A Mean-Field Game network model for urban planning

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Workshop

Some Mathematical Approaches to Climate Change and its Impacts

Pisa, 22/04/24

joint work with F. Camilli and A. Festa

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We study the model on a **network** Γ rather than in $\mathbb{T}^d$ $\Rightarrow$ more suitable for an urban planning structure.

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    $\pi_\alpha(y) = \ell_\alpha^{-1}(y\nu_j + (\ell_\alpha - y)\nu_i)$, $y \in [0, \ell_\alpha]$

    with ℓ_α is the length of the edge. $\mathcal{A}_i = \{\alpha \in \mathcal{A} : \nu_i \in \Gamma_\alpha\}$ denotes the set of indices of edges that are adjacent to the vertex ν_i

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$$\partial_\alpha v\left(\pi^{-1}(\nu_i)\right) := \begin{cases} \lim\limits_{h\to 0+} \dfrac{v_\alpha(0) - v_\alpha(h)}{h}, & \text{if } \nu_i = \pi_\alpha(0), \\ \lim\limits_{h\to 0+} \dfrac{v_\alpha(\ell_\alpha) - v_\alpha(\ell_\alpha - h)}{h}, & \text{if } \nu_i = \pi_\alpha(\ell_\alpha). \end{cases}$$

    the **outward derivative** at the vertices.

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$$r(t, x) = \max_{y \in \Gamma} \{w(t, y) - c(x, y)\}.$$

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In the same way, at time t firms located at y hire workers to minimize the wage, i.e.

$$w(t,y) = \min_{x \in \Gamma} \{r(t,x) + c(x,y)\}.$$

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This condition can be expressed in the following way: the couple of continuous functions $(w(t,\cdot), r(t,\cdot))$ induces an equilibrium in the labour market at time $t \in (0,T)$ if there is a **transport plan** γ between m_1 and m_2, i.e. γ has marginals m_1 and m_2 such that

$$w(t,y) - r(t,x) = c(x,y) \quad \text{on} \quad \mathrm{supp}(\gamma).$$

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[Villani et. al, works on optimal transport]: the equilibrium condition above is related to the following **Optimal Transport problem**:

$$C(m_1(t,\cdot), m_2(t,\cdot)) = \inf_{\gamma \in \Pi(m_1,m_2)} \int_{\Gamma\times\Gamma} c(x,y)d\gamma(x,y).$$

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Kantorovich duality: since Γ is compact, the cost $C(m_1(t,\cdot), m_2(t,\cdot))$ can be equivalently rewritten in the dual form as

$$C(m_1(t,\cdot), m_2(t,\cdot)) = \sup \left\{ \int_\Gamma w(t,y) dm_2(t)(y) - \int_\Gamma r(t,x) dm_1(t)(x) : \right.$$

$$\left. w, r \text{ continuous and } w(t,y) - r(t,x) \leq c(x,y) \text{ for every } x, y \in \Gamma \right\}$$

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$\Rightarrow$ at any t, the equilibrium condition in the labour market is equivalent to find a pair of continuous functions $(g_1, g_2) = (-r(t,\cdot), w(t,\cdot))$ satisfying $g_1(x) + g_2(y) \leq c(x,y)$ and optimal for the dual problem.

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Workers. The dynamics of the representative agent of the workers population m_1 is given by a Markov process (X_s, α_s) with $X_s \in \Gamma_{\alpha_s}$, characterized by the SDE

$$dX_s = u^1_{\alpha_s}(X_s)ds + \sqrt{2\mu^1_{\alpha_s}}dB^1_s, \quad X_t = x \in \Gamma,$$

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$$\mathbb{E}_{x,t} \int_t^T \left[L^1(u^1_{\alpha_s}, X_s) - r(s, X_s) + R^1[m_1(t), m_2(t)](X_s) \right] ds$$

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where L^1 represents the cost of motion, r the revenue (that individuals bring home) and $R^1[m_1, m_2]$ the rent cost.

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where L^2 represents the mobility cost, w the wage (that firms pay to workers) and $R^2[m_1, m_2]$ the rent cost.

