Climate, macroweather and statistical properties for a spatially heterogeneous 1D-EBM

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Talk overview

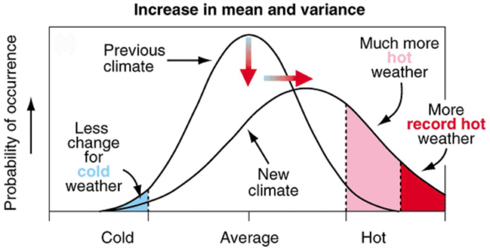


Fig. IPCC report, 2001.

1. Towards a definition of (macro)-weather and climate
2. Energy Balance Models (EBMs)
    - Intro: 0D-EBMs
    - Space heterogeneous 1D-EBM with local bistability
    - Early warning indicators

Weather, climate and macroweather

Weather vs climate

> *'The climate is what you expect; the weather is what you get'* — Mark Twain

Weather is the state of the atmosphere during a short period of time, it involves variables as temperature, humidity, rain, wind.

Climate is a statistical description of relevant quantities over a period of time ranging from decades to thousands of years.

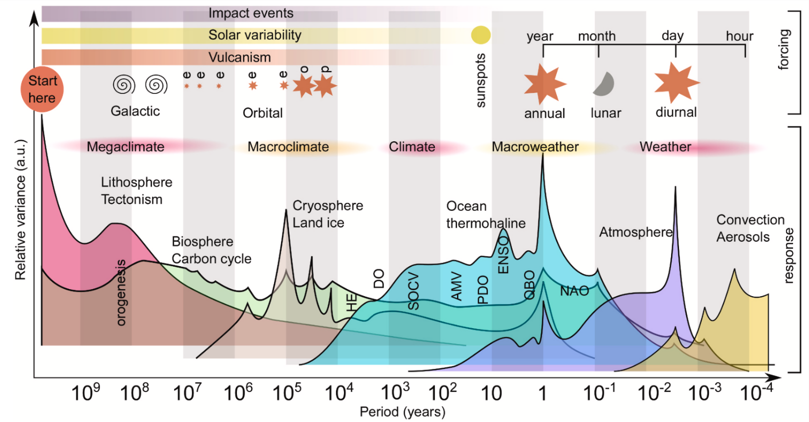


Fig. 1. Periodicities in Earth dynamics.

Random dynamical systems, Macroweather, Climate

- Weather. Fast scale, deterministic, typical timescale $\tau = 1$ day.

- **Macroweather**. Weather component has been averaged, stochastic, $\tau = 6$ months.

- Climate. Long time behaviour, $\tau = 30 - 100$ years.

- Weather. Fast scale, deterministic, typical timescale $\tau = 1$ day.
- **Macroweather**. Weather component has been averaged, stochastic, $\tau = 6$ months.
- Climate. Long time behaviour, $\tau = 30 - 100$ years.

Definition

(i) The macro-weather dynamics is $U_{q,\omega}(s,t)$

(ii) The climate dynamics is $\mathbb{P}^q_{s,t}$

(iii) The climate is $\mu_q(t)$

Weather dynamics

Definition

(i) The weather dynamics is $U_{q,\omega}(s,t)$

(X,d) metric space, $(U_{q,\omega}(s,t))_{s\leq t}$ non-autonomous random dynamical system, $q = q(\varepsilon t)$ (slow) and $\omega = \omega(t)$ (fast):

- $U_{q,\omega}(s,t)\colon X \to X$
- $U_{q,\omega}(s,s)(x) = x$, $\forall x \in X$
- $\left(U_{q,\omega}(r,t) \circ U_{q,\omega}(s,r)\right)(x) = U_{q,\omega}(s,t)(x)$

Example: $u = u(x,t)$ temperature on monthly timescale

$$\partial_t u = \kappa \Delta u + q(\varepsilon t) + R_a(u) - R_e(u) + \xi(t,\omega)$$

ξ space-time white noise (fast time-scale), $q(\varepsilon t)$ greenhouse effect due to CO_2. For $s < t$, $u(x,t) = u_t(x)$ depends on: u_s, ω, q. Set:

$$u_t = U_{q,\omega}(s,t)(u_s)$$

EBMs as models for macroweather - Is the timescale correct?

