The Strange Case of Continuous-Spin Particles

Brando Bellazzini

“About Some Future directions in Fundamental Physics”, Scuola Normale di Pisa 2024

BARBIERI'S LEGACY

'69
PhD Barbieri

Giudice

Mangano

Strumia

Dvali

Creminelli

Trincherini

Cacciapaglia

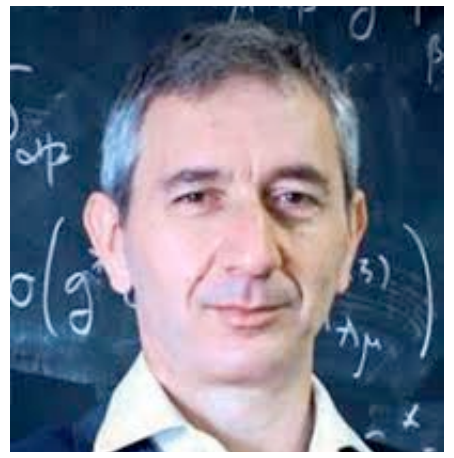
Contino

80'-90's

early 2000

Ridolfi

Rattazzi



Romanino

Riotto

Nicolis

Senatore

Cirelli

Papucci

Rychkov

Bellazzini

Franceschini

Carcamo

Vichi

Torre

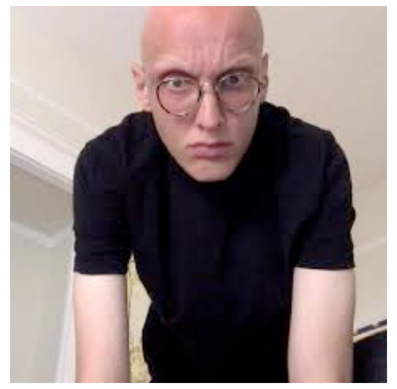
D'Eramo

Gori

Sala

Buttazzo

Pappadopulo



Tesi

late 2000, '10's

2005

+ many others

“nella misura in cui il mio livello di percezione e’ rimasto constante, non credo di aver visto un momento così incerto come l’attuale”

from “Vite di fisici tra atomi e particelle”, vol. II

Riccardo Barbieri

The Strange Case of Continuous-Spin Particles

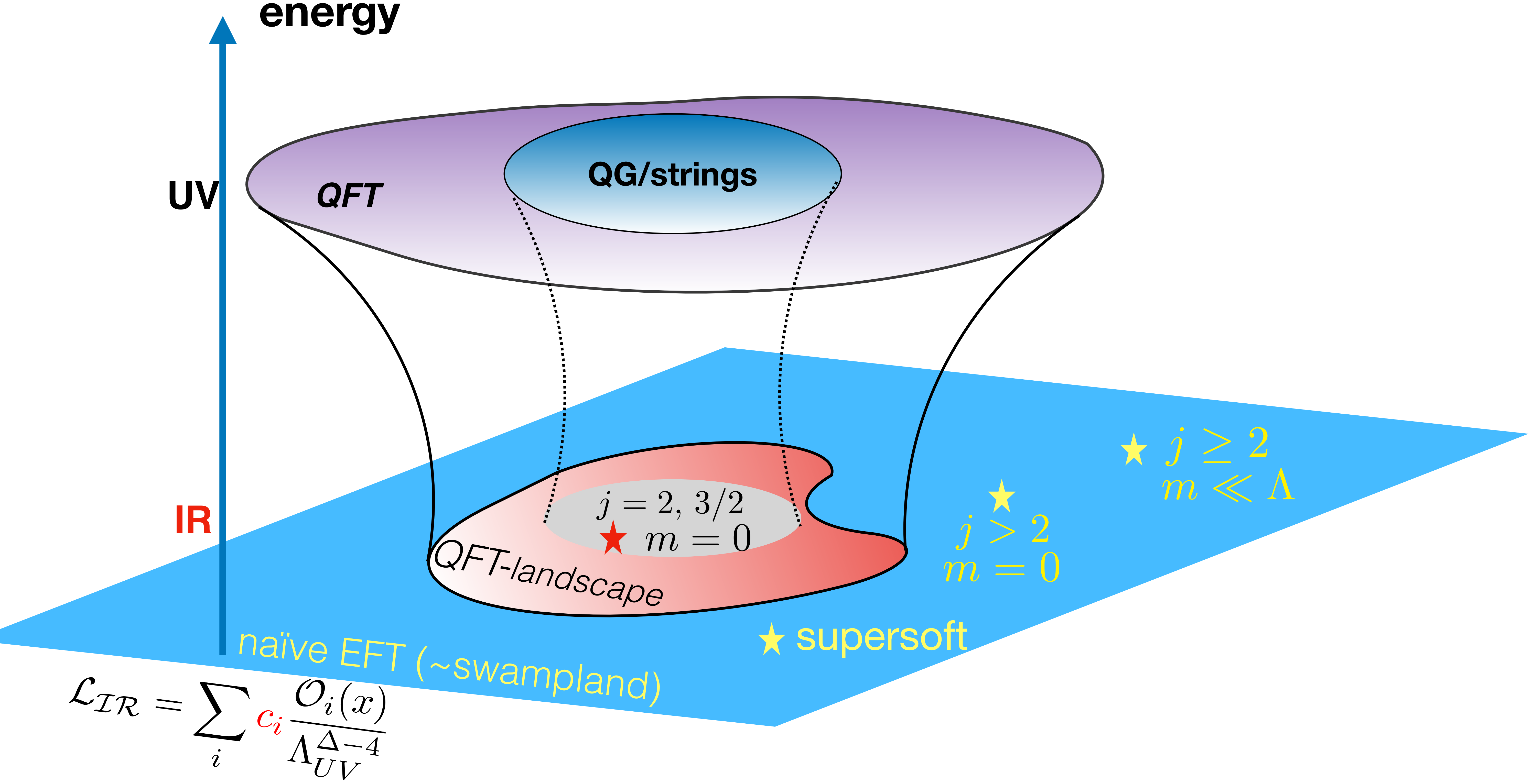
Brando Bellazzini

based on 2406.17017 B.B., S. De Angelis & M. Romano

"About Some Future directions in Fundamental Physics", Scuola Normale di Pisa 2024

Goals & Motivations

Landscape of Effective theories?



