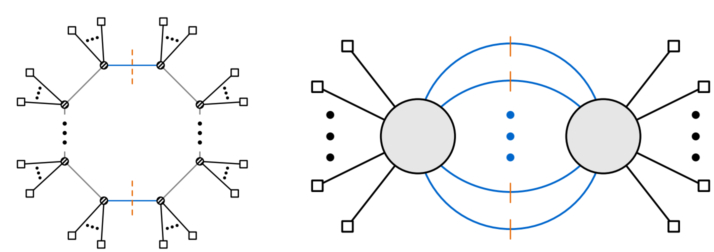


1-loop bubble graphs
spectral decomposition

2. Nonlocal signals:

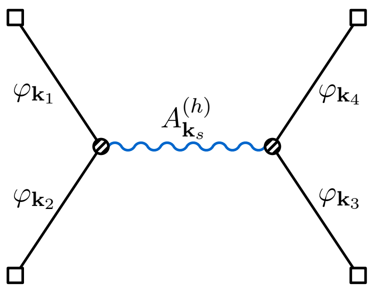


Nonlocal signal
Explicit result, PMB, improved bootstrap



Nonlocal signal
factorization theorem / PMB

Outlooks

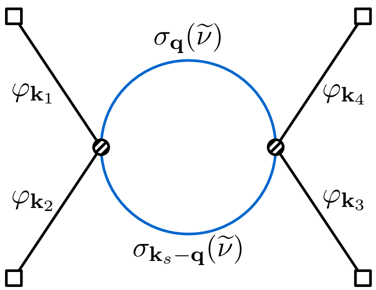


Tree-level correlators

Useful techniques developed in recent years

Partial MB / bootstrap

Essential problems solved, many potential applications; More internal legs?



Loop correlators

Full result for dS obtained by dS covariant bubble

Nonlocal signals found for general 1-loop

Local signals?

Spinning fields / derivative couplings / boost-breaking dispersions

More methods? dispersion integrals? CFT techniques?

Thank you!

back up

Time integrals have right poles only

Arbitrarily nested multi-layered time integral:

$$\int_{-\infty}^{0} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_V \, (-\tau_1)^{\widehat{p}_1 - 2s_{1\bar{1}}} \cdots (-\tau_V)^{\widehat{p}_V - 2s_{V\bar{V}}} e^{\mathrm{i}(\mathsf{a}_1 E_1 \tau_1 + \cdots + \mathsf{a}_V E_V \tau_V)}$$

$$\times \mathcal{N}(\tau_1, \tau_2)\mathcal{N}(\tau_2, \tau_3) \cdots \mathcal{N}(\tau_V - 1, \tau_V)\mathcal{N}(\tau_V, \tau_1) \qquad \mathcal{N}(\tau_i, \tau_j) = 1, \; \theta(\tau_i - \tau_j), \; \textit{or} \; \theta(\tau_j - \tau_i)$$

Can always be put into a sum of monotonically nested integrals using:

$$\theta(\tau_i - \tau_j) + \theta(\tau_j - \tau_i) = 1$$

Consider one monotonically nested integral at a time:

$$\int_{-\infty}^{0} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_\ell \, (-\tau_1)^{\widehat{p}_1 - 2s_{1\bar{1}}} \cdots (-\tau_\ell)^{\widehat{p}_\ell - 2s_{\ell\bar{\ell}}} e^{\mathrm{i}\mathsf{a}(E_1 \tau_1 + \cdots + E_\ell \tau_\ell)} \theta(\tau_1 - \tau_2) \cdots \theta(\tau_{\ell-1} - \tau_\ell)$$

UV convergent (BD vacuum); Singularities in s must be from IR

Isolate the IR part and do Taylor expansion

$$\int_{-\epsilon}^{0} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_\ell\, (-\tau_1)^{\widehat{p}_1 - 2s_{1\bar{1}}} \cdots (-\tau_\ell)^{\widehat{p}_\ell - 2s_{\ell\bar{\ell}}} e^{\mathrm{ia}(E_1\tau_1 + \cdots + E_\ell \tau_\ell)} \theta(\tau_1 - \tau_2) \cdots \theta(\tau_{\ell-1} - \tau_\ell)$$

$$\sum_{n_1, \cdots, n_\ell = 0}^{\infty} \frac{(-\mathrm{ia})^{n_1 + \cdots + n_\ell} E_1^{n_1} \cdots E_\ell^{n_\ell}}{n_1! \cdots n_\ell!} \int_{-\epsilon}^{0} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_\ell\, (-\tau_1)^{\widehat{p}_1 - 2s_{1\bar{1}} + n_1} \cdots (-\tau_\ell)^{\widehat{p}_\ell - 2s_{\ell\bar{\ell}} + n_\ell}$$

$$\times\, \theta(\tau_1 - \tau_2) \cdots \theta(\tau_{\ell-1} - \tau_\ell)$$

This can be integrated layer by layer

1st layer:
$$\int_{-\epsilon}^{0} \mathrm{d}\tau_1\, (-\tau_1)^{\widehat{p}_1 - 2s_{1\bar{1}} + n_1} \theta(\tau_1 - \tau_2) = \frac{(-\tau_2)^{\widehat{p}_1 - 2s_{1\bar{1}} + n_1 + 1}}{\widehat{p}_1 - 2s_{1\bar{1}} + n_1 + 1}$$