Let's simplify the models and consider

$$C\frac{dT}{dT} = Q\alpha - A - B(T - 273). \qquad (2.15)$$

Given an initial condition $T(0)$, the solution converges to the equilibrium with a *relaxation time*

$$\tau_0 = C/B \approx 30 \text{ days.}$$

> Now imagine that the temperature $T(t = 0)$ is out of equilibrium. The differential equation (2.15) has a solution
>
> $$T(t) = T_{eq} + (T(0) - T_{eq})e^{-t/\tau_0} \qquad (2.16)$$
>
> that can be demonstrated by substitution into (2.15). The decay time constant is given by $\tau_0 = C/B$. The perturbed climate relaxes to the equilibrium solution with a decay time of τ_0, which, for the all-land planet, is about $2.5 \times 10^7\text{s} \approx 30$ days as shown in Figure 2.2. It

Fig. 2. Pag. 32, [NK17].

Climate dynamics

Definition

(ii) Climate dynamics is $(\mathbb{P}^q_{s,t})_{s\leq t}$

- We call $\mu_s \in Pr(X, \mathcal{B}(X))$ *state*.

The linear operator $\mathbb{P}^q_{s,t}$ is given by:

$$\int_X \phi(y)(\mathbb{P}^q_{s,t}\mu_s)(dy) := \mathbb{E}\left[\phi\left(U_{q,\omega}(s,t)(x)\right)\mu_s(dx)\right],$$

with $\mu_s \in Pr(X, \mathcal{B}(X))$.

- $\mathbb{P}^q_{s,t}$ doesn't depend on ω.

What is climate

Definition

(iii) The climate is $(\mu_q(t))_{t\geq 0}$

- $\mu_q(t)$ is the invariant measure on (X, d) for $\mathbb{P}^q_{s,t}$, i.e.

$$\mu_q(t) = \mathbb{P}^q_{s,t}\mu_q(s), \quad \forall s \leq t, \, \forall q$$

- Existence is easy. Uniqueness is difficult, when it doesn't hold:

$$\mu_q(t) = \lim_{s \to -\infty} \mathbb{P}^q_{s,t}\lambda,$$

where λ is a "natural measure" (e.g. $X =$ torus, $\lambda =$ uniform measure).

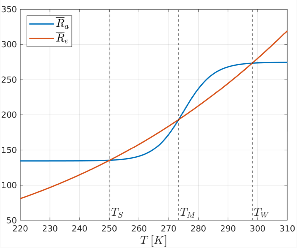
Energy balance Models (EBMs)

Zero-dimensional energy balance model (0D-EBM)

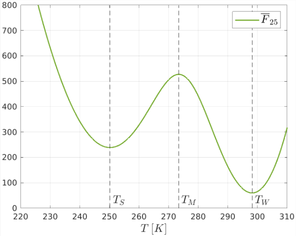
$T = T(t)$ global mean temperature evolves as:

$$\frac{dT}{dt} = \overline{R}_a(T) - \overline{R}_e(T) = \overline{Q}_0\beta(T) + q - \varepsilon_0\sigma_0 T^4, \; T(0) = T_0,$$

- $\overline{Q}_0$ solar radiation, β co-albedo, $q > 0$ CO_2 concentration
- The number of fixed points depends on q
- Fixed points are local extremum points of $\overline{F}_q$ s.t $\overline{F}'_q = \overline{R}_e - \overline{R}_a$.



Radiation balance, $q = 25$.



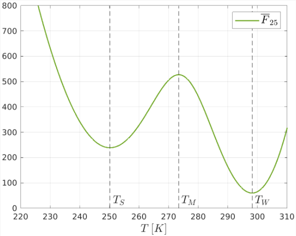
Potential $\overline{F}_{25}$.

Stochastic 0D-EBM

Consider the SDE

$$dT_t = \left(\overline{R}_a(T) - \overline{R}_e(T)\right) dt + \sigma dW_t,$$

with $(W_t)_t$ BM and $\sigma > 0$.



Theorem

Under coercivity and regularity assumptions, ∃! *invariant measure*

$$\overline{\nu}(dT) = \frac{1}{Z} \exp\left(-\frac{2}{\sigma^2}\overline{F}_q(T)\right) dT.$$

Remark: $\overline{\nu}$ is concentrated on global minimum points of $\overline{F}_q$.