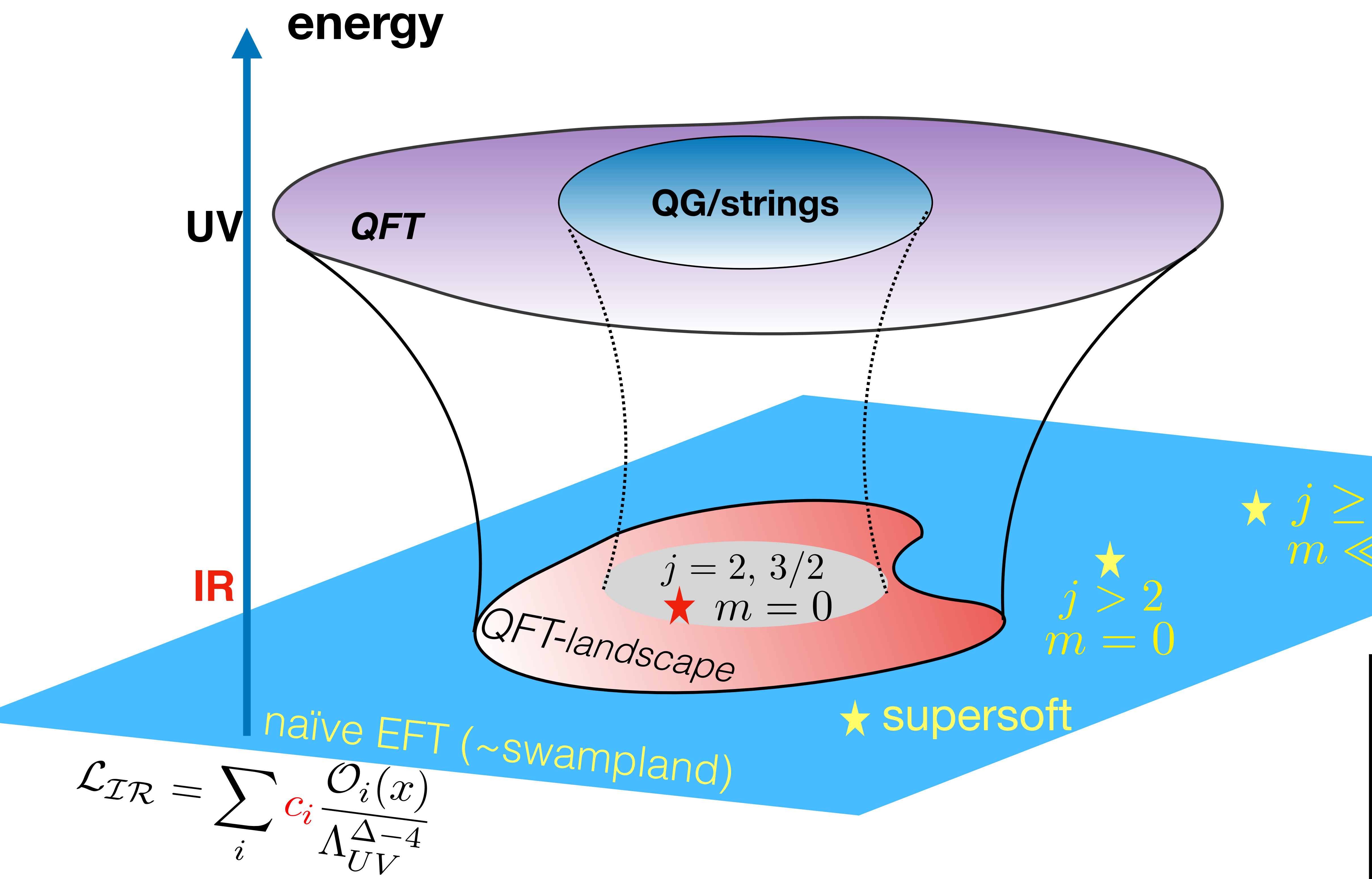
- Consistency conditions
  (soft-theorems/bootstrap/positivity/…)
  shape the space of EFT

- even m=0 not finished yet!
  Goal: formulate effective theory for CSP

- Pheno-opportunity?
  Motivation: missing something in the IR?

Schuster, Toro, Zhou
2303.04816, 2308.16218

Landscape of Effective theories?



$$\mathcal{L}_{\mathcal{IR}} = \sum_i c_i \frac{\mathcal{O}_i(x)}{\Lambda_{UV}^{\Delta-4}}$$

- Consistency conditions (soft-theorems/bootstrap/positivity/…) shape the space of EFT
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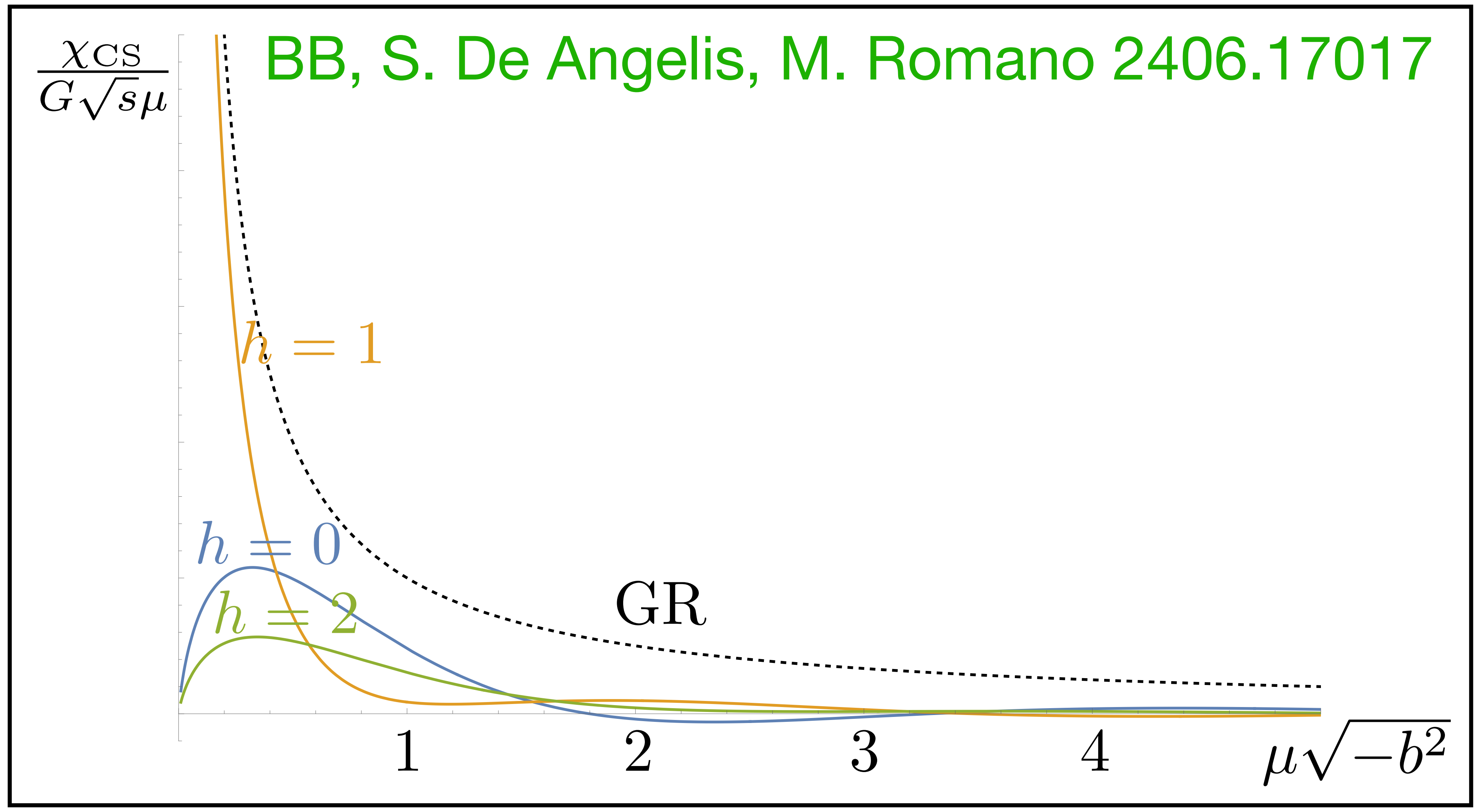
BB, S. De Angelis, M. Romano 2406.17017

