Analysis of a two-layer energy balance climate model

Joint work with P. Cannarsa, V. Lucarini, P. Martinez, J. Vancostenoble

Cristina Urbani
cristina.urbani@unimercatorum.it

Kick-off meeting of PRIN project "Some Mathematical Approaches to Climate Change and its Impacts" 22-23/04/2024

Università Mercatorum

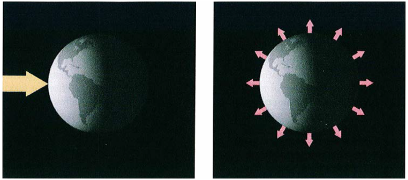
Università telematica delle Camere di Commercio Italiane

Outline

Outline

Energy balance models

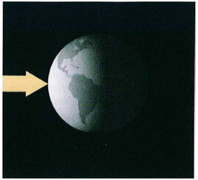
Description of the energy budget which causes the temperature variation

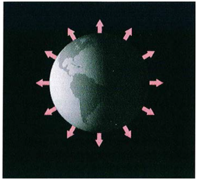


variation of T = absorbed energy - reflected energy

Energy balance models

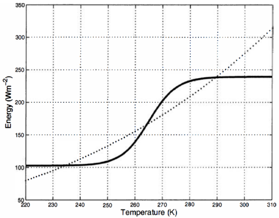
Description of the energy budget which causes the temperature variation





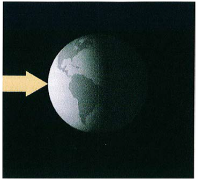
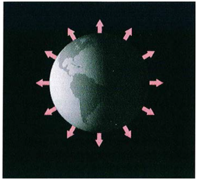
variation of T = absorbed energy - reflected energy

Multistabilty of the climate system:



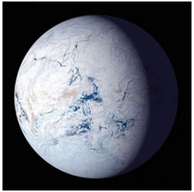
Energy balance models

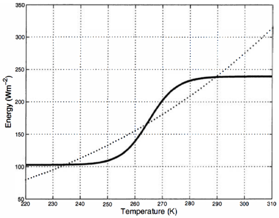
Description of the energy budget which causes the temperature variation

variation of T = absorbed energy - reflected energy

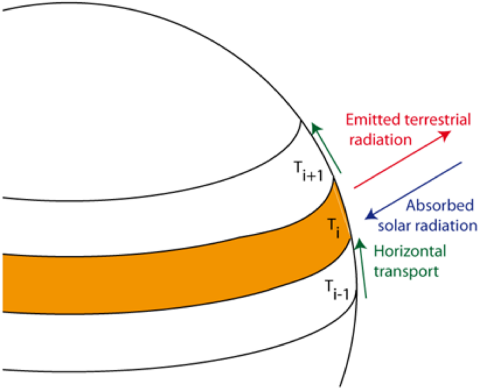
Multistabilty of the climate system:





Energy balance models

Description of the energy budget which causes the temperature variation

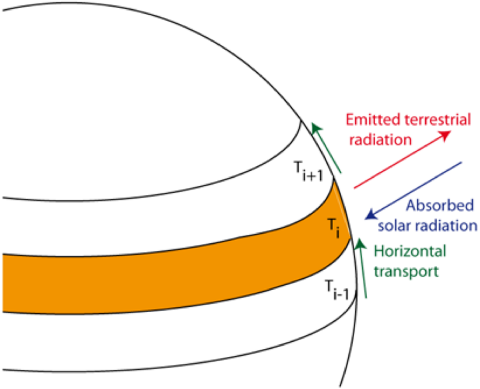


Class of climate models:

- 0-D: $T(t)$ is the mean Earth temperature,
- 1-D: $T(t, \varphi)$, $\varphi \in (-\pi/2, \pi/2)$, is the mean temperature on the latitude circles around the Earth
- 2-D: $T(t, m)$, $m \in S^2$, is the temperature on the Earth surface
- 3-D: $T(t, m, h)$ is the temperature described by the General Circulation Models

Energy balance models

Description of the energy budget which causes the temperature variation



Class of climate models:

- 0-D: $T(t)$ is the mean Earth temperature,
- 1-D: $T(t, \varphi)$, $\varphi \in (-\pi/2, \pi/2)$, is the mean temperature on the latitude circles around the Earth
- 2-D: $T(t, m)$, $m \in S^2$, is the temperature on the Earth surface
- 3-D: $T(t, m, h)$ is the temperature described by the General Circulation Models

Budyko-Sellers models (1969)

The average surface temperature T of the Earth satisfies

$$\text{variation of T} = \text{absorbed energy - reflected energy} + \text{diffusion}$$

Budyko-Sellers models (1969)

The average surface temperature T of the Earth satisfies

variation of T = absorbed energy - reflected energy + diffusion

that is, if $t \geq 0$ and $x = \sin\varphi \in (-1,1)$ (φ colatitude), then

$$\gamma\left[\frac{\partial T(t,x)}{\partial t} - k\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T(t,x)}{\partial x}\right)\right] = \mathcal{R}_a - \mathcal{R}_e$$

Budyko-Sellers models (1969)

The average surface temperature T of the Earth satisfies

variation of T = absorbed energy - reflected energy + diffusion

that is, if $t \geq 0$ and $x = \sin\varphi \in (-1,1)$ (φ colatitude), then

$$\gamma\Big[\frac{\partial T(t,x)}{\partial t} - k\frac{\partial}{\partial x}\Big((1-x^2)\frac{\partial T(t,x)}{\partial x}\Big)\Big] = \mathcal{R}_a - \mathcal{R}_e$$

where:

