Bootstrapping Mesons at Large N

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Our Goal

Mimic the bootstrap approach to constrain the space of QFT which describe QCD at large N.

Why large N?

- the theory maintains important features of real world QCD
- it simplifies (EFT is weakly coupled)
- it shares many similarities with string theory

QCD at large N_c

QCD with N_f massless quarks, in the 't Hooft limit: $N_c \to \infty$, fixed $\lambda = g^2 N_c$

Spectrum:

- Barions decouple: mass grows with N_c
- Mesons and Glueball marginally affected
- Confinement and chiral symmetry breaking $\Rightarrow$ Goldstone bosons π^a

Interactions:

- Planar diagram dominates (see later)
- Interactions $O(1/N) \Rightarrow$ Mesons decay widths $1/N$
  $\Rightarrow$ loops suppressed

Scattering amplitude of mesons is meromorphic!

Bootstrapping scattering amplitudes

Mesons scattering amplitude: an infinite tower of tree level exchanges

Parameters space:

- masses, spin of resonances m_i^2, J_i
- on-shell couplings g_{ijk}

Constraints:

- Crossing
- Analiticity
- Unitarity (positivity)
- Regge behavior

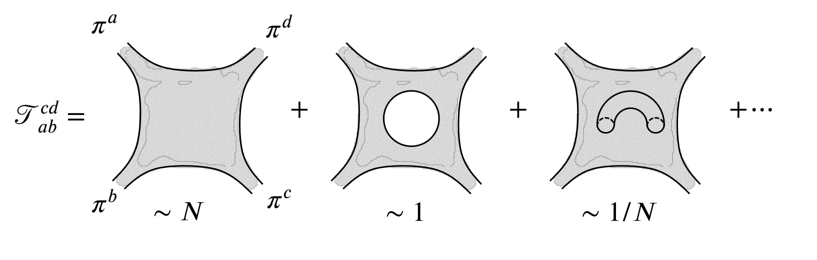
Pion scattering at Large N

$$\pi^a \pi^b \longrightarrow \pi^c \pi^d$$

Scattering amplitude:

$$\mathscr{F}_{ab}^{cd} = Tr[\sigma^a\sigma^b\sigma^c\sigma^d]\mathcal{A}(s,u) + Tr[\sigma^a\sigma^b\sigma^d\sigma^c]\mathcal{A}(s,t) + Tr[\sigma^a\sigma^c\sigma^b\sigma^d]\mathcal{A}(u,t)$$

parametrized by a single function $\mathcal{A}(s,t)$



$q\bar{q}$-mesons loops & tetraquarks *glueballs*

Pion scattering at Large N - 2

$$\pi^a \pi^b \longrightarrow \pi^c \pi^d$$

- Crossing: $\mathcal{A}(s,t) = \mathcal{A}(t,s)$
- Analyticity: only poles on the real axis
- OZI rule: no poles on the negative real axis

Isospin: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

$q\bar{q}$ - mesons $\in 0, 1$

tetraquark - mesons $\in 2$ (suppressed at large N_c)

poles at negative $s \leftrightarrow$ poles in Isospin 2 channel

Pion scattering at Large N - 3

$$\pi^a \pi^b \longrightarrow \pi^c \pi^d$$

- Regge boundedness:

$$\lim_{|s|\to\infty} \mathcal{A}(s,t) \sim s^{\alpha(t)} \qquad (\text{fixed } t<0)$$

Eventually we want to expand for $t \simeq 0$ – important to know $\alpha(0)$:

Finite N_c: $\alpha(0) \simeq 1.08$ (Pomeron)

Large N_c: $\alpha(0) \simeq 0.5$ (Pomeron suppressed, ρ-trajecrtory)

- Unitarity

$$\mathcal{A}(s,t) = \sum_{J \text{ even}} n_J^{(4)} f_J(s) \mathcal{P}_J \left(1 + \frac{2t}{s}\right), \qquad n_J^{(d)} : \text{normalizations}$$

$$Im[\mathcal{A}(s,t)] = \sum_J n_J^{(4)} \, \rho_J(s) \mathcal{P}_J \left(1 + \frac{2t}{s}\right),$$