WHAT ARE
CONTINUOUS-SPIN PARTICLES?

They are Infinite-dim Poincare' irreps

	P_μ^2	W_μ^2	Little group Generators $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma$		$\begin{cases} W_\mu P^\mu = 0 \\ [W_\mu, P_\nu] = 0 \\ [W_\mu, W_\nu] = -\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \end{cases}$	
massless	0	0	$U(1)$	$W_0 = W_3 = J_3$	$W_\mu = hP_\mu$	$1 - dim$
massive	m^2	$-m^2 j(j+1)$	$SU(2)$	$W_i = mJ_i$	$W_0 = 0$	$(2j+1) - dim$

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CSP	0	$-\mu^2$	$ISO(2)$	$W_0 = W_3 = J_3$	$W_\pm \propto (J_1 \pm iK_1)) \pm i\,(J_2 \pm iK_2) \propto J^\pm_{L/R}$	

"spin scale"

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"spin scale"

$$\begin{cases} [H, W_\pm] = \pm W_\pm \\ [W_+, W_-] = 0 \\ W_\mu^2 = -W_+W_- < 0 \\ W_+ = W_-^\dagger \end{cases}$$

$$\ldots \underset{W_-}{\overset{W_+}{\rightleftarrows}} |h-1\rangle \underset{W_-}{\overset{W_+}{\rightleftarrows}} |h\rangle \underset{W_-}{\overset{W_+}{\rightleftarrows}} |h+1\rangle \underset{W_-}{\overset{W_+}{\rightleftarrows}} \ldots$$

$H|h\rangle = h|h\rangle \qquad h \in \mathbb{Z}$ or $\mathbb{Z}/2$

$\infty - dim$

@ fixed p

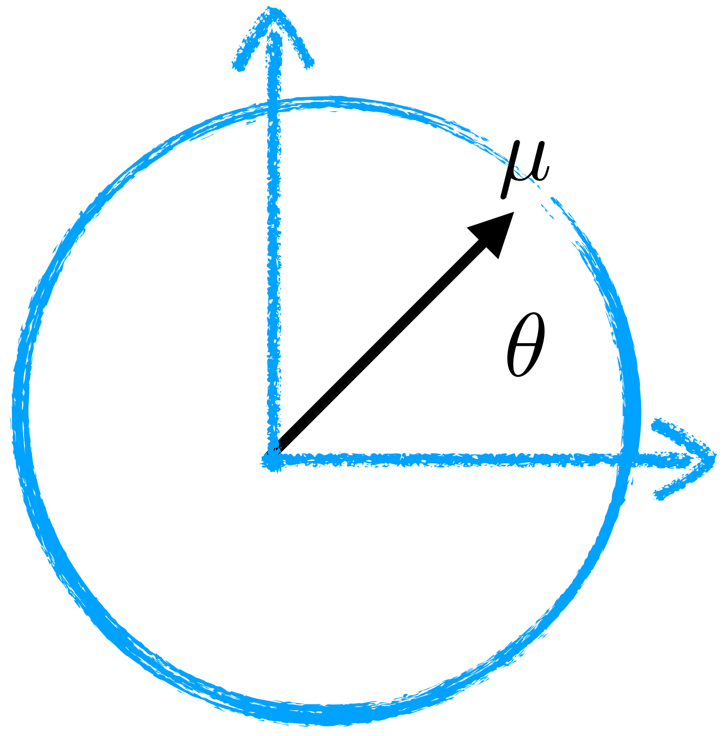
H-Basis vs θ-Basis

$$\begin{cases} [H, W_\pm] = \pm W_\pm \\ [W_+, W_-] = 0 \\ W_\mu^2 = -W_+ W_- < 0 \\ W_+ = W_-^\dagger \end{cases}$$

$$\begin{tikzcd} \dots \arrow[r, shift left, "W_+"] & {|h-1\rangle} \arrow[l, shift left, "W_-"] \arrow[r, shift left, "W_+"] & {|h\rangle} \arrow[l, shift left, "W_-"] \arrow[r, shift left, "W_+"] & {|h+1\rangle} \arrow[l, shift left, "W_-"] \arrow[r, shift left, "W_+"] & \dots \arrow[l, shift left, "W_-"] \end{tikzcd}$$

$$H|h\rangle = h|h\rangle$$

$$\boxed{|h\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ih\theta} |\theta\rangle \quad \xleftrightarrow{\text{Fourier}} \quad |\theta\rangle = \sum_{h=-\infty}^{\infty} e^{-ih\theta} |h\rangle}$$



