PRIN Workshop

On response theory for climate models

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Some Mathematical Approaches to Climate Change and its Impacts
April, 22, 2024

This work

What?

Dependence of the long-time average behaviour of solutions of nonlinear dissipative SPDEs from deterministic additive **forcings**.

Why?

To give a mathematical insight into whether statistical properties derived under current conditions will be valid under **different forcing scenarios** in physically relevant models (e.g. GFD models, climate models).

How?

Establishing regularity of **observable averages** against the invariant measure with respect to changes in a **time-independent forcing**

Example: two-layer quasi-geostrophic model

Consider the **streamfunction** $\psi = (\psi_1(x,y,t), \psi_2(x,y,t))^t$ for $(x,y) \in \mathbb{T}^2$, $t \geq 0$,

$$\begin{aligned} dq_1 + (\nabla^\perp \psi_1 \cdot \nabla q_1)\, dt &= (-\beta \partial_x \psi_1 + \nu \Delta^2 \psi_1 + f(a))dt + dW \\ \partial_t q_2 + \nabla^\perp \psi_2 \cdot \nabla q_2 &= -\beta \partial_x \psi_2 + \nu \Delta^2 \psi_2 - r \Delta \psi_2 \end{aligned} \tag{1}$$

where $\mathbf{q} = (q_1, q_2)$ is the so-called **QG potential vorticity**

$$q_1 = \Delta \psi_1 - F_1(\psi_1 - \psi_2), \quad q_2 = \Delta \psi_2 - F_2(\psi_2 - \psi_1)$$

with F_1, F_2 positive constants depending on the density of the layers.

Quest for ergodic properties

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Transition probabilities: $\mathbf{q}_0 \in \mathcal{H}$, $\Gamma \subset \mathcal{H}$

$$P_t(\mathbf{q}_0, \Gamma) := \mathrm{Law}(\mathbf{q}(t, \mathbf{q}_0))(\Gamma) = \mathbb{P}(\mathbf{q}(t;\mathbf{q}_0) \in \Gamma),$$

Markov semigroup acting on observables $\varphi : \mathcal{H} \to \mathbb{R}$

$$(\mathcal{P}_t\,\varphi)(\mathbf{q}_0) = \mathbb{E}\,\varphi(\mathbf{q}(t, \mathbf{q}_0)) = \int_{\mathcal{H}} \varphi(\zeta) P_t(\mathbf{q}_0, d\zeta)$$

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Invariant measure i.e. for observables $\varphi : \mathcal{H} \to \mathbb{R}$

$$\int_{\mathcal{H}} \varphi\, d\mu = \int_{\mathcal{H}} \varphi\, d(\mathcal{P}_t^*\mu) = \int_{\mathcal{H}} \mathcal{P}_t\varphi d\mu$$

If an invariant measure μ is unique, it is ergodic.

1. Exponential ergodicity

- temporal averages of an observable converge to averages of the observables with respect to the stationary distribution, i.e. given $\varphi \in L^1(\mathcal{H}, \mu)$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (\mathcal{P}_s \varphi)(\mathbf{q}_0)\, ds = \int_{\mathcal{H}} \varphi \, d\mu =: \langle \varphi, \mu \rangle, \quad \mu\text{-a.e.}$$

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- transition probabilities converge to a unique stationary distribution with an exponential rate, i.e. there exists $\gamma > 0$, $C = C(\mathbf{q}_0) > 0$ such that

$$d\left(P_t(\mathbf{q}_0, \cdot), \mu\right) \leq e^{-\gamma t} C(\mathbf{q}_0) \quad \text{for all } \mathbf{q}_0 \in \mathcal{H}.$$

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For example when d is a Wasserstein distance

$$|\mathcal{P}_t \varphi(\mathbf{q}_0) - \langle \varphi, \mu \rangle| = |\langle \varphi, P_t(\mathbf{q}_0, \cdot) \rangle - \langle \varphi, \mu \rangle| \leq d\,(P_t(\mathbf{q}_0, \cdot), \mu)$$

For stochastic fluid equations

When the noise acts on a minimum number of degrees of freedom, recent results ensure ergodicity and exponential ergodicity:

Butkovsky et al 2020,
Glatt-Holtz et al 2017:

- 2D Navier-Stokes
- 2D Hydrostatic Navier-Stokes
- Fractionally Dissipative Euler
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C., Bröcker, Kuna (2022):
2LQG + stochastic wind forcing

- exists invariant measure μ
- for r large enough, μ is unique and transitions probabilities converge exponentially to it.