Unitarity guarantees positivity of the spectral density:

$$\rho_J(s) = Im \; f_J(s) \sim \sum_X g_\pi^2 \delta(s - m_X^2) \geq 0$$

Alternative prospective: Effective Field Theory

Integrating out all resonances at tree level produces an effective theory of pions

$$\mathcal{A}(s,t) = g_{0,1}(s+t) + g_{1,1}(s+t)(st) + g_{1,0}ts + g_{0,2}(s+t)^2 + \dots$$

No states below cut off M: no poles.

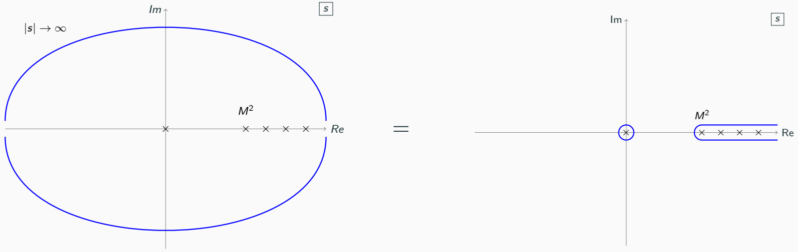
EFT also weakly coupled: all interactions $O(1/N)$: no logs

One can match the coefficients $g_{n,\ell}$ with the parameters of the Chiral Lagrangian

$$\mathscr{L} = -\frac{f_\pi^2}{4}\mathrm{Tr}[\partial^\mu U^\dagger \partial_\mu U] + \text{higher derivatives}, \qquad U = e^{i\pi^a\sigma^a}$$

Dispersion relations at large N_c

$$\oint_{\infty} \frac{ds}{2\pi i s} \frac{\mathcal{A}(s,t)}{s^k} = 0 \quad (\text{at large } N_c, k = 1, 2, \ldots)$$



low energy data $g_{n,\ell}$ $\longleftrightarrow$ high energy data $\rho_J(s)$

[Arkani-Hamed, Huang, Huang] , [Tolley, Wang, Zhou] ,
[Bellazzini, Mirò Rattazzi, Riembau,Riva] [Caron-Hout, Van Duong]

Sum rules and null constraints

$$\left.\begin{array}{l} g_{0,1} = \langle \frac{1}{m^2} \rangle \\ \\ g_{1,1} = \ldots \end{array}\right\} \leftarrow \text{sum rules} \qquad g_{0,1} > 0 \ (\sim 1/f_\pi^2)$$

$$\left.\begin{array}{l} 0 = \langle \frac{(J-2)J(J+1)(J+3)}{m^6} \rangle \\ \\ 0 = \ldots \end{array}\right\} \leftarrow \text{null constraints } \mathcal{X}_{k,\ell}$$

Notation:

$$\langle F(m^2, J) \rangle \equiv \sum_{J \text{ even}} n_J^{(4)} \int_{M^2}^{\infty} \frac{dm^2}{\pi m^2} \rho_J(m^2) \left[F(m^2, J) \right].$$

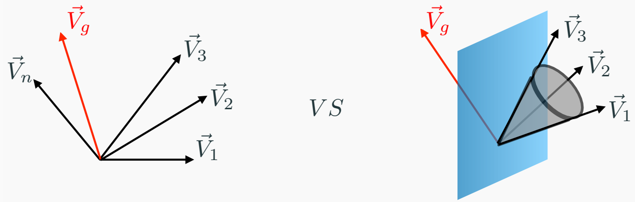
Unitarity $\Rightarrow \rho_J(s) \geq 0$

Bootstrap equations

Schematic form of equations:

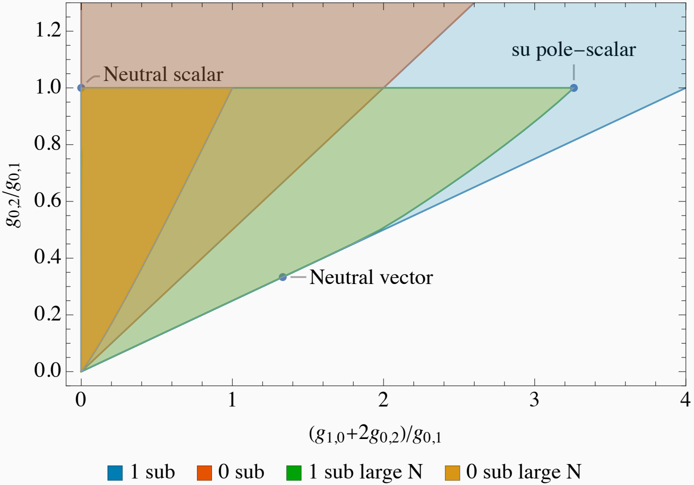
$$\underbrace{\begin{pmatrix} g_{0,1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{V}_g} = \sum_X g^2_{\pi\pi X} \underbrace{\begin{pmatrix} \cdots \\ \mathcal{X}_{3,1} \\ \cdots \\ \mathcal{X}_{4,1} \\ \cdots \end{pmatrix}}_{\vec{V}_X}$$

($X \equiv$ quantum numbers of states exchanged in $\pi\pi \to \pi\pi$)



Feasibility can be recast in a semi-definite positive problem and tested numerically

Regge behaviour & subtractions



[Albert, Rastelli '22] [Fernandez, Pomarol, Riva, Sciotti '22]

[McPeak, Venuti, AV '23]

Simple amplitudes

- Scalar exchange:

$$\mathcal{A}(s,t) \sim \frac{g^2}{s-M^2} + \frac{g^2}{t-M^2}$$

- su -pole:

$$\mathcal{A}(s,t) \sim \frac{M^2 g^2}{(s-M^2)(t-M^2)} \longrightarrow \text{infinite tower of states at } M^2$$

- Vector exchange:

$$\mathcal{A}(s,t) \sim \frac{2t+M^2}{s-M^2} + \frac{2s+M^2}{t-M^2} \longrightarrow s^1 \quad (\text{ violates Regge?})$$

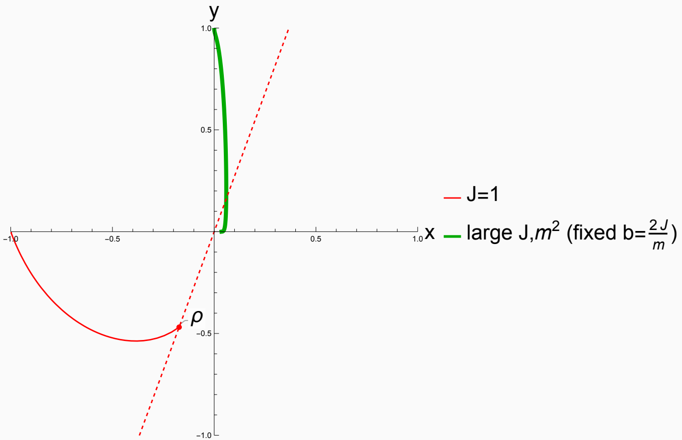
- Improved Vector exchange:

$$\mathcal{A}(s,t) \sim \frac{2s+M^2}{t-M^2}\left(\frac{m_\infty^2}{m_\infty^2 - s}\right) + crossed \longrightarrow s^0$$