states labelled by an angle

H-Basis vs θ-Basis

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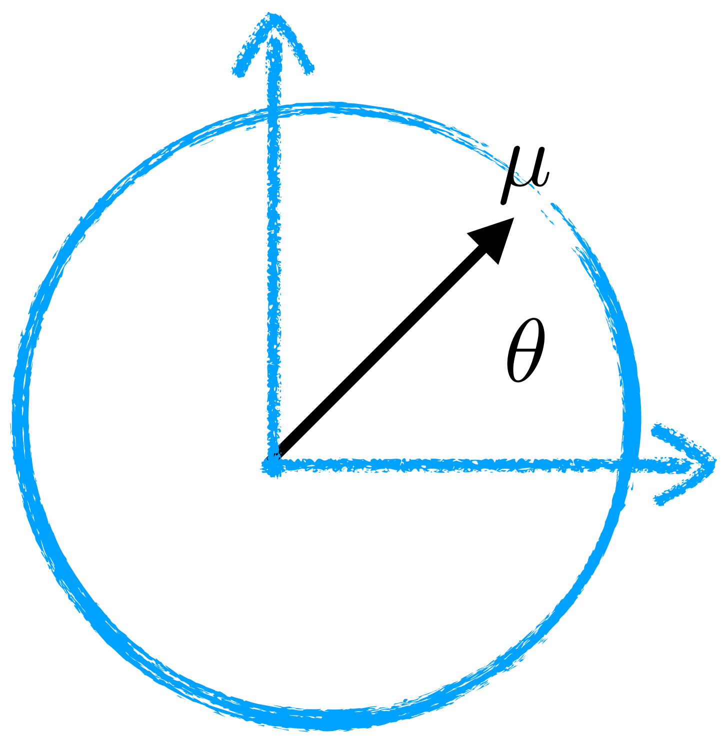

states labelled by an angle

$$W_\pm|\theta\rangle = \mu e^{\pm i\theta}|\theta\rangle = \mu_\pm|\theta\rangle$$

$$H|\theta\rangle = i\frac{d}{d\theta}|\theta\rangle$$

$$W_+W_-|\theta\rangle = \mu^2|\theta\rangle$$

ISO2-translat.=phase mult.

$$e^{i\alpha_\mp W_\pm}|\theta\rangle = e^{i(\alpha_\mp\cdot\mu_\pm)}|\theta\rangle$$

ISO2-rotat.=rotate

$$e^{-i\omega H}|\theta\rangle = |\theta+\omega\rangle$$

ON-SHELL AMPLITUDES

What is a Scattering Amplitude?

1. Lorentz invariance/Little group covariance
2. Unitarity

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e.g. ordinary massless particles

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2. Unitarity

$$\mathcal{M}_{h_1\ldots h_n}(\Lambda k_1 \ldots \Lambda k_n) = e^{ih_1\theta(\Lambda,k_1)} \ldots e^{ih_n\theta(\Lambda,k_n)}\mathcal{M}_{h_1\ldots h_n}(k_1 \ldots k_n)$$

$$\mathcal{M} - \mathcal{M}^\dagger = i\mathcal{M}\mathcal{M}^\dagger$$

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Step.1 *trivialize kinematics by choosing right variables*

$$\lambda_\alpha(p_i)=|i\rangle \qquad \tilde{\lambda}_{\dot\alpha}(p_i)=[i| \qquad \lambda_\alpha(p)\tilde{\lambda}_{\dot\alpha}(p)=p_{\alpha\dot\alpha}=|i\rangle[i|$$

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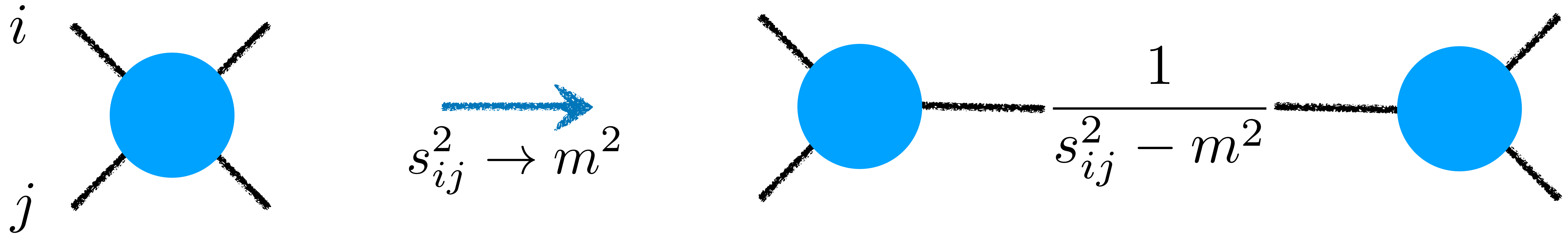
$$\begin{cases}\Lambda_\alpha^{\ \beta}\lambda_\beta(p)=e^{i\theta(\Lambda,p)/2}\lambda_\alpha(\Lambda p)=w(\Lambda,p)\lambda_\alpha(\Lambda p)\\ \tilde{\Lambda}_{\dot\alpha}^{\ \dot\beta}\tilde{\lambda}_{\dot\beta}(p)=e^{-i\theta(\Lambda,p)/2}\tilde{\lambda}_{\dot\alpha}(\Lambda p)=w(\Lambda,p)^{-1}\tilde{\lambda}_{\dot\alpha}(\Lambda p)\end{cases} \longrightarrow \begin{cases}\mathcal{M}(\lambda_\alpha(p_i),\tilde{\lambda}_{\dot\alpha}(p_j))=\mathcal{M}(\langle ij\rangle,[ij]) & SL(2,\mathbb{C})\text{-invariant}\\ \mathcal{M}(w_i^{-1}\langle ij\rangle, w_i[i,j])=w_i^{2h_i}\mathcal{M}(\langle ij\rangle,[ij]) & 2h_i\text{-homogenous}\end{cases}$$

What is a Scattering Amplitude?

Step.2 *factorization+weak coupling*



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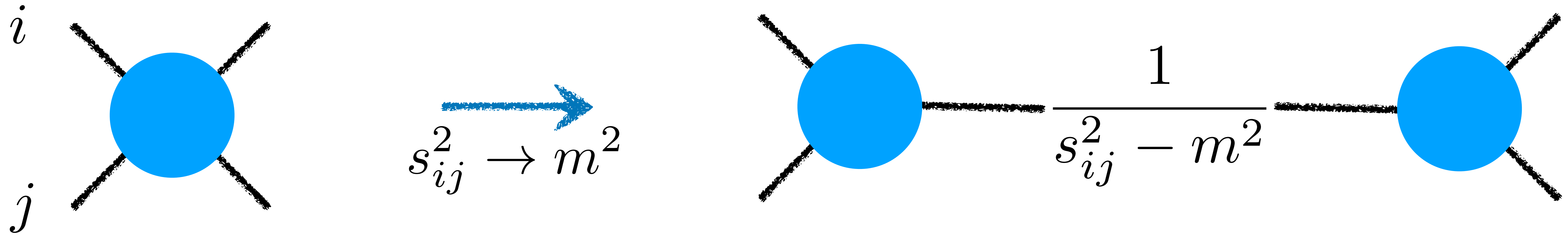

Example

$$\mathcal{M}_{c.o.m.}(s,t,u) = g^2 \frac{s-u}{\sqrt{-su}}$$

consistent????
masses, spin, tree or loop?