$\mathcal{R}_a$: average amount of solar radiation per unit area flowing to the surface of the Earth

Budyko-Sellers models (1969)

The average surface temperature T of the Earth satisfies

variation of T = absorbed energy - reflected energy + diffusion

that is, if $t \geq 0$ and $x = \sin\varphi \in (-1, 1)$ (φ colatitude), then

$$\gamma\Big[\frac{\partial T(t,x)}{\partial t} - k\frac{\partial}{\partial x}\Big((1-x^2)\frac{\partial T(t,x)}{\partial x}\Big)\Big] = \mathcal{R}_a - \mathcal{R}_e$$

where:

$\mathcal{R}_a$: average amount of solar radiation per unit area flowing to the surface of the Earth

$\mathcal{R}_e$: average amount of solar radiation per unit area emitted from the Earth (depends on the amount of greenhouse gases, clouds and water vapour in the atmosphere)

Budyko-Sellers models (1969)

The average surface temperature T of the Earth satisfies

$$\text{variation of T} = \text{absorbed energy - reflected energy} + \text{diffusion}$$

that is, if $t \geq 0$ and $x = \sin\varphi \in (-1, 1)$ (φ colatitude), then

$$\gamma\left[\frac{\partial T(t,x)}{\partial t} - k\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T(t,x)}{\partial x}\right)\right] = \mathcal{R}_a - \mathcal{R}_e$$

where:

$\mathcal{R}_a$: average amount of solar radiation per unit area flowing to the surface of the Earth

$\mathcal{R}_e$: average amount of solar radiation per unit area emitted from the Earth (depends on the amount of greenhouse gases, clouds and water vapour in the atmosphere)

γ: effective heat capacity, that is, the required energy to raise the temperature by 1 Kelvin

Budyko-Sellers models (1969)

The average surface temperature T of the Earth satisfies

variation of T = absorbed energy - reflected energy + diffusion

that is, if $t \geq 0$ and $x = \sin\varphi \in (-1, 1)$ (φ colatitude), then

$$\gamma\left[\frac{\partial T(t,x)}{\partial t} - k\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T(t,x)}{\partial x}\right)\right] = \mathcal{R}_a - \mathcal{R}_e$$

where:

$\mathcal{R}_a$: average amount of solar radiation per unit area flowing to the surface of the Earth

$\mathcal{R}_e$: average amount of solar radiation per unit area emitted from the Earth (depends on the amount of greenhouse gases, clouds and water vapour in the atmosphere)

γ: effective heat capacity, that is, the required energy to raise the temperature by 1 Kelvin

$D = \gamma \cdot k$: effective thermal conductivity that controls the efficacy of the latitudinal diffusion

The albedo function

Assumptions:

- the Earth emits as a black body:

$$\mathcal{R}_e(T) = \sigma_B T^4$$

The albedo function

Assumptions:

- the Earth emits as a black body:

$$\mathcal{R}_e(T) = \sigma_B T^4$$

- the absorbed energy is a fraction of the solar radiation Q

$$\mathcal{R}_a(T) = Q(t,x)\beta(T)$$

The albedo function

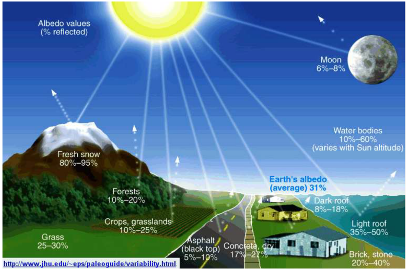
Assumptions:

- the Earth emits as a black body:

$$\mathcal{R}_e(T) = \sigma_B T^4$$

- the absorbed energy is a fraction of the solar radiation Q

$$\mathcal{R}_a(T) = Q(t,x)\beta(T)$$



The albedo function

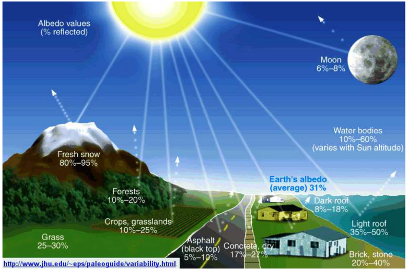
Assumptions:

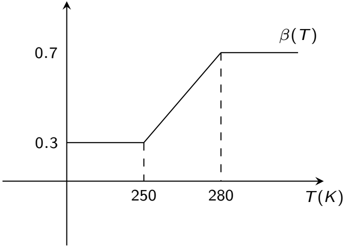
- the Earth emits as a black body:

$$\mathcal{R}_e(T) = \sigma_B T^4$$

- the absorbed energy is a fraction of the solar radiation Q

$$\mathcal{R}_a(T) = Q(t,x)\beta(T)$$





The albedo function

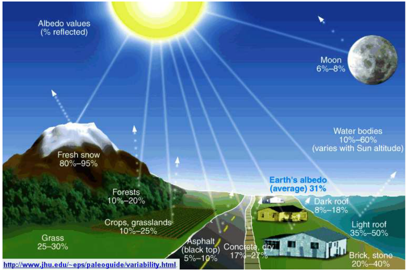
Assumptions:

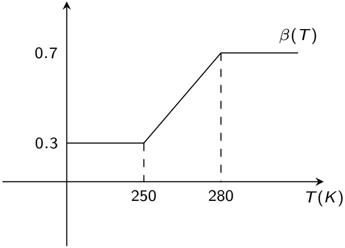
- the Earth emits as a black body:

$$\mathcal{R}_e(T) = \varepsilon_a \sigma_B T^4 \qquad \rightsquigarrow \varepsilon_a \sim 0.6$$