Simple lessons from null constraints

Consider two particular combinations of null constraints

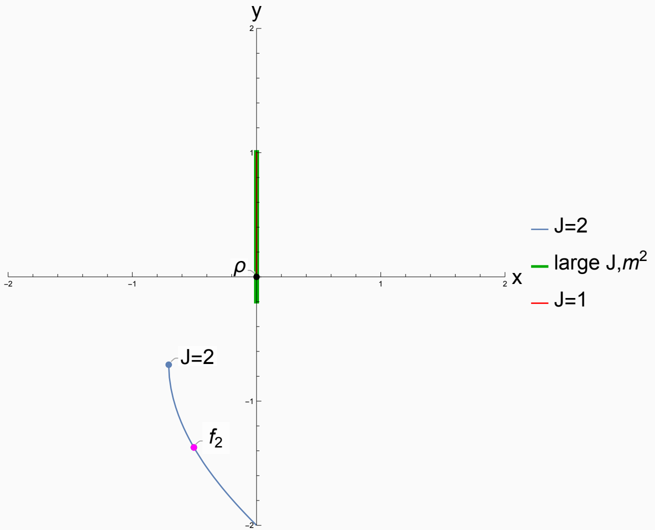
$$x : \vec{n}_1 \cdot \vec{V}_J(m^2)\,, \qquad y : \vec{n}_2 \cdot \vec{V}_J(m^2)$$



- Spin-1 alone are inconsistent
- A Spin-1 can be "fixed" by adding resonances at ∞

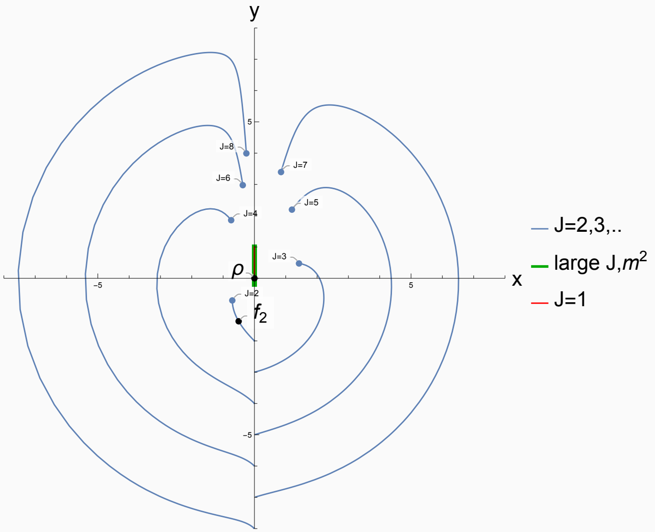
Geometry of null constraints - 2

Consider a *different* combination



- A Spin-2 cannot be "fixed" by adding resonances at ∞ or vectors

Geometry of null constraints - 2



- A Spin-2 cannot be "fixed" by adding resonances at ∞ or vectors
- A Spin-2 requires higher (odd) spins

General lessons

- $J = 1$ must be present at finite mass
- $J \geq 2$ do not need to be present at finite mass
- If we introduce a $J > 1$ spin, it must come with a whole tower of states of increasing spin ($\sim$ *CEMZ*)

Intermezzo: importance of Regge behavior

Complex scalar+gravity+gauge

In absence of gravity one can obtain "forward sum rules"

$$g_{0,2} \sim < \frac{1}{m^4} > \qquad \text{positive!}$$

Graviton exchange

$\Rightarrow 1/t$ pole in the 2-Subtracted Dispersion Relation (2SDR), regular in 1SDR

Photon exchange

$\Rightarrow 1/t$ pole in 1SDR, doesn't contribute to 2SDR

Loss of positivity in 2SDR: [Caron-Huot,Mazac,Rastelli,Simmons-Duffin '21]

$$g_{0,2} \geq -\# G \,,$$

Adding subtractions

- Gravity implies softer Regge behaviour $\Rightarrow$ can impose 1SDRs [Haring,Zhiboedov '22]
- Problem: 1SDR are not sign definite at large mass, spin (charge 0,1 and charge 2 contribute in opposite ways)
- Generically this would make 1SDR useless.
- Let us restrict to (large-N)-like theories and assume (t-channel dominance)

$$\rho_J^{Q=2}(s) = 0 \qquad \text{no states in charge 2-sector}$$

- with this assumption 1SDR can be used:

$$g_{0,2} \geq -\# e^2 \quad \text{(independent of } G\text{!)}$$

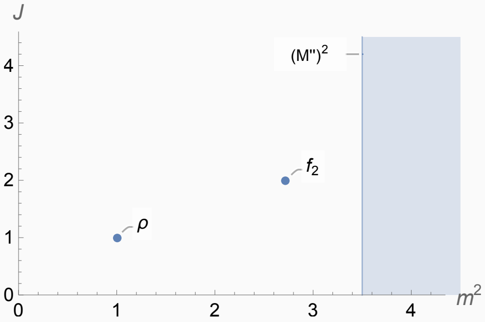
In the limit $e^2 \to 0$ we recover positivity!
[McPeak, Venuti, AV '23]