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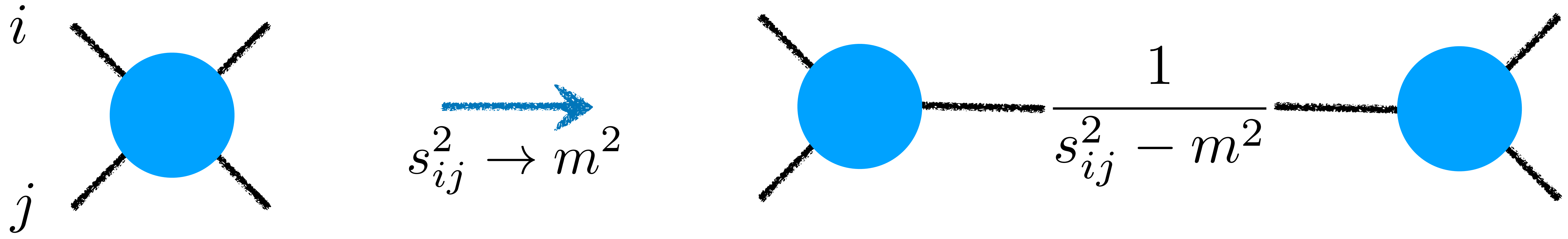
Step.2 *factorization+weak coupling*



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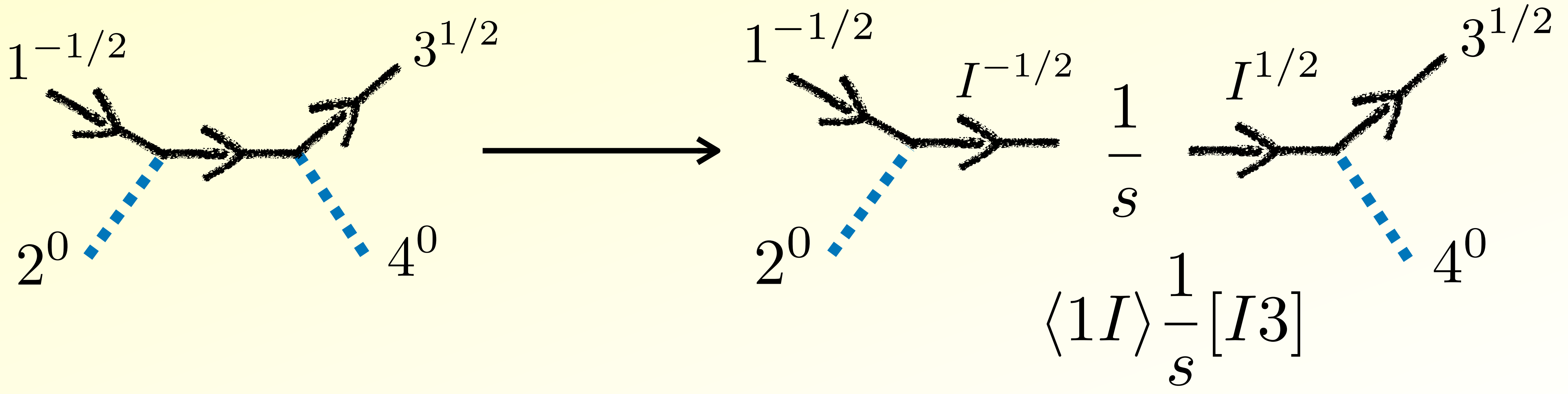
$$=$$

$$\mathcal{M} = g^2 \langle 12\rangle [23] \left(\frac{1}{s} - \frac{1}{u}\right)$$

yes! m=0, h=-1/2, 0, 1/2, 0; tree



$\mathscr{M}$ is solution of a linear differential problem

$$\mathbb{H}_{i=1,2,\ldots} = -\frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial\lambda_\alpha} - \tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial\tilde{\lambda}_{\dot{\alpha}}}\right)_{i=1,2,\ldots}$$

$$\mathbb{H}_i \mathcal{M} = h_i \mathcal{M} = \begin{cases} -\frac{1}{2} \times \mathcal{M} & i=1 \\ 0 \times \mathcal{M} & i=2 \\ \frac{1}{2} \times \mathcal{M} & i=3 \\ 0 \times \mathcal{M} & i=4 \end{cases}$$

$\mathbb{H}^i$ *represented on the space of functions of spinors*

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CSP: *represent 2 more diff.op.* —$\mathbb{W}_\pm$— *on functions of spinors!*

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$$\left.\begin{aligned} e^{i\alpha_\mp W_\pm}|\theta\rangle &= e^{i(\alpha_\mp\cdot\mu_\pm)}|\theta\rangle \\ e^{-i\omega H}|\theta\rangle &= |\theta+\omega\rangle\end{aligned}\right\} \longrightarrow \mathcal{M}_{\theta_1\ldots\theta_n}(\Lambda k_1\ldots) = \mathcal{M}_{\theta_1-\omega\ldots\theta_n}(k_1\ldots)\prod_i e^{-i\alpha^+\mu_1^- - i\alpha^-\mu_1^+}$$

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$$\mathbb{W}^i_\pm\mathcal{M} = \mu^i_\pm\mathcal{M} \qquad \mathbb{H}^i\mathcal{M} = i\frac{d}{d\theta_i}\mathcal{M} \qquad \begin{aligned}[][\mathbb{W}_+,\mathbb{W}_-] &= 0\\ [\mathbb{H},\mathbb{W}_\pm] &= \pm\mathbb{W}_\pm\end{aligned}$$

$\forall i = 1,2,\ldots$

The right variables, again

$$\mathcal{M}(\lambda^i_\alpha(p), \lambda^j_{\dot\alpha}(p)) = \mathcal{M}(\langle ij\rangle, [ij]) \qquad \lambda_\alpha(p)\tilde\lambda_{\dot\alpha}(p) = p_{\alpha\dot\alpha}$$

can't be right for CSP!