Consider for $a \in \mathbb{R}$

$$\begin{aligned} dq_1 + (\nabla^\perp \psi_1 \cdot \nabla q_1)\, dt &= (\nu \Delta^2 \psi_1 + f(a))dt + dW \\ \frac{\partial q_2}{\partial t} + \nabla^\perp \psi_2 \cdot \nabla q_2 &= \nu \Delta^2 \psi_2 - r \Delta \psi_2 \end{aligned} \tag{2}$$

with invariant measure μ_a.

2. Response theory

Consider for $a \in \mathbb{R}$

$$\begin{aligned} dq_1 + (\nabla^\perp \psi_1 \cdot \nabla q_1)\, dt &= (\nu \Delta^2 \psi_1 + f(a))dt + dW \\ \frac{\partial q_2}{\partial t} + \nabla^\perp \psi_2 \cdot \nabla q_2 &= \nu \Delta^2 \psi_2 - r \Delta \psi_2 \end{aligned} \tag{2}$$

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How does μ_a change with a? One typically looks for a response formula i.e.

$$\frac{\mathrm{d}}{\mathrm{d}a} \langle \varphi, \mu_a \rangle \Big|_{a=a_0} = F(\mathcal{P}_t^{a_0}, \mu_{a_0}, \varphi, \partial_a \mathcal{P}_t^{a_0})$$

so that

$$\langle \varphi, \mu_a \rangle \sim \langle \varphi, \mu_{a_0} \rangle + (a - a_0) F(\mathcal{P}_t^{a_0}, \mu_{a_0}, \varphi, \partial_a \mathcal{P}_t^{a_0})$$

Linear response in finite dimension

Most mathematical results are for SRB measure in finite dimensional deterministic context for hyperbolic or partially hyperbolic systems (e.g. work of Ruelle, Baladi, Dolgopyat).

Chaotic Hypothesis (Gallavotti-Cohen): chaotic systems are hyperbolic.

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Under this hypothesis, theory of linear response showed enormous potential in the applications to climate (climate sensitivity) and geophysical fluid dynamics (GFD) models (e.g. work of Majda, Lucarini, Gottwald and many more)

Linear response in infinite dimension

For a large class of **stochastic** systems in a **infinite dimensional**: Hairer and Majda, 2010

- applies even with highly degenerate noise (via Hairer Mattingly 2008)
- requires sophisticated techniques (asymptotic strong Feller, Malliavin calculus)
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In C., Kuna, Bröcker 2022, having the 2LQG model in mind, we addressed the following questions:

Q: Can we get away with simpler tools for a less degerate noise?

Q: Can we provide a toolbox for Navier-Stokes type equations?

Formal idea (Hairer, Majda, 2010; C., Kuna, Bröcker, 2022)

Fix $t \geq 0$ and a reference parameter $a_0 \in \mathbb{R}$. It is easy to see that

$$\langle (1 - \mathcal{P}_t^{a_0})\psi, \mu_a - \mu_{a_0} \rangle = \langle (\mathcal{P}_t^{a} - \mathcal{P}_t^{a_0})\psi, \mu_a \rangle \quad \text{for all } \psi \in \mathcal{O}.$$

If $\varphi = (1 - \mathcal{P}_t^{a_0})\psi$ then

$$\frac{\langle \varphi, \mu_a - \mu_{a_0} \rangle}{a - a_0} = \left\langle \frac{(\mathcal{P}_t^{a} - \mathcal{P}_t^{a_0})\psi}{a - a_0}, \mu_a \right\rangle$$

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- for any φ there exists ψ with $\varphi = (1 - P_t^{a_0})\psi$

Spectral gap

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If $\mathcal{P}_t^{a_0}$ has a spectral gap in $\mathcal{O}$, namely there exists $\rho < 1$ s.t.

$$\|\mathcal{P}_t^{a_0}\varphi - \langle \varphi, \mu_{a_0} \rangle\|_{\mathcal{O}} \le \rho \|\varphi - \langle \varphi, \mu_{a_0} \rangle\|_{\mathcal{O}},$$

i.e. $\mathcal{P}_t^{a_0}$ is a bounded operator on $\mathcal{O}/\ker \mu_{a_0}$ with $\|\mathcal{P}_t^{a_0}\| < 1$, then $(1 - \mathcal{P}_t^{a_0})$ is invertible on $\mathcal{O}/\ker \mu_{a_0}$.