Back to pions

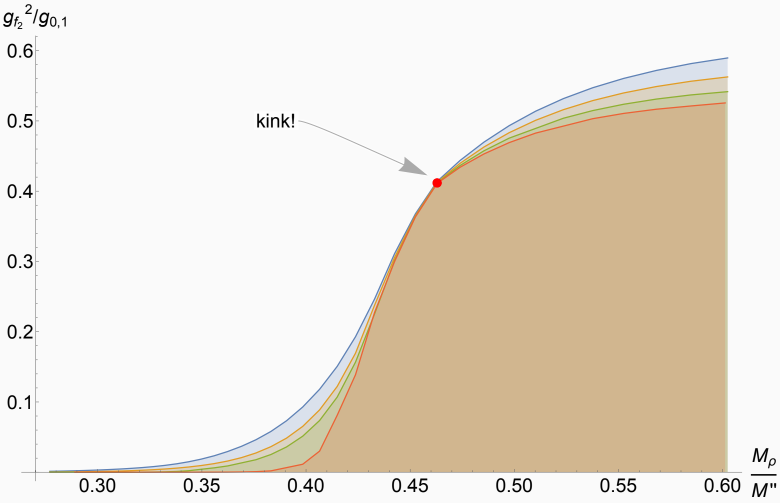
Forcing the f_2

Hot to force the presence of a spin $J \geq 2$ in the system?

Maximize the residue of a spin-2 resonance (aka f_2) [Albert, Henriksson, Rastelli, AV - '23]

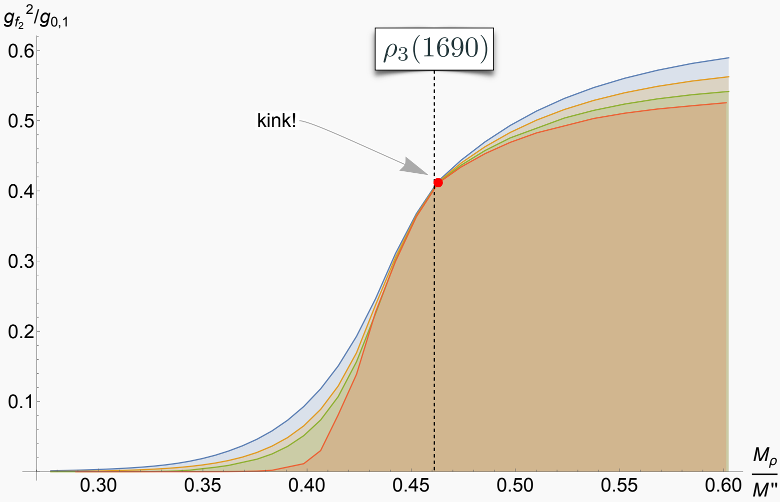


Forcing the f_2



(# null constraints = 56, 79, 106, 137)

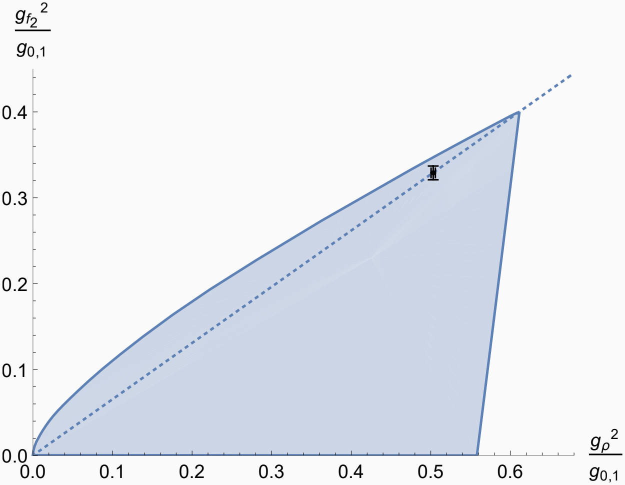
Forcing the f_2



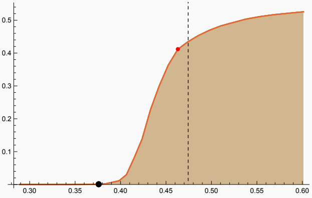
(# null constraints = 56, 79, 106, 137)