LG generators:

$$W_{\alpha\dot\alpha} = \frac{-i}{2}\left(J_\alpha{}^\beta P_{\beta\dot\alpha} - J^{\dot\beta}_{\dot\alpha} P_{\alpha\dot\beta}\right) \qquad J_{\alpha\beta} = i\lambda_{(\alpha}\frac{\partial}{\partial\lambda^{\beta)}} \qquad J_{\dot\alpha\dot\beta} = i\tilde\lambda_{(\dot\alpha}\frac{\partial}{\partial\tilde\lambda^{\dot\beta)}} \longrightarrow \begin{cases} W_{\alpha\dot\alpha} = P_{\alpha\dot\alpha}\mathbb{H} \\ W_{\alpha\dot\alpha}W^{\alpha\dot\alpha} = 0 \end{cases}$$

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Solution: $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ acts on **2-dim** space of spinors

$$\begin{cases} \lambda^i_\alpha = |i\rangle \quad \rho^i_\alpha = |\mathbb{i}\rangle \quad \langle\lambda^i\rho^i\rangle = \langle i\mathbb{i}\rangle = \text{const} \\ \tilde\lambda^i_{\dot\alpha} = [i| \quad \tilde\rho^i_{\dot\alpha} = [\mathbb{i}| \quad [\tilde\lambda^i\tilde\rho^i] = [i\mathbb{i}] = \text{const} \end{cases}$$

example

$\lambda_\alpha = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_\alpha$ $\rho_\alpha \propto \frac{1}{\sqrt{2E}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_\alpha$

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1. Defined up to $ISO(2)$: $|i\rangle \to w|i\rangle\ , \quad |\mathbb{i}\rangle \to w^{-1}|\mathbb{i}\rangle + 2i\alpha_+ w^{-1}|i\rangle\ , \quad [i| \to w^{-1}[i|\ , \quad [\mathbb{i}| \to w[\mathbb{i}| - 2i\alpha_- w[i|\ ,$

2. non-vanishing W^2_μ : $W_{\alpha\dot\alpha}W^{\alpha\dot\alpha} = -\langle\lambda\rho\rangle[\tilde\lambda\tilde\rho]\lambda_\alpha\frac{\partial}{\partial\rho_\alpha}\tilde\lambda^{\dot\alpha}\frac{\partial}{\partial\tilde\rho^{\dot\alpha}} \neq 0$

ISO(2) acting on Amplitudes

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~~2. non-vanishing W_μ^2 : $W_{\alpha\dot{\alpha}}W^{\alpha\dot{\alpha}} = -\langle\lambda\rho\rangle[\tilde{\lambda}\tilde{\rho}]\lambda_\alpha \frac{\partial}{\partial\rho_\alpha}\tilde{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\rho}^{\dot{\alpha}}} \neq 0$~~

$\mathbb{H}^i\,, \mathbb{W}^i_\pm$ *represented on bi-spinor functions*

$$\mathbb{W}_i^- = \langle i\mathbb{i}\rangle|i\rangle\frac{\partial}{\partial|\mathbb{i}\rangle}\ , \qquad \mathbb{W}_i^+ = [i\mathbb{i}]|i]\frac{\partial}{\partial|\mathbb{i}]}\ , \qquad \mathbb{H}_i = -\frac{1}{2}\left(|i\rangle\frac{\partial}{\partial|i\rangle} - |\mathbb{i}\rangle\frac{\partial}{\partial|\mathbb{i}\rangle} - |i]\frac{\partial}{\partial|i]} + |\mathbb{i}]\frac{\partial}{\partial|\mathbb{i}]}\right)$$

$$[\mathbb{W}_+, \mathbb{W}_-] = 0$$

$$[\mathbb{H}, \mathbb{W}_\pm] = \pm\mathbb{W}_\pm$$

CSP-Amplitudes

$$\mathbb{W}_i^- = \langle i\mathbb{i}\rangle |i\rangle \frac{\partial}{\partial|\mathbb{i}\rangle}\ , \qquad \mathbb{W}_i^+ = [i\mathbb{i}]|i]\frac{\partial}{\partial|\mathbb{i}]}\ , \qquad \mathbb{H}_i = -\frac{1}{2}\left(|i\rangle\frac{\partial}{\partial|i\rangle} - |\mathbb{i}\rangle\frac{\partial}{\partial|\mathbb{i}\rangle} - |i]\frac{\partial}{\partial|i]} + |\mathbb{i}]\frac{\partial}{\partial|\mathbb{i}]}\right)$$

$$\mathcal{M}_{\theta_1\dots\theta_n} = \mathcal{M}_{\theta_1\dots\theta_n}(\langle ij\rangle, [ij], \langle i\mathbb{j}\rangle, [i\mathbb{j}], , \langle\mathbb{i}\mathbb{j}\rangle, [\mathbb{i}\mathbb{j}])$$

$$\mathbb{W}^i_\pm\mathcal{M} = \mu^i_\pm\mathcal{M}$$

$$\mathbb{H}^i\mathcal{M} = i\frac{d}{d\theta_i}\mathcal{M}$$

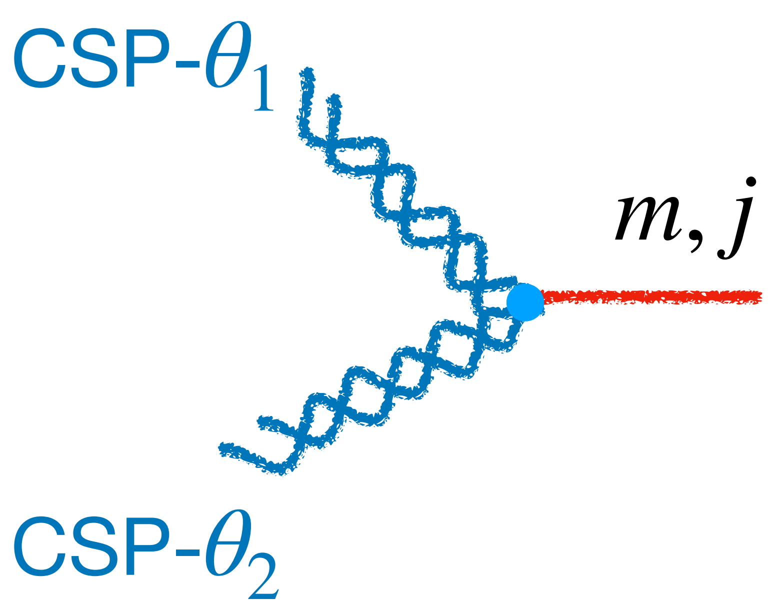
$$\langle i\mathbb{i}\rangle\sum_{j=1}^{n}\left(\langle ji\rangle\frac{\partial}{\partial\langle j\mathbb{i}\rangle} + \langle\mathbb{j}i\rangle\frac{\partial}{\partial\langle\mathbb{j}\mathbb{i}\rangle} + \dots\right)\log\mathcal{M}_{\theta_1\dots\theta_n} = \mu_i e^{-i\theta_i}$$