- the absorbed energy is a fraction of the solar radiation Q

$$\mathcal{R}_a(T) = Q(t,x)\beta(T)$$





Outline

A two layer energy balance model

T_a: temperature of an atmosphere layer

T_s: surface temperature of the Earth

A two layer energy balance model

T_a: temperature of an atmosphere layer

T_s: surface temperature of the Earth

$$\begin{cases} \gamma_a \left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x} \left((1-x^2) \frac{\partial T_a}{\partial x} \right) \right] = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \left[\frac{\partial T_s}{\partial t} - k_0 \frac{\partial}{\partial x} \left((1-x^2) \frac{\partial T_s}{\partial x} \right) \right] = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ (1-x^2) \frac{\partial T_a}{\partial x}_{|x=\pm 1} = 0 = (1-x^2) \frac{\partial T_s}{\partial x}_{|x=\pm 1}, \qquad T_a(0,x) = T_a^{(0)}(x), \quad T_s(0,x) = T_s^{(0)}(x) \end{cases}$$

A two layer energy balance model

T_a: temperature of an atmosphere layer

T_s: surface temperature of the Earth

$$\begin{cases} \gamma_a \left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_a}{\partial x}\right)\right] = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \left[\frac{\partial T_s}{\partial t} - k_0 \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_s}{\partial x}\right)\right] = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ (1-x^2)\frac{\partial T_a}{\partial x}_{|x=\pm 1} = 0 = (1-x^2)\frac{\partial T_s}{\partial x}_{|x=\pm 1}, \qquad T_a(0,x) = T_a^{(0)}(x), \quad T_s(0,x) = T_s^{(0)}(x) \end{cases}$$

Exchange of energy between the layers:

- linear terms: non-radiative vertical exchanges of energy due to the action of the geophysical fluids

A two layer energy balance model

T_a: temperature of an atmosphere layer

T_s: surface temperature of the Earth

$$\begin{cases} \gamma_a \left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_a}{\partial x}\right)\right] = -\lambda(T_a - T_s) + \varepsilon_a\sigma_B|T_s|^3T_s - 2\varepsilon_a\sigma_B|T_a|^3T_a + \mathcal{R}_a, \\ \gamma_s \left[\frac{\partial T_s}{\partial t} - k_0 \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_s}{\partial x}\right)\right] = -\lambda(T_s - T_a) - \sigma_B|T_s|^3T_s + \varepsilon_a\sigma_B|T_a|^3T_a + \mathcal{R}_s, \\ (1-x^2)\frac{\partial T_a}{\partial x}_{|x=\pm1} = 0 = (1-x^2)\frac{\partial T_s}{\partial x}_{|x=\pm1}, \qquad T_a(0,x) = T_a^{(0)}(x), \quad T_s(0,x) = T_s^{(0)}(x) \end{cases}$$

Exchange of energy between the layers:

- linear terms: non-radiative vertical exchanges of energy due to the action of the geophysical fluids
- nonlinear terms: emission of infrared radiation by one level being captured by the other layer

A two layer energy balance model

T_a: temperature of an atmosphere layer

T_s: surface temperature of the Earth

$$\begin{cases} \gamma_a \left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_a}{\partial x}\right)\right] = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \left[\frac{\partial T_s}{\partial t} - k_0 \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_s}{\partial x}\right)\right] = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ (1-x^2)\frac{\partial T_a}{\partial x}_{|x=\pm 1} = 0 = (1-x^2)\frac{\partial T_s}{\partial x}_{|x=\pm 1}, \qquad T_a(0,x) = T_a^{(0)}(x), \quad T_s(0,x) = T_s^{(0)}(x) \end{cases}$$

Exchange of energy between the layers:

- linear terms: non-radiative vertical exchanges of energy due to the action of the geophysical fluids
- nonlinear terms: emission of infrared radiation by one level being captured by the other layer

Relevant constants:

- $\lambda \geq 0$: coupling parameter for vertical exchanges
- $\varepsilon_a \in [0,1]$: absorptivity (depends on greenhouse gases CO_2, CH_4)

Well-posedness of the ODE problem

$$
\text{(2LEBM)} \quad \begin{cases} \gamma_a \frac{\partial T_a}{\partial t} = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \frac{\partial T_s}{\partial t} = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ T_a(0) = T_a^{(0)}, \quad T_s(0) = T_s^{(0)} \end{cases}
$$

Well-posedness of the ODE problem

$$
\text{(2LEBM)} \quad \begin{cases}
\gamma_a \frac{\partial T_a}{\partial t} = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\
\gamma_s \frac{\partial T_s}{\partial t} = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\
T_a(0) = T_a^{(0)}, \quad T_s(0) = T_s^{(0)}
\end{cases}
$$

Assumptions:

- $\lambda \geq 0, \quad q > 0, \quad \sigma_B > 0, \quad \varepsilon_a \in (0, 2)$

Well-posedness of the ODE problem

$$(2LEBM) \quad \begin{cases} \gamma_a \frac{\partial T_a}{\partial t} = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \frac{\partial T_s}{\partial t} = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ T_a(0) = T_a^{(0)}, \quad T_s(0) = T_s^{(0)} \end{cases}$$

Assumptions:

- $\lambda \geq 0, \quad q > 0, \quad \sigma_B > 0, \quad \varepsilon_a \in (0,2)$
- $\beta_a, \beta_s : \mathbb{R} \to \mathbb{R}$ globally Lipschitz, $\beta_a \geq 0$ and $\beta_s > 0$, and

$$\mathcal{R}_a = q\beta_a(T_a), \quad \mathcal{R}_s = q\beta_s(T_s).$$

Well-posedness of the ODE problem

$$
\text{(2LEBM)} \quad \begin{cases}
\gamma_a \frac{\partial T_a}{\partial t} = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\
\gamma_s \frac{\partial T_s}{\partial t} = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\
T_a(0) = T_a^{(0)}, \quad T_s(0) = T_s^{(0)}
\end{cases}
$$

Assumptions:

- $\lambda \geq 0, \quad q > 0, \quad \sigma_B > 0, \quad \varepsilon_a \in (0, 2)$
- $\beta_a, \beta_s : \mathbb{R} \to \mathbb{R}$ globally Lipschitz, $\beta_a \geq 0$ and $\beta_s > 0$, and
$$\mathcal{R}_a = q\beta_a(T_a), \quad \mathcal{R}_s = q\beta_s(T_s).$$
- $T_a^{(0)} \geq 0, \quad T_s^{(0)} \geq 0$

Well-posedness of the ODE problem

$$
\text{(2LEBM)} \quad \begin{cases} \gamma_a \frac{\partial T_a}{\partial t} = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \frac{\partial T_s}{\partial t} = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ T_a(0) = T_a^{(0)}, \quad T_s(0) = T_s^{(0)} \end{cases}
$$

Assumptions:

- $\lambda \geq 0, \quad q > 0, \quad \sigma_B > 0, \quad \varepsilon_a \in (0, 2)$
- $\beta_a, \beta_s : \mathbb{R} \to \mathbb{R}$ globally Lipschitz, $\beta_a \geq 0$ and $\beta_s > 0$, and
$$\mathcal{R}_a = q\beta_a(T_a), \quad \mathcal{R}_s = q\beta_s(T_s).$$
- $T_a^{(0)} \geq 0, \quad T_s^{(0)} \geq 0$

Proposition

$(2LEMB)$ has unique solution, defined and bounded for any $t \in [0, +\infty)$. Moreover

$$\forall\, t \in (0, +\infty), \quad T_a(t) > 0 \quad \text{and} \quad T_s(t) > 0.$$

Asymptotic behaviour of solutions

Definition

A C^1 system of ODE on $\mathbb{R}^n$ $\quad \frac{d}{dt}x_i = F_i(x_1, \ldots, x_n) = F_i(x), \; i = 1, \ldots, n \quad$ is *competitive* if

$$\frac{\partial}{\partial x_j} F_i(x) \leq 0 \qquad i \neq j$$

and cooperative if the reverse inequalities hold.

Asymptotic behaviour of solutions

Definition

A C^1 system of ODE on $\mathbb{R}^n$ $\quad \frac{d}{dt}x_i = F_i(x_1, \dots, x_n) = F_i(x), \;\; i = 1, \dots, n$ $\quad$ is *competitive* if

$$\frac{\partial}{\partial x_j} F_i(x) \leq 0 \qquad i \neq j$$

and cooperative if the reverse inequalities hold.

For $\varepsilon_a \in (0, 2)$ and $\lambda \geq 0$

$$T_s \mapsto \frac{1}{\gamma_a}\left[\lambda T_s + \varepsilon_a \sigma_B |T_s|^3 T_s - \lambda T_a - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a(T_a)\right]$$

$$T_a \mapsto \frac{1}{\gamma_s}\left[\lambda T_a - \sigma_B |T_s|^3 T_s - \lambda T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s(T_s)\right]$$

are nondecreasing on $[0, +\infty)$

Asymptotic behaviour of solutions

Definition

A C^1 system of ODE on $\mathbb{R}^n$ $\quad \frac{d}{dt}x_i = F_i(x_1, \ldots, x_n) = F_i(x), \; i = 1, \ldots, n \quad$ is *competitive* if

$$\frac{\partial}{\partial x_j} F_i(x) \leq 0 \qquad i \neq j$$

and cooperative if the reverse inequalities hold.

For $\varepsilon_a \in (0,2)$ and $\lambda \geq 0$

$$T_s \mapsto \frac{1}{\gamma_a}\left[\lambda T_s + \varepsilon_a \sigma_B |T_s|^3 T_s - \lambda T_a - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a(T_a)\right]$$

$$T_a \mapsto \frac{1}{\gamma_s}\left[\lambda T_a - \sigma_B |T_s|^3 T_s - \lambda T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s(T_s)\right]$$

are nondecreasing on $[0, +\infty)$

$\Longrightarrow$ our system is cooperative

Asymptotic behaviour of solutions

Definition

A C^1 system of ODE on $\mathbb{R}^n$ $\quad \frac{d}{dt}x_i = F_i(x_1, \ldots, x_n) = F_i(x), \;\; i = 1, \ldots, n \quad$ is *competitive* if

$$\frac{\partial}{\partial x_j} F_i(x) \leq 0 \qquad i \neq j$$

and cooperative if the reverse inequalities hold.

For $\varepsilon_a \in (0,2)$ and $\lambda \geq 0$

$$T_s \mapsto \frac{1}{\gamma_a}\left[\lambda T_s + \varepsilon_a \sigma_B |T_s|^3 T_s - \lambda T_a - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a(T_a)\right]$$

$$T_a \mapsto \frac{1}{\gamma_s}\left[\lambda T_a - \sigma_B |T_s|^3 T_s - \lambda T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s(T_s)\right]$$

are nondecreasing on $[0, +\infty)$

$\Longrightarrow$ our system is cooperative

$\Longrightarrow$ [Smith] any initial condition $(T_a^{(0)}, T_s^{(0)})$ converges to an equilibrium.