Can it be large-N QCD?

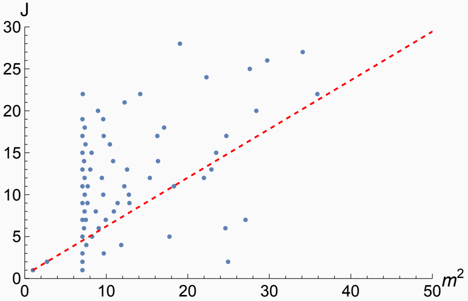
Fix the cut-off $M'' = M_{\text{kink}}$: $\rho-$coupling vs f_2-coupling



Ratio $\frac{g_{f_2}^2}{g_\rho^2}$ close to real word QCD!

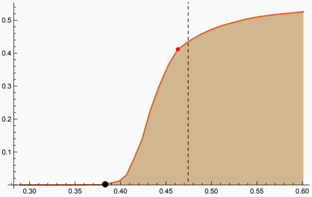


$$\frac{M_\rho}{M''} = 0.376$$

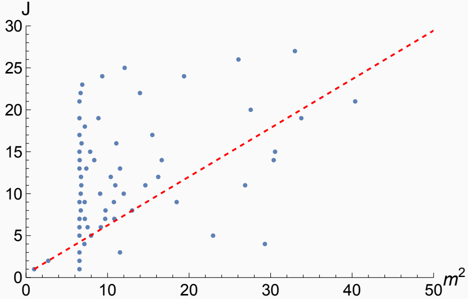


Kink seems to be related to the formation of a Regge trajectory

Chew-Frautschi Plot



$$\frac{M_\rho}{M''}=0.391$$



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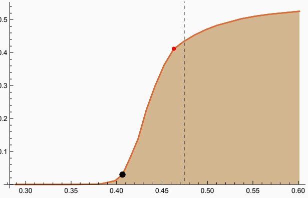
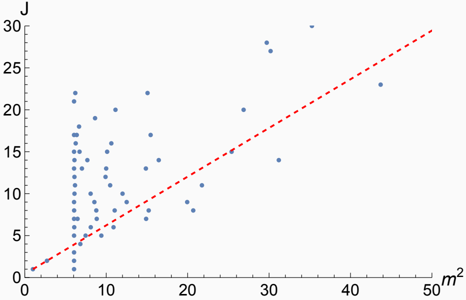


$$\frac{M_\rho}{M''}=0.407$$



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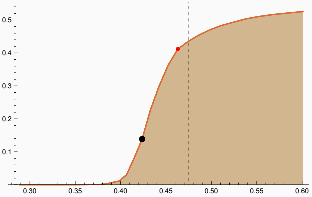

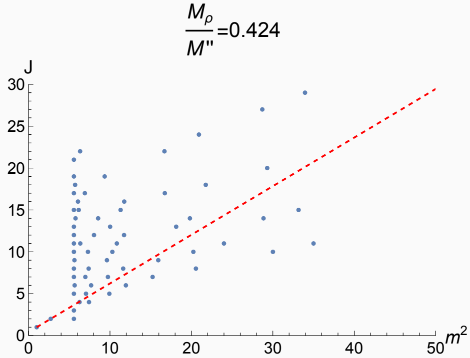

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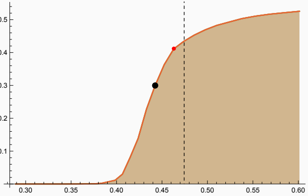
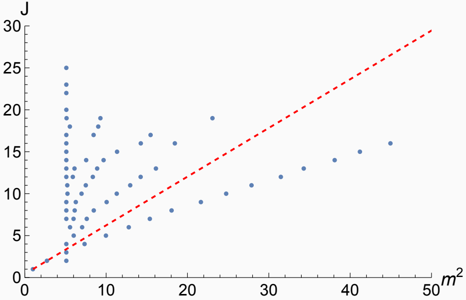