$$[i\mathbb{i}]\sum_{j=1}^{n}\left([ji]\frac{\partial}{\partial[j\mathbb{i}]} + [\mathbb{j}i]\frac{\partial}{\partial[\mathbb{j}\mathbb{i}]} + \dots\right)\log\mathcal{M}_{\theta_1\dots\theta_n} = \mu_i e^{+i\theta_i}$$

$$\mathrm{Exp}\left(-i\omega\mathbb{H}_i\right)\mathcal{M}_{\theta_1\dots\theta_n} = \mathcal{M}_{\theta_1\dots(\theta_i+\omega)\dots\theta_n}$$

CSP-Amplitudes: explicit solutions

3-pts, Examples



$$|\mathbf{3}\rangle[\mathbf{3}| = -|1\rangle[1| - |2\rangle[2|$$

$$\mathcal{M}(\langle i\mathbb{j}\rangle, \cancel{\langle \mathbb{i}\mathbb{j}\rangle}, \langle ij\rangle, \ldots)$$

$$\langle \mathbb{1}\mathbb{2}\rangle\langle 12\rangle + \langle \mathbb{1}1\rangle\langle 2\mathbb{2}\rangle + \langle \mathbb{1}2\rangle\langle \mathbb{2}1\rangle = 0$$

translations $\mathbb{W}_{\pm}$
$$\begin{cases} \langle 21\rangle \dfrac{\partial}{\partial\langle 2\mathbb{1}\rangle} \log \mathcal{M}_{\theta_1\theta_2} = \dfrac{\mu}{\langle 1\mathbb{1}\rangle} e^{-i\theta_1} \\ [21] \dfrac{\partial}{\partial[2\mathbb{1}]} \log \mathcal{M}_{\theta_1\theta_2} = \dfrac{\mu}{[1\mathbb{1}]} e^{i\theta_1} \end{cases}$$

$\&\,, 1 \leftrightarrow 2$

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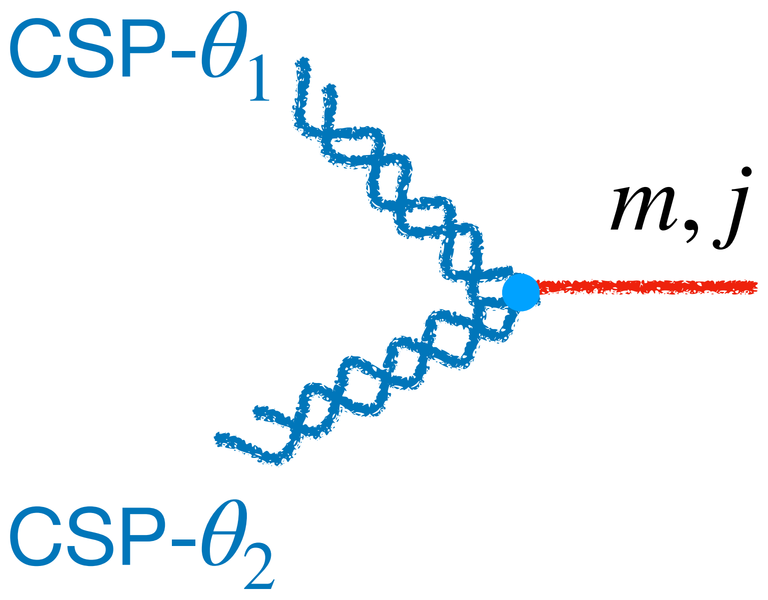

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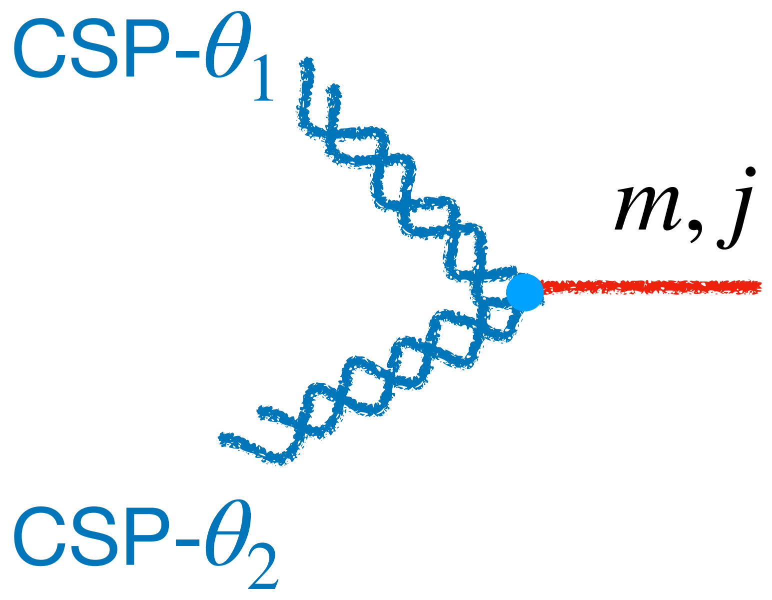
$$\mathcal{M}_{\theta_1\theta_2} = \mathrm{Exp}\left(\mu e^{-i\theta_1} \frac{\langle 2\mathbb{1}\rangle}{\langle 1\mathbb{1}\rangle\langle 21\rangle} + \mu e^{i\theta_1} \frac{[2\mathbb{1}]}{[1\mathbb{1}][21]}\right) \mathrm{Exp}\,(1 \leftrightarrow 2)\, \widetilde{\mathcal{M}}(\langle ij\rangle, [ij], \theta_i)$$