Case $\lambda = 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} \varepsilon_a \sigma_B T_s^4 - 2\varepsilon_a \sigma_B T_a^4 = 0, \\ -\sigma_B T_s^4 + \varepsilon_a \sigma_B T_a^4 + \mathcal{R}_s(T_s) = 0 \end{cases}$$

Case $\lambda = 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} \varepsilon_a \sigma_B T_s^4 - 2\varepsilon_a \sigma_B T_a^4 = 0, \\ -\sigma_B T_s^4 + \varepsilon_a \sigma_B T_a^4 + \mathcal{R}_s(T_s) = 0 \end{cases}$$

that is equivalent to solve

$$\sigma_B(1 - \frac{\varepsilon_a}{2}) T_s^4 = q\beta_s(T_s)$$

Case $\lambda = 0$ and $\mathcal{R}_a = 0$

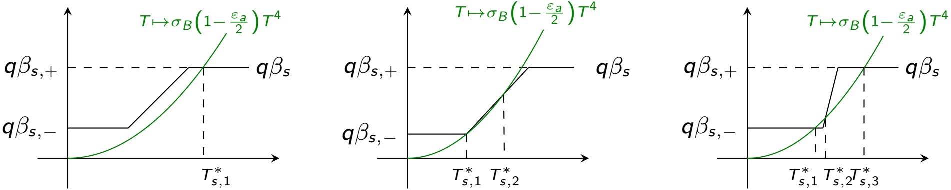
Equilibrium points are solutions of

$$\begin{cases} \varepsilon_a \sigma_B T_s^4 - 2\varepsilon_a \sigma_B T_a^4 = 0, \\ -\sigma_B T_s^4 + \varepsilon_a \sigma_B T_a^4 + \mathcal{R}_s(T_s) = 0 \end{cases}$$

that is equivalent to solve

$$\sigma_B(1 - \frac{\varepsilon_a}{2}) T_s^4 = q\beta_s(T_s)$$

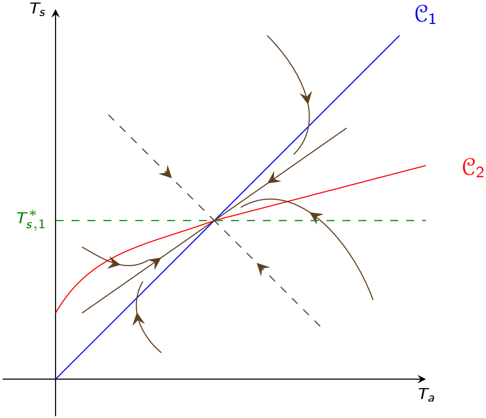
$\implies$ one, two or three possible equilibria



depending on parameters σ_B, ε_a, q.

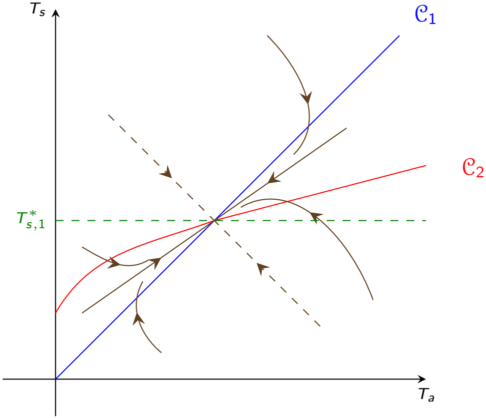
Nature of the equilibrium points ($\lambda = 0$, $\mathcal{R}_a = 0$)

One equilibrium: $(T^*_{a,1}, T^*_{s,1})$ **asymptotically stable**

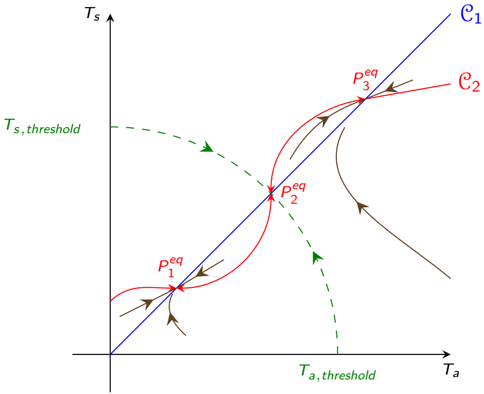


Nature of the equilibrium points ($\lambda = 0$, $\mathcal{R}_a = 0$)

One equilibrium: $(T^*_{a,1}, T^*_{s,1})$ **asymptotically stable**



Three equilibria: $(T^*_{a,1}, T^*_{s,1})$, $(T^*_{a,3}, T^*_{s,3})$ **asymptotically exponentially stable**, $(T^*_{a,2}, T^*_{s,2})$ **unstable**



Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

There exists at most one warm and one cold equilibrium.

Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

There exists at most one warm and one cold equilibrium.

Lemma

If $\lambda = 0$ and $\varepsilon_a \in (0, 2)$ or if $\lambda > 0$ and $\varepsilon_a \in (0, 1.99)$, there exist at most three equilibrium points.

Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

There exists at most one warm and one cold equilibrium.

Lemma

If $\lambda = 0$ and $\varepsilon_a \in (0, 2)$ or if $\lambda > 0$ and $\varepsilon_a \in (0, 1.99)$, there exist at most three equilibrium points.

Lemma

If there exists a warm (resp. a cold) equilibrium, then it is asymptotically exponentially stable.

