Pollution model on network

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Introduction

- Our aim is to develop some models where pollution is included into economic decisions.
- Several papers contributed on modeling and studying this kind of phenomena both in a static and a dynamic framework.
- In a series of paper by Fabbri, Boucekkine, Gozzi, Federico [EJOR (2019), JME (2021), GEB (2021)] the problem is modeled in a continuous times-space framework; Similar formulation in De Frutos et al. [JEEM (2019),Automatica (2020)]
- Pollution is a typical example of negative externalities: it is generated in a certain place and it may move integrally or partially into another distant place.

Three possible perspectives on the problem:

- The world-Centralized point of view. There is one planner acting in the economy and maximizing a given payoff

$$J(K, P)$$

depending on economic variable **(for example: capital K)** and on the environmental state **(for example: pollution P)** (Optimal Control).

- Nations-Decentralized point of view. There is a small number of agents. Each agents maximized a given payoff

$$J^i(K, P)$$

depending on some economic variables **(K)** and on the environmental state **(P)** and also interact with the other agents (Differential Games).

- Individuals-Decentralized point of view II. Same as the case above, but when the number of agents is large. (Differential Games, Mean Field Games)

We are going to present a model where space is modeled as a Network.

Motivation:

- Spatial economic data related to economic variable (capital) are of discrete nature. We may have data for cities, districts, nations. More specifically for capital and people moving along different center of interests.
- We can interpret these geographic locations as *nodes* of a graph and these paths as *weighted arcs* of a graph.

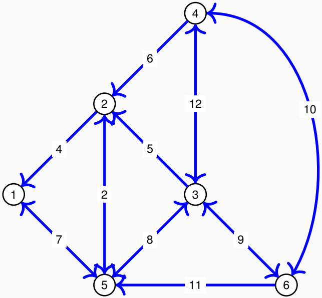
Time/space structure

The model is in continuous time $t \in \mathbb{R}^+$.

- Space: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,

$$\mathcal{V} = \{1, 2, \ldots, N\}, \quad \mathcal{E} = \{(i,j) : i \sim j\}$$

- Graph is simple (no self-loops, no multiple links), finite and weighted. Weights are collected in a matrix W: $w_{ij} \geq 0$, $i, j = 1, \ldots, n$, $w_{ii} = 0$.



We introduce $L := \{\ell_{ij}\}$ as

$$\ell_{ij} := \begin{cases} w_{ij}, & \text{if } i \neq j, \\ -\sum_{j=1}^{n} w_{ij}, & \text{if } i = j, \end{cases}$$

Main assumption on the graph structure

- L is a Metzler matrix, i.e. $\ell_{i,j} \geq 0$ for each $i \neq j \Rightarrow$ the semigroup $e^{t(L)}$ is positive.
- L satisfies the following: $L + L^*$ is negative semidefinite, i.e. $\langle x, L + L^* x \rangle \leq 0 \Rightarrow L$ is a dissipative operator

We consider the following variables:

- $P_i(t) =$ pollution at time t and location i;
- $C_i(t) =$ consumption rate at time t and location i;
- $I_i(t) =$ investment at time t and location i;
- $B_i(t) =$ abatment level at time t and location i;
- and others..

All the variables we use are spatially distributed: they have different values at different locations i

At time t and at any location $i \in \mathcal{V}$, there is a single individual producing through a linear production function

$$Y_i(t) = A_i(t) I_i(t),$$

- $Y_i(t)$ is the output;
- $I_i(t)$ is the capital input;
- $A_i(t) > 1$ is technological level at location i.

- At any location, the produced output is consumed, invested, and used for de-pollution, implying:

$$C_i(t) + I_i(t) + B_i(t, x) = Y_i(t),$$

$$\Rightarrow C_i(t) = (A_i(t) - 1)I_i(t) - B_i(t, x)$$

where $C_i(t)$ and $B_i(t)$ are, respectively, the consumption rate and the resources devoted to de-pollution policies at location i and at time t.

- Net emissions are given by

$$N_i(t) = I_i(t) - \varphi_i(B_i(t))^{\theta}, \quad \theta \in (0, 1),$$

where $\varphi_i \geq 0$ is the efficiency of abatement and $\theta \in (0, 1)$ is the return to scale of abatement .