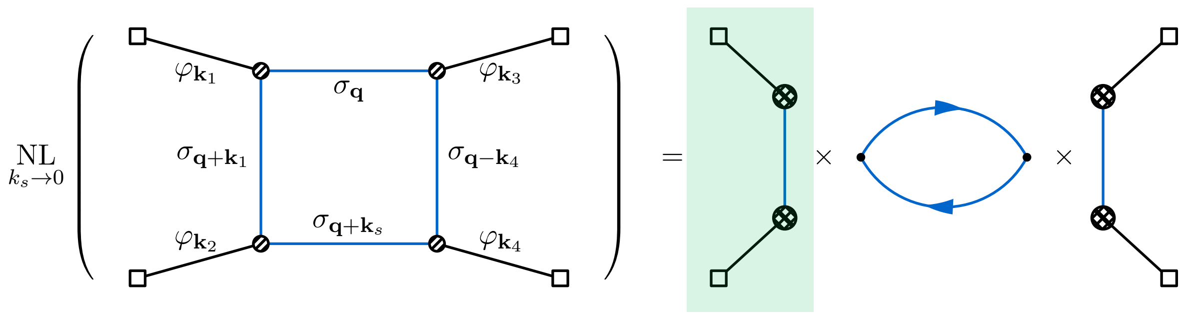
$$\longrightarrow \quad s_{1\bar{1}} = \frac{\widehat{p}_1 + n_1 + 1}{2}. \quad (n_1 = 0, 1, 2, \cdots)$$

2nd layer:
$$\int_{-\epsilon}^{0} \mathrm{d}\tau_2\, (-\tau_2)^{\widehat{p}_{12} - 2s_{1\bar{1}2\bar{2}} + n_{12} + 1} \theta(\tau_2 - \tau_3) = \frac{(-\tau_3)^{\widehat{p}_{12} - 2s_{1\bar{1}2\bar{2}} + n_{12} + 2}}{\widehat{p}_{12} - 2s_{1\bar{1}2\bar{2}} + n_{12} + 2}$$

$$\longrightarrow \quad s_{1\bar{1}2\bar{2}} = \frac{\widehat{p}_{12} + n_{12} + 2}{2}. \quad (n_1, n_2 = 0, 1, 2, \cdots)$$

This can be done recursively; Only right poles are generated.

Computation of the left subgraph



$$\mathcal{T}_{\mathsf{c}}^{(\mathrm{L})}(k_1,k_2) = \frac{1}{4k_1k_2}\sum_{\mathsf{a}_1,\mathsf{a}_2=\pm}(-\mathsf{a}_1\mathsf{a}_2)\int_{-\infty}^{0}\frac{\mathrm{d}\tau_1}{\tau_1^2}\frac{\mathrm{d}\tau_2}{\tau_2^2}e^{\mathsf{a}_1 \mathrm{i}k_1\tau_1+\mathsf{a}_2\mathrm{i}k_2\tau_2}D^{(\widetilde{\nu})}_{\mathsf{a}_1\mathsf{a}_2}(k_1;\tau_1,\tau_2)$$
$$\times\,(-\tau_1)^{3/2+\mathsf{c}\mathrm{i}\widetilde{\nu}}(-\tau_2)^{3/2+\mathsf{c}\mathrm{i}\widetilde{\nu}}$$

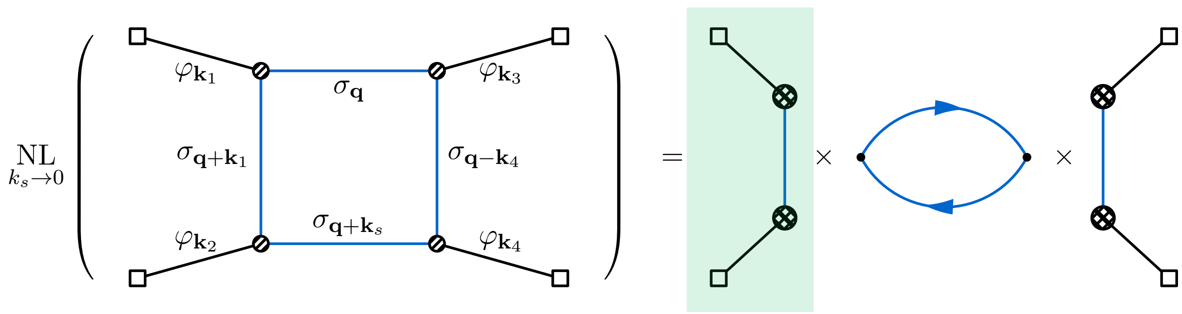
$$\mathcal{T}_{\mathsf{c}}^{(\mathrm{L})}(k_1,k_2) = \mathcal{P}_{\mathsf{c}}^{(2)}(\widetilde{\nu})\frac{k_{12}^{-4-2\mathsf{c}\mathrm{i}\widetilde{\nu}}}{4k_1k_2}$$