Dependence of equilibria on ε_a and λ

Proposition

Fixed $\lambda \geq 0$, let $(T_a^{eq,\varepsilon_a^*}, T_s^{eq,\varepsilon_a^*})$ be an asymptotically exponentially stable warm [resp. cold] equilibrium point with $\varepsilon_a = \varepsilon_a^*$.

Then, there exists a unique asymptotically exponentially stable warm [resp. cold] equilibrium $(T_a^{eq,\varepsilon_a}, T_s^{eq,\varepsilon_a})$ for ε_a close to ε_a^* and the following monotonicity property holds

$\varepsilon_a \mapsto T_s^{eq,\varepsilon_a}$ is increasing: the surface temperature of the equilibrium increases as ε_a increases.

Dependence of equilibria on ε_a and λ

Proposition

Fixed $\lambda \geq 0$, let $(T_a^{eq,\varepsilon_a^*}, T_s^{eq,\varepsilon_a^*})$ be an asymptotically exponentially stable warm [resp. cold] equilibrium point with $\varepsilon_a = \varepsilon_a^*$.

Then, there exists a unique asymptotically exponentially stable warm [resp. cold] equilibrium $(T_a^{eq,\varepsilon_a}, T_s^{eq,\varepsilon_a})$ for ε_a close to ε_a^* and the following monotonicity property holds

$\varepsilon_a \mapsto T_s^{eq,\varepsilon_a}$ is increasing: the surface temperature of the equilibrium increases as ε_a increases.

Proposition

Fixed $\varepsilon_a \in (0,1)$, let $(T_a^{eq,\lambda^*}, T_s^{eq,\lambda^*})$ be an asymptotically exponentially stable warm [resp. cold] equilibrium with $\lambda = \lambda^* \geq 0$.

Then, there exists a unique asymptotically exponentially stable equilibrium point $(T_a^{eq,\lambda}, T_s^{eq,\lambda})$ for λ close to λ^* and the following monotonicity properties are satisfied:

- $\lambda \mapsto T_s^{eq,\lambda}$ is decreasing
- $\lambda \mapsto T_a^{eq,\lambda}$ is increasing

Outline

Well-posedness of the PDE problem

$$
\begin{cases}
\gamma_a \left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_a}{\partial x}\right)\right] = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\
\gamma_s \left[\frac{\partial T_s}{\partial t} - k_0 \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_s}{\partial x}\right)\right] = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\
(1-x^2)\frac{\partial T_a}{\partial x}_{|x=\pm 1} = 0 = (1-x^2)\frac{\partial T_s}{\partial x}_{|x=\pm 1}, \qquad T_a(0,x) = T_a^{(0)}(x), \quad T_s(0,x) = T_s^{(0)}(x)
\end{cases}
\tag{2LEBM*}
$$

Well-posedness of the PDE problem

$$
\begin{cases}
\gamma_a\left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_a}{\partial x}\right)\right] = -\lambda(T_a - T_s) + \varepsilon_a\sigma_B|T_s|^3T_s - 2\varepsilon_a\sigma_B|T_a|^3T_a + \mathcal{R}_a, \\
\gamma_s\left[\frac{\partial T_s}{\partial t} - k_0 \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_s}{\partial x}\right)\right] = -\lambda(T_s - T_a) - \sigma_B|T_s|^3T_s + \varepsilon_a\sigma_B|T_a|^3T_a + \mathcal{R}_s, \\
(1-x^2)\frac{\partial T_a}{\partial x}_{|x=\pm1} = 0 = (1-x^2)\frac{\partial T_s}{\partial x}_{|x=\pm1}, \qquad T_a(0,x) = T_a^{(0)}(x), \quad T_s(0,x) = T_s^{(0)}(x)
\end{cases}
\tag{2LEBM*}
$$

Assumptions:

- $\lambda \geq 0, \quad q_a, q_s > 0, \ q_a, q_s \in L^\infty(I), \quad \sigma_B > 0, \quad \varepsilon_a \in (0,1)$
- $\beta_a, \beta_s : \mathbb{R} \to \mathbb{R}$ globally Lipschitz, $\beta_a \geq 0$ and $\beta_s > 0$, and

$$\mathcal{R}_a = q_a(x)\beta_a(T_a), \quad \mathcal{R}_s = q_s(x)\beta_s(T_s).$$

Well-posedness of the PDE problem

$$\begin{cases} \gamma_a \left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_a}{\partial x}\right)\right] = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \left[\frac{\partial T_s}{\partial t} - k_0 \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial T_s}{\partial x}\right)\right] = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ (1-x^2)\frac{\partial T_a}{\partial x}_{|x=\pm 1} = 0 = (1-x^2)\frac{\partial T_s}{\partial x}_{|x=\pm 1}, \qquad T_a(0,x) = T_a^{(0)}(x), \quad T_s(0,x) = T_s^{(0)}(x) \end{cases} \tag{2LEBM*}$$

Assumptions:

- $\lambda \geq 0$, $\quad q_a, q_s > 0,\ q_a, q_s \in L^\infty(I)$, $\quad \sigma_B > 0$, $\quad \varepsilon_a \in (0,1)$
- $\beta_a, \beta_s : \mathbb{R} \to \mathbb{R}$ globally Lipschitz, $\beta_a \geq 0$ and $\beta_s > 0$, and

$$\mathcal{R}_a = q_a(x)\beta_a(T_a), \quad \mathcal{R}_s = q_s(x)\beta_s(T_s).$$