We introduce in the previous context a *green* technology and thus its corresponding technological level A_i^R and capital input $R_i(t)$. Production at time t and at location $i \in \mathcal{V}$, becomes

$$Y_i(t) = A_i(t) I_i(t) + A_i^R(t) R_i(t),$$

- $Y_i(t)$ is the output;
- $I_i(t)$ is the capital input, $R_i(t)$ is the capital green input;
- $A_i(t) > 1$ is technological level at location i, $A_i^R(t) > 1$ is technological level of renewable energy at location i.

The economic/environmental model

- At any location, the produced output is consumed, invested, and used for de-pollution, implying:

$$C_i(t) + I_i(t) + R_i(t) + B_i(t,x) = Y_i(t),$$

$$\Rightarrow C_i(t) = (A_i(t) - 1)I_i(t) + (A_i^R(t) - 1)R_i(t) - B_i(t,x)$$

- Net emissions are given by

$$N_i(t) = I_i(t) + \epsilon_i R_i(t) - \varphi_i (B_i(t))^\theta, \quad \theta \in (0,1),$$

  where $\varphi_i \geq 0$ is the efficiency of abatement, $\theta \in (0,1)$ is the return to scale of abatement, $\epsilon_i << 1$ is the pollution effect of the renewable energy.

The pollution at location *i* evolves according to:

$$\begin{cases} \dfrac{d}{dt}P_i(t) = \underbrace{\sum_{j=1}^{n} w_{ij}P_j(t)}_{inflow} - \underbrace{\sum_{j=1}^{n} w_{ji}P_i(t)}_{outflow} - \underbrace{\delta_i P_i(t)}_{decay} + N_i(t) \\ P_i(0) = p_i(0) \geq 0. \end{cases}$$

In a vector formulation,

$$\begin{cases} \frac{d}{dt}P(t) = (L - \delta)P(t) + N(t), \\ P(0) = p \in \mathbb{R}^n_+. \end{cases}$$

where $P(t) := (P_1(t), \dots, P_n(t))^T$, $N(t) := (N_1(t), \dots, N_n(t))^T$,

The economic/environmental model

Consider a social planner, who aims at controlling consumption level to maximize the following social welfare:

$$J((I,B,R),p) := \int_0^{+\infty} e^{-\rho t} \left(\sum_{i=1}^{n} \left(\frac{C_i(t)^{1-\gamma}}{1-\gamma} - \omega_i P_i(t) - f_i(R_i(t)) \right) \right) dt,$$

where $\omega_i > 0$ represents local environmental awareness at the location i, $\rho > 0$ is a discount factor and $\gamma \in (0,1) \cup (1,+\infty)$ is a preference parameter.

In summary, the social planner has to solve the optimal control problem,

$$J((I,B,R);p) := \int_0^{+\infty} e^{-\rho t}\left(\langle \frac{C(t)^{1-\gamma}}{1-\gamma}, \mathbb{1}\rangle - \langle \omega, P(t)\rangle - \langle f(R(t)), \mathbb{1}\rangle\right) dt,$$

over

$$\begin{cases} \frac{d}{dt}P(t) = (L-\delta)P(t) + N(t), \\ P(0) = p \in \mathbb{R}^n_+. \end{cases}$$

where

$$C(t) = (A(t) - \mathbb{1})I(t) + (A^R(t) - 1)R(t) - B(t)$$

$$N(t) = I(t) + \epsilon R(t) - \varphi(B(t))^\theta,$$

The problem is linear!

$$
\begin{aligned}
&J(p,(I,R,B)) = -\langle \alpha, p\rangle + \\
&+\int_0^{+\infty} e^{-\rho t}\left[\left\langle \frac{((A^I(t)-1)I(t)+(A^R(t)-1)R(t)-B(t))^{1-\gamma}}{1-\gamma}, \mathbf{1}\right\rangle\right. \\
&\left. -\int_0^{+\infty} e^{-\rho t}\left[\langle f(R(t)),\mathbf{1}\rangle - \langle \alpha, I(t)+\varepsilon R(t)-\varphi(t)B(t)^{\theta}\rangle\right]\right. dt.
\end{aligned}
$$

The problem is reduced to a static problem.