$$\frac{M_\rho}{M''}=0.442$$



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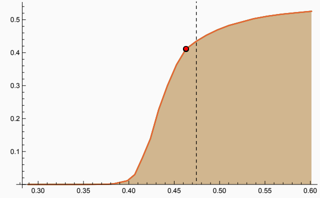
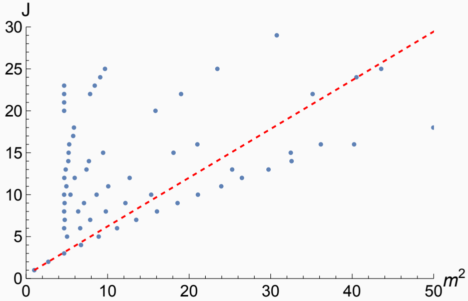


$$\frac{M_\rho}{M''} = 0.463$$



Kink seems to be related to the formation of a Regge trajectory

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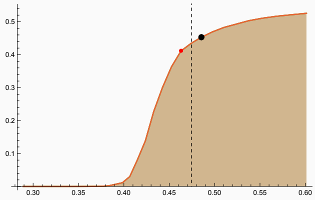
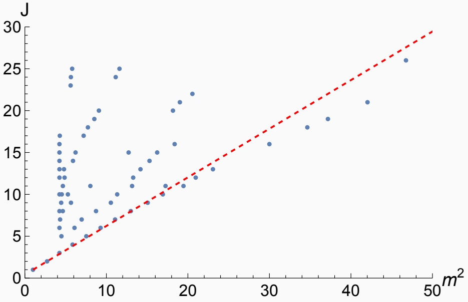
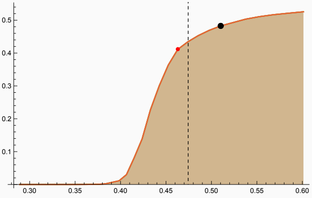


$$\frac{M_\rho}{M''} = 0.485$$

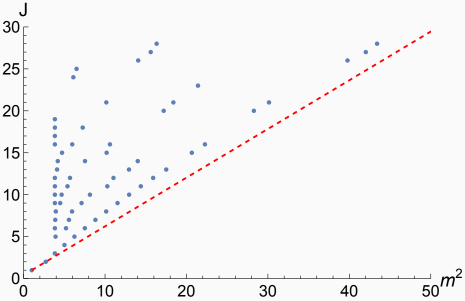


Kink seems to be related to the formation of a Regge trajectory

Chew-Frautschi Plot



$$\frac{M_\rho}{M''} = 0.510$$



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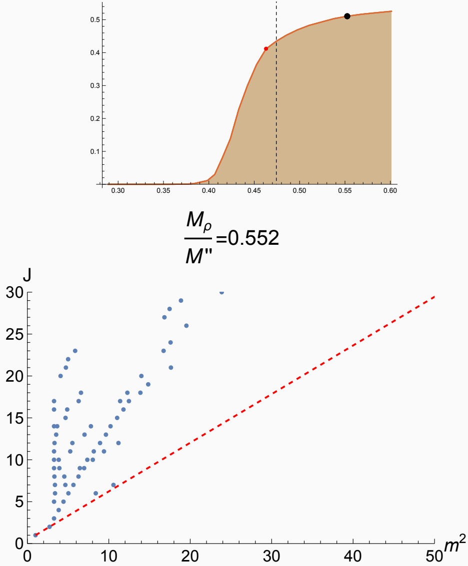