We can rewrite the system in vectorial form

$$\begin{cases} \boldsymbol{T}'(t,x) = \boldsymbol{A}\boldsymbol{T}(t,x) + \boldsymbol{F}(t,x,\boldsymbol{T}(t,x)), \\ (1-x^2)\boldsymbol{T}(t,x)\big|_{x=\pm 1} = 0, \\ \boldsymbol{T}(0,x) = \boldsymbol{T}^{(0)}(x). \end{cases}$$

Well-posedness of the PDE problem

Spaces:

- $\boldsymbol{H} := H \times H,$ $\qquad H := L^2(I)$
- $\boldsymbol{V} := V \times V,$ $\qquad V := \{u \in H : \text{ abs. cont. in } I, \sqrt{1-x^2}u_x \in L^2(I)\}$
- $\boldsymbol{D(\mathcal{A})} := D(A) \times D(A),$ $\qquad D(A) := \{u \in V : (1-x^2)u_x \in H^1(I)\}.$

Well-posedness of the PDE problem

Spaces:

- $\boldsymbol{H} := H \times H$, $\quad H := L^2(I)$
- $\boldsymbol{V} := V \times V$, $\quad V := \{u \in H : \text{ abs. cont. in } I, \sqrt{1-x^2}u_x \in L^2(I)\}$
- $\boldsymbol{D(\mathcal{A})} := D(A) \times D(A)$, $\quad D(A) := \{u \in V : (1-x^2)u_x \in H^1(I)\}$.

Proposition

$\boldsymbol{T}^{(0)} \in \boldsymbol{H} \implies$ (2LEBM*) has a unique local mild solution

$$\boldsymbol{T} \in C((0, \mathfrak{T}(\boldsymbol{T}^{(0)}), \boldsymbol{V}).$$

Moreover, either $\mathfrak{T}(\boldsymbol{T}^{(0)}) = +\infty$ or

$$\lim_{t \to \mathfrak{T}(\boldsymbol{T}^{(0)})^-} \|\boldsymbol{T}\|_{\boldsymbol{V}} = +\infty.$$

Furthermore, $\exists\, \theta \in (0,1)$ such that for any $0 < \eta < \tau < \mathfrak{T}(\boldsymbol{T}^{(0)})$

$$\boldsymbol{T} \in C^{\theta}([\eta, \tau], \boldsymbol{D(\mathcal{A})}) \cap C^{1+\theta}([\eta, \tau], \boldsymbol{H}).$$

Comparison principles for the PDE problem

Bounds:

- $\boldsymbol{H}$: $\boldsymbol{T}^{(0)} \in \boldsymbol{H} \Longrightarrow \boldsymbol{T}$ bounded in $\boldsymbol{H}$ (in the existence time inteval)

Comparison principles for the PDE problem

Bounds:

- $\boldsymbol{H}$: $\boldsymbol{T}^{(0)} \in \boldsymbol{H} \Longrightarrow \boldsymbol{T}$ bounded in $\boldsymbol{H}$ (in the existence time inteval)
- $\boldsymbol{L}^{\infty}$: $\boldsymbol{T}^{(0)} \in \boldsymbol{D}(\mathcal{A}) \subset \boldsymbol{L}^{\infty}(\boldsymbol{I}) \Longrightarrow \boldsymbol{T}$ bounded in $\boldsymbol{L}^{\infty}(\boldsymbol{I})$ (in the existence time inteval)

Comparison principles for the PDE problem

Bounds:

- $\boldsymbol{H}$: $\boldsymbol{T^{(0)}} \in \boldsymbol{H} \implies \boldsymbol{T}$ bounded in $\boldsymbol{H}$ (in the existence time inteval)
- $\boldsymbol{L^\infty}$: $\boldsymbol{T^{(0)}} \in \boldsymbol{D(\mathcal{A})} \subset \boldsymbol{L^\infty(I)} \implies \boldsymbol{T}$ bounded in $\boldsymbol{L^\infty(I)}$ (in the existence time inteval)
- $\boldsymbol{V}$: $\boldsymbol{T^{(0)}} \in \boldsymbol{H} \implies \boldsymbol{T} \in \boldsymbol{D(\mathcal{A})}$ in small time $\implies \boldsymbol{T}$ bounded in $\boldsymbol{V} \implies$ global existence.

Comparison principles for the PDE problem

Bounds:

- H: $\boldsymbol{T}^{(0)} \in \boldsymbol{H} \implies \boldsymbol{T}$ bounded in $\boldsymbol{H}$ (in the existence time inteval)
- L^∞: $\boldsymbol{T}^{(0)} \in \boldsymbol{D}(\mathcal{A}) \subset \boldsymbol{L}^\infty(\boldsymbol{I}) \implies \boldsymbol{T}$ bounded in $\boldsymbol{L}^\infty(\boldsymbol{I})$ (in the existence time inteval)
- V: $\boldsymbol{T}^{(0)} \in \boldsymbol{H} \implies \boldsymbol{T} \in \boldsymbol{D}(\mathcal{A})$ in small time $\implies \boldsymbol{T}$ bounded in $\boldsymbol{V} \implies$ global existence.