Spectral gap and observables

Hairer and Majda (2010): $(\mathcal{O}, \|\cdot\|_{\mathcal{O}})$ is the closure of C_0^∞ wrt

$$\|\varphi\|_{V_1,V_2} = \sup_{x\in\mathcal{H}} \left(\frac{|\varphi(x)|}{V_1(x)} + \frac{\|D\varphi(x)\|}{V_2(x)} \right)$$

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We consider approach from Butkovsky, Kulik, Scheutzow, 2020, and C., Bröcker, Kuna, 2022 based on the generalized Harris' theorem (Hairer, Mattingly, Scheutzow, 2011).

With this approach observables are Hölder-type functions.

Differentiability of the semigroup

Is $a \mapsto \mathcal{P}_t^a \psi = \mathbb{E}\psi(\mathbf{q}(t;\cdot,a))$ still differentiable when ψ is not?

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Yes, if Q is the covariance operator of the noise W, we ask for:

(i) $a \mapsto f(a)$ to be differentiable as a map with values in range Q

(ii) $\sup_{a\in(a_0-\varepsilon,a_0+\varepsilon)} |D_a Q^{-1/2} f| < \infty$

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One can then show:

$$\frac{\mathrm{d}}{\mathrm{d}a} \langle \varphi, \mu_a \rangle \Big|_{a = a_0} = \langle D_a \mathcal{P}_t^{a_0} (1 - \mathcal{P}_t^{a_0})^{-1} (\varphi - \langle \varphi, \mu_{a_0} \rangle), \mu_{a_0} \rangle.$$

Without these restrictions we can nevertheless establish weak local Hölder continuity of $a \mapsto \mu_a$ (fractional response).

Two-layer energy balance model

$T_{a,s}(t,.) : [-1,1] \to \mathbb{R}$ are the zonally averaged temperature in the atmosphere (T_a) and at the surface (T_s).

$$dT_a = \Big(AT_a + \kappa\sigma T_s|T_s|^3 - 2\kappa\sigma T_a|T_a|^3 - \lambda(T_a - T_s) + R_a(T_a)\Big)\,dt + dW_a,$$
$$dT_s = \Big(AT_s + \kappa\sigma T_a|T_a|^3 - \sigma T_s|T_s|^3 - \lambda(T_s - T_a) + R_s(T_s)\Big)\,dt + dW_s.$$

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The Wiener processes both have the form $W_{a,s}(t) = \sum_{n\le N_0} \sigma_n e_n B_t^{(n)}$

- $\{B^{(n)}\}$ are 1D Wiener processes on some joint probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $(e_n)_{n\ge 0}$ eigenfunctions of $-A$ forming a complete ONS of $L^2([-1,1])$
- $\sigma_k > 0$ for $k = 1, \ldots, N_0$ (the σ_k may differ between the surface and the atmosphere).

with Bröcker, Cannarsa, Kuna, Urbani:

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   In particular we want the solution to generate a Markov semigroup in an appropriate Banach space (not necessarily Hilbert here).

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   The generalised coupling method as in C., Bröcker, Kuna 2022, showed potential in one-layer model and simplified versions of the 2LEBM.

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   The generalised coupling method as in C., Bröcker, Kuna 2022, showed potential in one-layer model and simplified versions of the 2LEBM.

3. Response theory or break of linear response.
   Approach presented should be adapted for "multplicative" parameters as

$$\lambda(T_a - T_s), \quad \mathbf{q}(\mathbf{x})\beta_s(T_s), \quad \kappa\sigma T_s|T_s|^3$$

Summary and more questions

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- The importance of spectral gap
- LR for forcings differentiable in the parameter in the range of the noise (on the same d.o.f.)
- 2 layer Energy Balance Model

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- the result for linear response should apply to all examples in Glatt-Holtz et al. 2016, Butkovsky et al. 2020
- different parameters e.g. given by numerical approximation?
- different forms of noise?
- correlations vs averages?

References

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Grazie!