Main Result

Proposition (Under suitable assumptions on the coefficients)

For convex cost,

- The optimal control problem admit a unique solution ;
- The optimal spatial pollution density $P(t)$ converges as $t \to \infty$ to the long-run pollution profile P_∞, unique solution to the following ODE:

$$(L - \delta)P_\infty + N^* = 0.$$

Main Results-Linear cost function

Proposition (Under suitable assumptions on the coefficients)

If $f_i(R_i) = \lambda_i R_i$,

- If $\lambda_i < \left[\frac{(a_i^R-1)}{(a_i^I-1)} - \varepsilon_i\right]\alpha_i$ for some $i \in \{1,\dots,n\}$

$$\begin{cases} B_i^* = \left(\frac{\lambda_i+\varepsilon_i\alpha_i}{\theta(A_i^R-1)\varphi_i\alpha_i}\right)^{\frac{1}{\theta-1}}, \quad I_i^* = 0 \\ R_i^* = (A_i^R-1)^{\frac{1-\gamma}{\gamma}}(\lambda_i+\varepsilon_i\alpha_i)^{-\frac{1}{\gamma}} + (A_i^R-1)^{-1}\left(\frac{\lambda_i+\varepsilon_i\alpha_i}{\theta(A_i^R-1)\varphi_i\alpha_i}\right)^{\frac{1}{\theta-1}}. \end{cases}$$

- If $\lambda_i > \left[\frac{(A_i^R-1)}{(A_i-1)} - \varepsilon_i\right]\alpha_i$ for some $i \in \{1,\dots,n\}$

$$\begin{cases} I_i^* = (a_i^I-1)^{\frac{1-\gamma}{\gamma}}\alpha_i^{-\frac{1}{\gamma}} + (A_i-1)^{\frac{\theta}{1-\theta}}(\theta\varphi_i\alpha_i)^{\frac{1}{1-\theta}} \\ B_i^* = ((A_i-1)\varphi_i\theta)^{\frac{1}{1-\theta}}, \quad R_i^* = 0. \end{cases}$$

- If $\lambda_i = \left[\frac{(A_i^R-1)}{(A_i-1)} - \varepsilon_i\right]\alpha_i$, infinite solutions.

Note: α represents the environmental aggregation between nodes

MAIN RESULTS-QUADRATIC COST FUNCTION

If $f_i(R_i) = \lambda_i R_i^2$, no explicit solution. But some numerical simulations are performed.

- No dichotomy.
- In this (very) simplified context, we can catch the influence of spatial heterogeneity into the optimal control.

Numerical simulation-Quadratic cost function

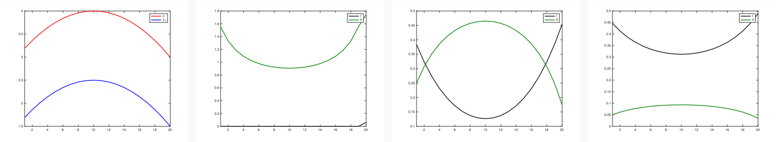


Figure 1: Impact of the Heterogeneity of A and A^R on I^*, R^*. On the left: distribution of A, A^R. From left to the right: I^*, R^* for $\lambda = 0.1, 1, 5$.

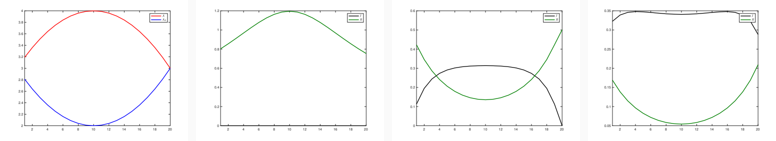


Figure 2: Impact of the Heterogeneity of A and A^R on I^*, R^*. On the left: distribution of A, A^R. From left to the right: I^*, R^* for $\lambda = 0.1, 1, 5$.