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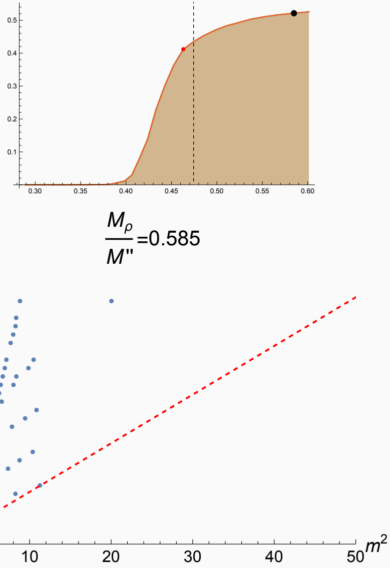
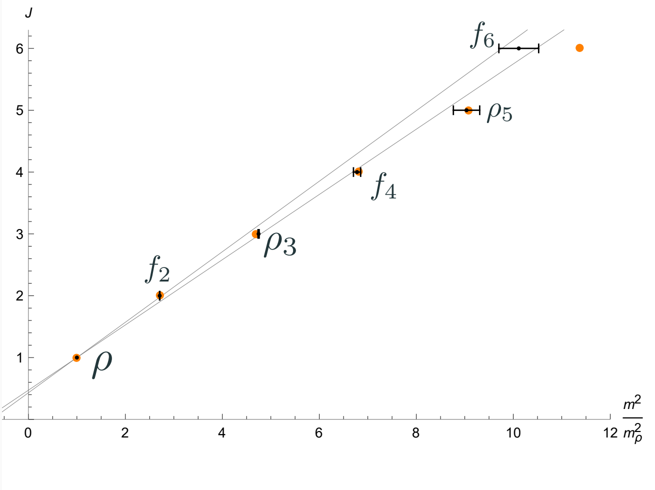


Kink seems to be related to the formation of a Regge trajectory

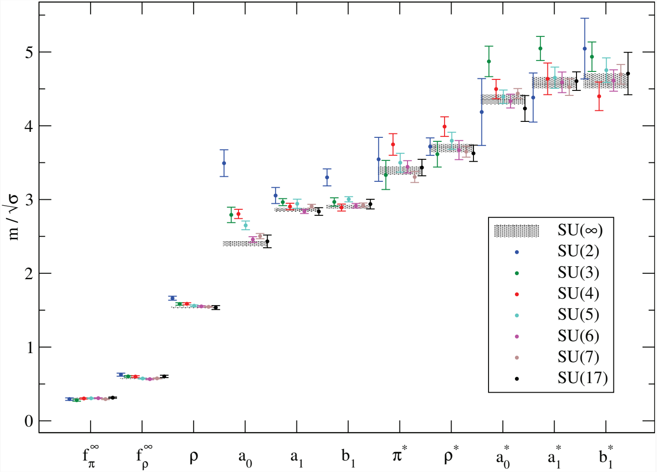
spectrum @ kink VS real world



Impressive agreement with real world QCD...

Is $3 \gg 1$?

Is it surprising that we agree with real world data?



[Bali, Bursa, Castagnini, Collins, Del Debbio, Lucini,Panero '13]

...perhaps not so much!

Can it be large-N QCD?

	spectrum VS real-word	Asymptotically Linear Regge trajectories	daughter trajectories	degenerate ρ, f trajectories
Large-N QCD	✓?	expected	suppressed(?)	✓
Kink solution	✓	X	not seen	✓

Where do we go next?

Imposing the presence of higher spin resonances creates Regge-like trajectories

Similar kinks and spectra for $J = 3, 4$

Maximising the ratio $g_{f_2}^2 / g_{0,1}$ pushes towards the right direction of parameter space.
Perhaps the right answer is behind the corner?

To do better we need to impose more constraints $\Rightarrow$ mixed amplitudes

- mix π^a with the first scalar meson
- glueball scattering (also compare with simulations at large-N)
  [Guerrieri, Hebbar, van Rees '23] [Haring, Zhiboedov '23]
- mix π and ρ-vector [Albert, Henriksson, Rastelli, AV - in progress]

Stay tuned!