Problem: this model is useful for paleoclimate, not for current climate change.

- No global bifurcation closeby
- Evidences for local bistability

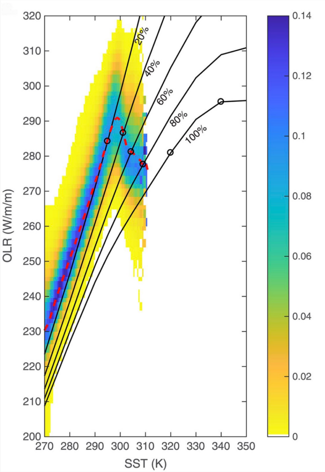


Fig. 1: Outgoing longwave radition (OLR) as a function of sea-surface temperature (SST) [DG18, Fig. 2].

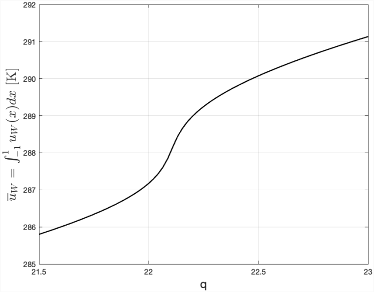
One-dimensional energy balance model (1D-EBM)

$u = u(x,t)$ temperature, $x = \sin(\phi)$, $\phi =$ latitude

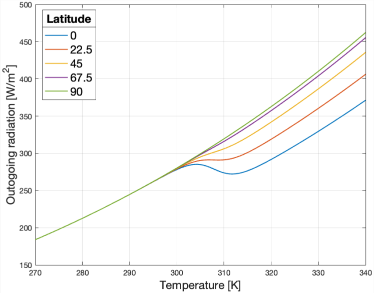
$$\partial_t u = \partial_x \left(\kappa(x) u_x\right) + R_a(x,u) + q - R_e(x,u)$$

$$u(x,0) = u_0, \quad \partial_x u(-1,t) = \partial_x u(1,t) = 0,$$

- R_e modelling bistability in tropical region
- Three steady-state solution u_S, u_M, u_W.



Bifurcation diagram around u_W in $(q, \bar{u}_W)$ plane, $\bar{u}_W = \|u_W\|_1$.



$R_e = R_e(x,u)$

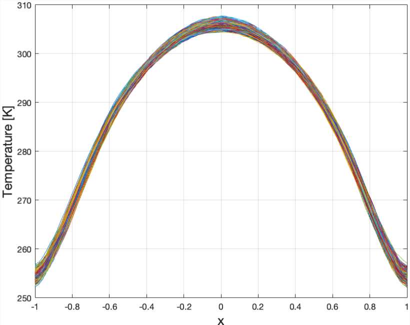
Stochastic 1D-EBM

Consider

$$\partial_t u = \partial_x (\kappa(x) u_x) + R_a(x,u) + q - R_e(x,u) + \sigma dW_t$$

$$u(x,0) = u_W, \quad \partial_x u(-1,t) = \partial_x u(1,t) = 0,$$

- u_W warm climate
- $(W_t)_t$ cylindrical Wiener process
- $\sigma \gg 0$ noise-intensity



Early warning indicators

$U = (u_{ij})_{ij} \in \mathbb{R}^{n\times m}$, $(x_i)_{i=1,\dots,n}$, $(t_j)_{j=1,\dots,m}$, $u_{ij} = u(x_i, t_j)$ numerical approximation of

$$\partial_t u = \partial_x \left(\kappa(x) u_x\right) + R_a(x,u) + q - R_e(x,u) + \sigma dW_t$$
$$u(0,x) = u_W \quad \partial_x u(t,-1) = \partial_x u(t,1) = 0,$$

u_w warm climate.