NUMERICAL SIMULATION-QUADRATIC COST FUNCTION

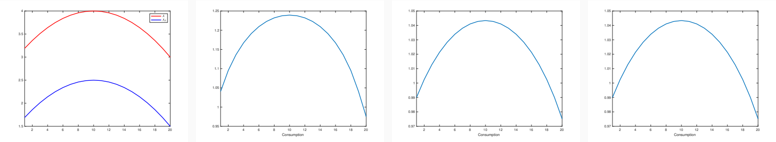


Figure 3: Impact of the Heterogeneity of A and A^R on C^*. On the left: distribution of A, A^R. From left to the right: C^* for $\lambda = 0.1, 1, 5$

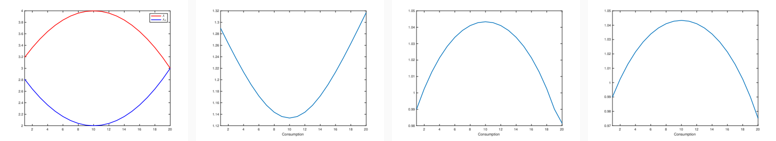


Figure 4: Impact of the Heterogeneity of A and A^R on C^*. On the left: distribution of A, A^R. From left to the right: C^* for $\lambda = 0.1, 1, 5$

Numerical simulation-Quadratic cost function

- R^*: when there is only green investment, the space heterogeneity of R^* show a spatial discrepancy. When it is optimal to do both type of investment, R^* follow the heterogenity of A^R.
- I^*: the space heterogeneity of I^* show a spatial discrepancy.
- C^* increases when it is optimal to do only green investment.

Numerical simulation-Quadratic cost function

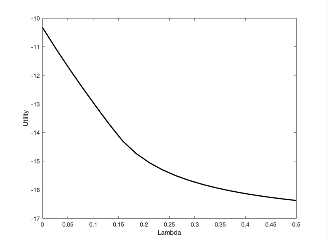


Figure 5: Value function with $\gamma = 6$ compared to λ

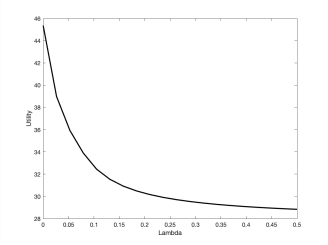


Figure 6: Value function with $\gamma = 0.5$ compared to λ

Numerical simulation-Quadratic cost function

Classical growth à la Stokey (1999): as the economy develops, it starts depolluting without breaking down growth. In other terms: Pollution goes down with production.

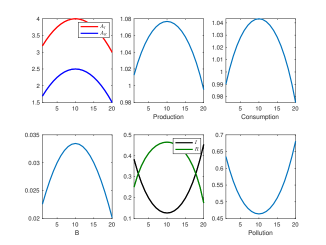


Figure 7: Optimal Production, Consumption, Abatment, Pollution, Investment.

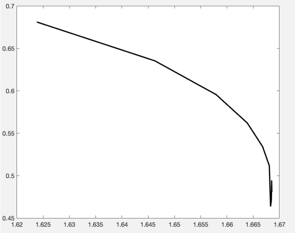


Figure 8: Production vs Pollution

Differentiation of technology generates Classical growth à la Stokey.

Future Research

It would be interesting to replace the equation for the pollution, with an EBCM. Some literature: Brock, Engström, Xepapadeas [EER, '13], Xepapadeas, Yannacopoulos [JEDC '14].

Possibile future research lines:

- Enrich the model with an equation for capital.
- Carbon tax
- EBCM in place of Pollution

Future Research

We could an EBCM with human impact h_t

$$\frac{\partial T(x,t)}{\partial t} = \underbrace{D\frac{\partial}{\partial x}\left[\left(1-x^2\right)\frac{\partial T(x,t)}{\partial x}\right]}_{\textit{Diffusion}} + \\ + \underbrace{R_a(T(x,t))}_{\text{Absorbed radiation}} - \underbrace{R_e(T(x,t))}_{\text{Outgoing radiation}} + \underbrace{h(x,t)}_{\text{human effect}}$$

In this way, we would deal with a quantity, T, where a natural mechanism inducing a non-homogenous distribution for the temperature!

Some questions:

- When and where it is optimal to invest in R?
- Space heterogeneity of carbon tax?
- How do different types of economy (closed/open economy) affect the distribution of taxation?

Thanks for the attention!