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Associated with the Langrangian L^1_α and L^2_α of workers and firms, we introduce the Hamiltonians $H^i : (\cup_{\alpha\in\mathcal{A}}\Gamma_\alpha \setminus \mathcal{V}) \times \mathbb{R} \longrightarrow \mathbb{R}$ which are defined on each edge by

$$H^i_\alpha(x,p) = \sup_{u\in U^i_\alpha} \{-up - L^i_\alpha(x,u)\}, \quad x \in \Gamma_\alpha \setminus \mathcal{V},\ p \in \mathbb{R}.$$

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(i) **Forward-Backward MFG:** for $(t,x) \in (0,T) \times (\Gamma_\alpha \setminus \mathcal{V})$, $\alpha \in \mathcal{A}$, $i = 1, 2$,

$$-\partial_t \phi_i - \mu_\alpha^i \partial^2 \phi_i + H^i(x, \partial\phi_i) = R^i[m_1(t), m_2(t)] + g_i,$$

$$\partial_t m_i - \mu_\alpha^i \partial^2 m_i - \partial(m_i \partial_p H^i(x, \partial\phi_i)) = 0.$$

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(ii) **Transition conditions:** for $(t,\nu_j) \in (0,T) \times \mathcal{V}$, $\alpha, \beta \in \mathcal{A}$, $i = 1,2$,

$$\sum_{\alpha \in \mathcal{A}_j} \gamma_{j\alpha}^i \mu_\alpha^i \partial_\alpha \phi_i(t,\nu_j) = 0,$$

$$\sum_{\alpha \in \mathcal{A}_j} \mu_\alpha^i \partial_\alpha m_i(t,\nu_j) + n_{j\alpha} \partial_p H_\alpha^i \left(\nu_j, \partial\phi_i(t,\nu_j)\right) m_i|_{\Gamma_\alpha}(t,\nu_j) = 0,$$

$$\phi_i|_{\Gamma_\alpha}(t,\nu_j) = \phi_i|_{\Gamma_\beta}(t,\nu_j), \qquad \frac{m_i|_{\Gamma_\alpha}(t,\nu_j)}{\gamma_{j\alpha}^i} = \frac{m_i|_{\Gamma_\beta}(t,\nu_j)}{\gamma_{j\beta}^i}$$

($n_{j\alpha} = 1$ if $\nu_j = \pi_\alpha(\ell_\alpha)$ and $n_{j\alpha} = 1$ if $\nu_j = \pi_\alpha(0)$).

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(iii) **Initial-terminal conditions:** for $x \in \Gamma$, $i = 1, 2$,

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(iv) **Optimal Transport problem:** for $t \in (0, T)$,

$$g_1(t, x) + g_2(t, y) \leq c(x, y) \quad \text{for all } (x, y) \in \Gamma,$$

$$C(m_1(t), m_2(t)) = \int_\Gamma g_1(t, x) dm_1(t)(x) + \int_\Gamma g_2(t, y) dm_2(t)(y).$$

Description of the model: the MFG-OT system

(iii) **Initial-terminal conditions:** for $x \in \Gamma$, $i = 1, 2$,

$$\phi_i(T, x) = 0, \quad m_i(0, x) = m_0^i, \quad x \in \Gamma.$$

(iv) **Optimal Transport problem:** for $t \in (0, T)$,

$$g_1(t,x) + g_2(t,y) \le c(x,y) \quad \text{for all } (x,y) \in \Gamma,$$
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A solution to (MFGOT) system is given by two triples

$$(\phi_i(t,x), m_i(t,x), g_i(t,x))_{i=1,2}$$

satisfying $(i) - (iv)$ in a suitable sense.

Theoretical results: existence and uniqueness

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Some functional spaces.

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$$H^m(\Gamma) := \{v : \Gamma \longrightarrow \mathbb{R} : \ v \in C(\Gamma) \text{ and } v_\alpha \in H^m(0,\ell_\alpha) \ \text{ for all } \alpha \in \mathcal{A}\},$$

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$$W := \left\{ w \in H_b^1(\Gamma) : \; \frac{w|_{\Gamma_\alpha}(\nu_j)}{\gamma_{j\alpha}} = \frac{w|_{\Gamma_\beta}(\nu_j)}{\gamma_{j\beta}} \text{ for all } j \in I, \; \alpha, \beta \in \mathcal{A}_j \right\},$$
$$PC := \left\{ v : [0,T] \times \Gamma \longrightarrow \mathbb{R} : \; v|_{[0,T]\times\Gamma_\alpha} \in C([0,T] \times \Gamma_\alpha) \text{ for all } \alpha \in \mathcal{A} \right\}.$$

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For functions in these spaces no continuity at the vertices is required.