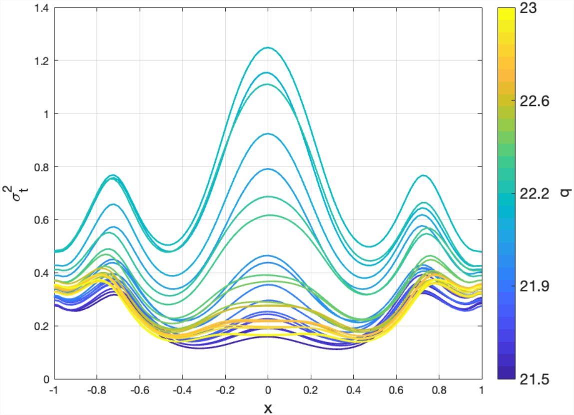
Time variance

$$\sigma_t^2(x_i) = \frac{1}{m}\sum_{j=1}^{m} \left(u_{ij} - \overline{u}_i\right)^2,$$

where $\overline{u}_i = \frac{1}{m}\sum_{j=1}^{m} u_{ij}$.

Time variance

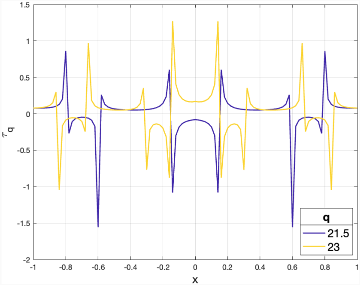
$$\sigma_t^2(x_i) = \frac{1}{m}\sum_{j=1}^{m}(u_{ij} - \overline{u}_i)^2$$



Relaxation time

Set $R(x,u) = R_a + q - R_e$. Define the *relaxation time* as:

$$\tau_q(x_i) := \frac{1}{\partial_u R_{|u=u_W(x_i) \atop x=x_i}}.$$



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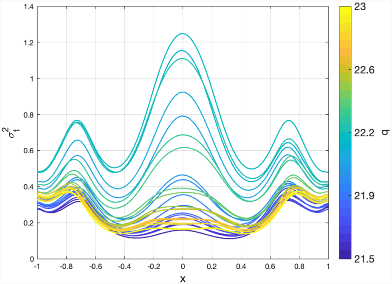

Potential $\overline{F}_{25}$.

Conclusions

- Macroweather as solution of stochastic equation (SDE or SPDE).
- Climate as invariant measure
- Space heterogeneous 1D-EBM is a dynamical system (DS) given by a continuum of interlaced DSs.
- The global system tends to a stable fixed point.
- The restriction of the stable fixed point can be locally unstable w.r.t. the uncoupled part.
- Instability areas $\longleftrightarrow$ variance increase.

Open problems

- Trichotmoy is a simplification.
- Rigorous mathematical analysis of the space-heterogeneous 1D-EBM.
- Explain why some areas lose stability.

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Thanks for the attention!

1D-EBM: variational setting

Let $\mathcal{R}$ s.t. $\partial_u \mathcal{R} = R_e - R_a - q$. Consider the functional

$$F_q(u) := \frac{\kappa}{2} \int_{-1}^{1} \left[u'(x)\right]^2 dx + \int_{-1}^{1} \mathcal{R}(x, u(x)) dx,$$

and the space

$$H^1 := \left\{ u(x) = u_{-1} + \int_{-1}^{x} v(y) dy, \quad u, v \in L^2(-1,1) \right\}.$$

The variational problem consists in studying

$$\inf \left\{ F_q(u) \;\mid u \in H^1,\ u \geq 0 \right\}.$$

Key fact: minimum points of F_q are steady-state solutions for the 1D-EBM, i.e.

$$0 = \kappa \Delta u + Q_0(x)\beta(u) + q - \sigma_0 \varepsilon_0 u^4,$$
$$0 = u'(-1) = u'(1), \quad u \geq 0.$$

Stochastic 1D-EBM: invariant measure

Consider the stochastic 1D-EBM

$$\partial_t u = \kappa \Delta u + R_a(x,u) - R_e(u) + \varepsilon \eta_t,$$

where $\varepsilon > 0$ and $(\eta_t)_{t \geq 0}$ is a space-time white noise.

Theorem (Da Prato, 2004)

If $\mathcal{R} = \mathcal{R}(x,u)$ is regular and coercive, then there exists an unique invariant measure ν. It is formally given by:

$$\nu(du) = \frac{1}{Z} \exp\left(-\frac{2}{\varepsilon^2} F_q(u)\right) du$$

Remark: the invariant measure is concentrated on global minimum points of F_q.