$$\mathcal{P}_{\mathsf{c}}^{(2)}(\widetilde{\nu}) \equiv 2^{4+2\mathsf{c}\mathrm{i}\widetilde{\nu}}\sum_{\mathsf{a}_1,\mathsf{a}_2=\pm}\widetilde{\mathcal{I}}^{-1/2+\mathsf{c}\mathrm{i}\widetilde{\nu},-1/2+\mathsf{c}\mathrm{i}\widetilde{\nu}}_{\widetilde{\nu}|\mathsf{a}_1\mathsf{a}_2}(1,1)$$

$$\widetilde{\mathcal{I}}^{p_1p_2}_{\widetilde{\nu}|\mathsf{ab}}(u_1,u_2) \equiv (-\mathsf{ab})k_s^{5+p_{12}}\int_{-\infty}^{0}\mathrm{d}\tau_1\mathrm{d}\tau_2(-\tau_1)^{p_1}(-\tau_2)^{p_2}e^{\mathrm{i}\mathsf{a}k_{12}\tau_1+\mathrm{i}\mathsf{b}k_{34}\tau_2}D^{(\widetilde{\nu})}_{\mathsf{ab}}(k_s;\tau_1,\tau_2)$$

$$u_1 \equiv 2k_s/(k_{12}+k_s),\quad u_2 \equiv 2k_s/(k_{34}+k_s)$$

Computation of the left subgraph



$$\widetilde{\mathcal{I}}^{p_1p_2}_{\widetilde{\nu}|\pm\pm}(1,1) = \frac{\pm \mathrm{i} e^{\mp \mathrm{i} p_{12}\pi/2} e^{-\pi\widetilde{\nu}}}{2^{5+p_{12}}} \Gamma\left[\begin{matrix} \frac{5}{2}+p_1-\mathrm{i}\widetilde{\nu}, \frac{5}{2}+p_1+\mathrm{i}\widetilde{\nu}, \frac{5}{2}+p_2-\mathrm{i}\widetilde{\nu}, \frac{5}{2}+p_2+\mathrm{i}\widetilde{\nu} \\ 3+p_1, 3+p_2 \end{matrix}\right]$$

$$-\frac{e^{\mp \mathrm{i} p_{12}\pi/2}}{2^{5+p_{12}}} \Gamma\left[5+p_{12}, \tfrac{5}{2}+p_1 \pm \mathrm{i}\widetilde{\nu}, \tfrac{5}{2}+p_2\pm \mathrm{i}\widetilde{\nu}\right] {}_3\widetilde{\mathrm{F}}_2\left[\begin{matrix} 5+p_{12}, \frac{1}{2}\pm \mathrm{i}\widetilde{\nu}, 1 \\ \frac{7}{2}+p_1\pm\mathrm{i}\widetilde{\nu}, \frac{7}{2}+p_2\pm\mathrm{i}\widetilde{\nu}\end{matrix}\middle| 1\right],$$

$$\widetilde{\mathcal{I}}^{p_1p_2}_{\widetilde{\nu}|\pm\mp}(1,1) = \frac{e^{\mp \mathrm{i} \bar{p}_{12}\pi/2}}{2^{5+p_{12}}} \Gamma\left[\begin{matrix} \frac{5}{2}+p_1-\mathrm{i}\widetilde{\nu}, \frac{5}{2}+p_1+\mathrm{i}\widetilde{\nu}, \frac{5}{2}+p_2-\mathrm{i}\widetilde{\nu}, \frac{5}{2}+p_2+\mathrm{i}\widetilde{\nu} \\ 3+p_1, 3+p_2 \end{matrix}\right]$$

$$2^{4+2\mathrm{c}\mathrm{i}\widetilde{\nu}} \sum_{\mathsf{a}_1,\mathsf{a}_2=\pm} \widetilde{\mathcal{I}}^{-1/2+\mathrm{c}\mathrm{i}\widetilde{\nu},-1/2+\mathrm{c}\mathrm{i}\widetilde{\nu}}_{\widetilde{\nu}|\mathsf{a}_1\mathsf{a}_2}(1,1) = -\frac{8(2+\mathrm{i}\mathrm{c}\widetilde{\nu})\Gamma(2+2\mathrm{i}\mathrm{c}\widetilde{\nu})\sin(\pi \mathrm{i}\mathrm{c}\widetilde{\nu})}{3+2\mathrm{i}\mathrm{c}\widetilde{\nu}}$$