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$$m_0^i \in L^2(\Gamma) \cap \mathcal{M}, \quad m_0^i \geq \delta > 0, \quad \int_\Gamma m_0^i(x)dx = 1, \; i = 1, 2$$

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Moreover, we assume that the Hamiltonian $H_\alpha^i(\cdot, p)$, $i = 1, 2$ satisfies

(i) $H_\alpha^i \in C^1(\Gamma_\alpha \times \mathbb{R})$,

(ii) $H_\alpha^i(x, \cdot)$ is convex in p for any $x \in \Gamma_\alpha$,

(iii) $H_\alpha^i(x, p) \leq C_0^i(|p| + 1)$ for any $(x, p) \in \Gamma_\alpha \times \mathbb{R}$,

(iv) $|\partial_p H_\alpha^i(x, p)| \leq C_0^i$ for any $(x, p) \in \Gamma_\alpha \times \mathbb{R}$,

(v) $|\partial_x H_\alpha^i(x, p)| \leq C_0^i$ for any $(x, p) \in \Gamma_\alpha \times \mathbb{R}$,

for constant C_0^i independent of α.

Theoretical results: existence and uniqueness

Concerning the viscosity and the Kirchhoff coefficients, we assume that

$$\mu^i_\alpha > 0,\ \gamma^i_{j,\alpha} > 0,\ \sum_{\alpha\in\mathcal{A}_j} \gamma^i_{j\alpha}\mu^i_\alpha = 1, \quad \alpha \in \mathcal{A},\ j \in I,\ i = 1, 2.$$

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The coupling costs R^i, $i = 1, 2$, are continuous and uniformly bounded in $\mathcal{M} \times \mathcal{M} \times \Gamma$ and

$$\max_{\alpha \in \mathcal{A}} \max_{x \in \Gamma_\alpha} \left| R_\alpha^i[m_1, m_2] - R_\alpha^i[\eta_1, \eta_2] \right| \leq L \max_{i=1,2} \mathbf{d}_1(m_i, \eta_i)$$

for all m_i, $\eta_i \in \mathcal{M}$, $i = 1, 2$, where $\mathbf{d}_1$ is the Wasserstein distance which metrises the topology of weak convergence of probability measures on Γ.

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$$\int_\Gamma (-\partial_t \phi_i w + \mu^i \partial \phi_i \partial w + H^i(x, \partial \phi_i) w dx = \int_\Gamma (R^i[m_1(t), m_2(t)] + g_i) w dx;$$

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(iii) $(g_1(t,\cdot), g_2(t,\cdot)) = (g(t,\cdot), g^c(t,\cdot))$ where, for any $t \in [0,T]$, $g(t,\cdot)$ is a c-concave Kantorovich potential, i.e.

$$C(m_1(t), m_2(t)) = \int_\Gamma g(t,x) dm_1(t)(x) + \int_\Gamma g^c(t,y) dm_2(t)(y),$$

such that $\int_\Gamma g(t,x) dx = 0$ $(g^c(t,y) := \inf_{x \in \Gamma}\{c(x,y) - g(t,x)\})$.

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Theorem. There exists a solution to the (MFGOT) system. Moreover, if

$$\sum_{i=1}^{2}\int_{\Gamma}(R^i[m_1,m_2]-R^i[\bar{m}_1,\bar{m}_2])(m_i-\bar{m}_i)dx \geq 0$$

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– **Stability and monotonicity properties** of Kantorovich potentials and **stability** of sol. of HJ and FP equations.

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Future perspectives.

– Study a **long-time behavior** of the system.

– Put a Dirichlet boundary condition, many agents **exit the game** and hence the number of workers and firms change so the city **evolves in different configurations**.

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Thank you for your attention