Theorem (comparison principle)

Let $\boldsymbol{T}^{(0)}, \tilde{\boldsymbol{T}}^{(0)} \in \boldsymbol{D}(\mathcal{A})$ be nonnegative. Then

$$\boldsymbol{T}^{(0)} \leq \tilde{\boldsymbol{T}}^{(0)} \quad \implies \quad \boldsymbol{T}(t) \leq \tilde{\boldsymbol{T}}(t), \qquad \text{for all } t \geq 0.$$

Comparison principles for the PDE problem

Bounds:

- H: $T^{(0)} \in H \implies T$ bounded in H (in the existence time inteval)
- L^∞: $T^{(0)} \in D(\mathcal{A}) \subset L^\infty(I) \implies T$ bounded in $L^\infty(I)$ (in the existence time inteval)
- V: $T^{(0)} \in H \implies T \in D(\mathcal{A})$ in small time $\implies T$ bounded in $V \implies$ global existence.

Theorem (comparison principle)

Let $T^{(0)}, \tilde{T}^{(0)} \in D(\mathcal{A})$ be nonnegative. Then

$$T^{(0)} \leq \tilde{T}^{(0)} \quad \implies \quad T(t) \leq \tilde{T}(t), \qquad \text{for all } t \geq 0.$$

Theorem (comparison principle, supersol. and subsol.)

Let $T^{(0)} \in D(\mathcal{A})$ be nonnegative and let T^+ and T^- be a supersolution and a subsolution of (2LEBM*), respectively. Then

$$T^{(0)} \leq T^+(0) \quad \implies \quad T(t) \leq T^+(t), \qquad \text{for all } t \geq 0,$$

and

$$T^{(0)} \geq T^-(0) \quad \implies \quad T(t) \geq T^-(t), \qquad \text{for all } t \geq 0.$$

Consequences of comparison principles for the PDE problem

- **positivity**:

$$q(x) \geq q_{min} > 0$$

and consider $\boldsymbol{T}^{min}$ solution on the ODE with

$$\boldsymbol{T}^{min,0} = (0,0), \qquad q = q_{min}.$$

Lemma (positivity)

Let $\boldsymbol{T}^{(0)} \in \boldsymbol{D}(\mathcal{A})$ and $\boldsymbol{T}$ the associated solution of (2LEBM*). Then

$$\boldsymbol{T}^{(0)} \geq 0 \quad \Longrightarrow \quad \boldsymbol{T}(t) \geq \boldsymbol{T}^{min}(t) > 0, \qquad \text{for all } t > 0.$$

Consequences of comparison principles for the PDE problem

- **positivity**:

$$q(x) \geq q_{min} > 0$$

and consider $\boldsymbol{T}^{min}$ solution on the ODE with

$$\boldsymbol{T}^{min,0} = (0,0), \qquad q = q_{min}.$$

Lemma (positivity)

Let $\boldsymbol{T}^{\mathbf{(0)}} \in \boldsymbol{D(\mathcal{A})}$ and $\boldsymbol{T}$ the associated solution of (2LEBM*). Then

$$\boldsymbol{T}^{\mathbf{(0)}} \geq 0 \quad \Longrightarrow \quad \boldsymbol{T}(t) \geq \boldsymbol{T}^{min}(t) > 0, \qquad \text{for all } t > 0.$$

- **existence of equilibria**:

Lemma (equilibria)

(2LEBM*) admits at least one equilibrium point $\boldsymbol{T}^{stat}$. There exist a minimal and a maximal equilibrium points, $\boldsymbol{T}^{stat,min}$ and $\boldsymbol{T}^{stat,max}$, respectively, such that any other equilibrium is such that

$$\boldsymbol{T}^{stat,min} \leq \boldsymbol{T}^{stat} \leq \boldsymbol{T}^{stat,max}.$$

Outline

Ongoing work and future directions

Ongoing work:

- Maximum principle ($\Longrightarrow$ [Smith] convergence to an equilibrium from almost every initial conditions)
- Stability of stationary points (principal eigenvalue)
- Uniqueness of the (coldest and warmest) equilibrium points
- Finite number of equilibria
- Stochastic 2-layer energy balance model (J. Broecker, P. Cannarsa, G. Carigi, T. Kuna)

Ongoing work and future directions

Ongoing work:

- Maximum principle ($\Longrightarrow$ [Smith] convergence to an equilibrium from almost every initial conditions)
- Stability of stationary points (principal eigenvalue)
- Uniqueness of the (coldest and warmest) equilibrium points
- Finite number of equilibria
- Stochastic 2-layer energy balance model (J. Broecker, P. Cannarsa, G. Carigi, T. Kuna)

Plan of future work:

- Inverse problems for parameter reconstruction (ε_a, λ, q)
- Extension to a variable solar radiation $Q(t,x) = r(t)q(x)$, with r positive and periodic - allowing for seasonal cycle
- Extension to a space-dependent absorptivity $\varepsilon_a(x)$

Ongoing work and future directions

Ongoing work:

- Maximum principle ($\Longrightarrow$ [Smith] convergence to an equilibrium from almost every initial conditions)
- Stability of stationary points (principal eigenvalue)
- Uniqueness of the (coldest and warmest) equilibrium points
- Finite number of equilibria
- Stochastic 2-layer energy balance model (J. Broecker, P. Cannarsa, G. Carigi, T. Kuna)

Plan of future work:

- Inverse problems for parameter reconstruction (ε_a, λ, q)
- Extension to a variable solar radiation $Q(t,x) = r(t)q(x)$, with r positive and periodic - allowing for seasonal cycle
- Extension to a space-dependent absorptivity $\varepsilon_a(x)$

P. Cannarsa, V. Lucarini, P. Matrinez, C.U., J. Vancostenoble, *Analysis of a two-layer energy balance model: long time behaviour and greenhouse effect*, CHAOS, vol. 33, pp. 113111 (2023)


GRAZIE! THANK YOU!
MERCI! DANKE!