3-pts, Examples



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$$\boxed{\mathcal{M}_{\theta_1\theta_2} = \mathrm{Exp}\left(\mu e^{-i\theta_1}\frac{\langle 2\mathbb{1}\rangle}{\langle 1\mathbb{1}\rangle\langle 21\rangle} + \mu e^{i\theta_1}\frac{[2\mathbb{1}]}{[1\mathbb{1}][21]}\right)\mathrm{Exp}\,(1 \leftrightarrow 2)\,\widetilde{\mathcal{M}}(\langle ij\rangle, [ij], \theta_i)}$$

$$e^{-i\omega} \qquad e^{i\omega}$$

$$\sum_{h_i} \widehat{\mathcal{M}}_{h_i}(\langle ij\rangle, [i,j]) e^{-ih_i\theta_i}$$

helicity $\mathbb{H}$ $\quad \mathrm{Exp}\left(-i\omega\mathbb{H}^i\right)$

n-pts & High-energy limit

$$\mathcal{M}_{\theta_1\ldots\theta_n} = \sum_{p_j^{\pm}} \exp\left(\sum_j \frac{\mu_j^+ \langle j|p_j^+|\mathring{\mathbb{j}}]}{[j\mathring{\mathbb{j}}]\langle j|p_j^+|j]} + \frac{\mu_j^- \langle \mathring{\mathbb{j}}|p_j^-|j]}{\langle j\mathring{\mathbb{j}}\rangle\langle j|p_j^-|j]}\right) \sum_{\{h_i\}} \mathcal{M}_{h_1\ldots h_n} e^{-i\sum_k \theta_k h_k}$$

n-pts & High-energy limit

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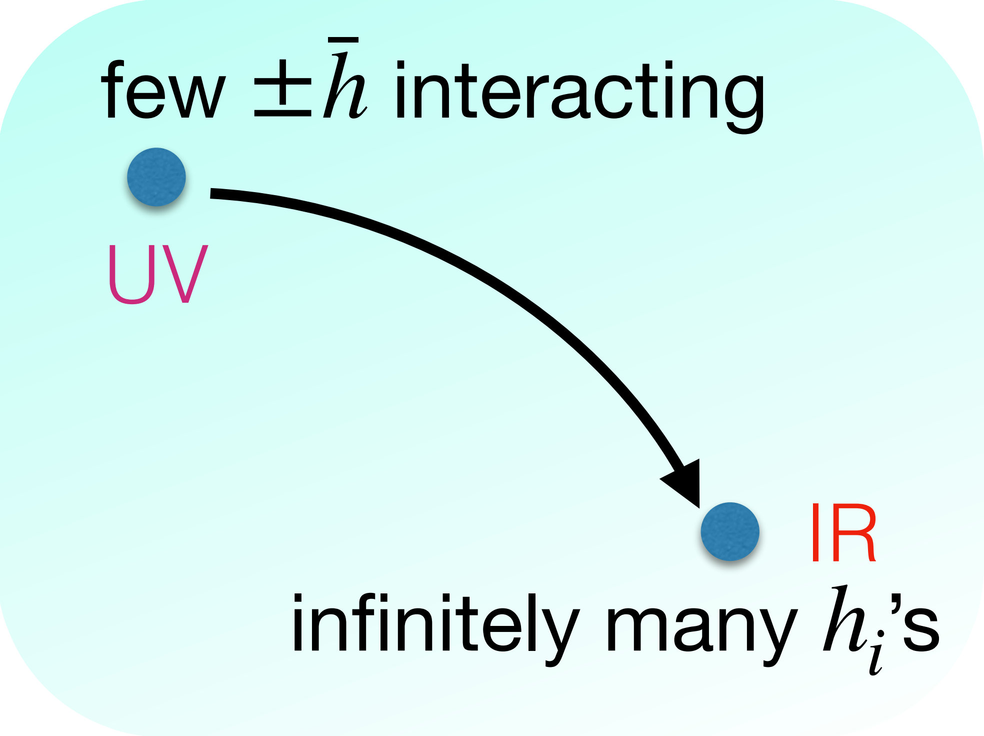
"mostly" helicity h_i

$\mathscr{M}$=Exp(…) × ($\mathscr{M}_{UV,\mu=0}$ × phases)

- IR-deformation —via W_μ^2 — of UV theories
- 3pt's uniquely fixed by UV



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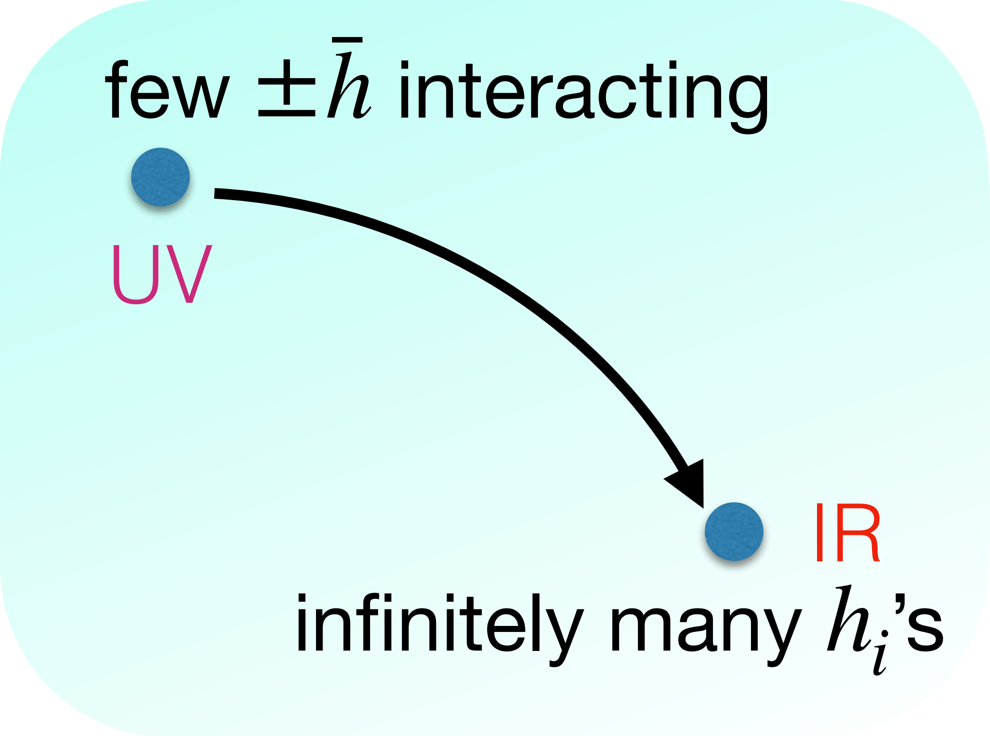

few $\pm\bar{h}$ interacting

UV

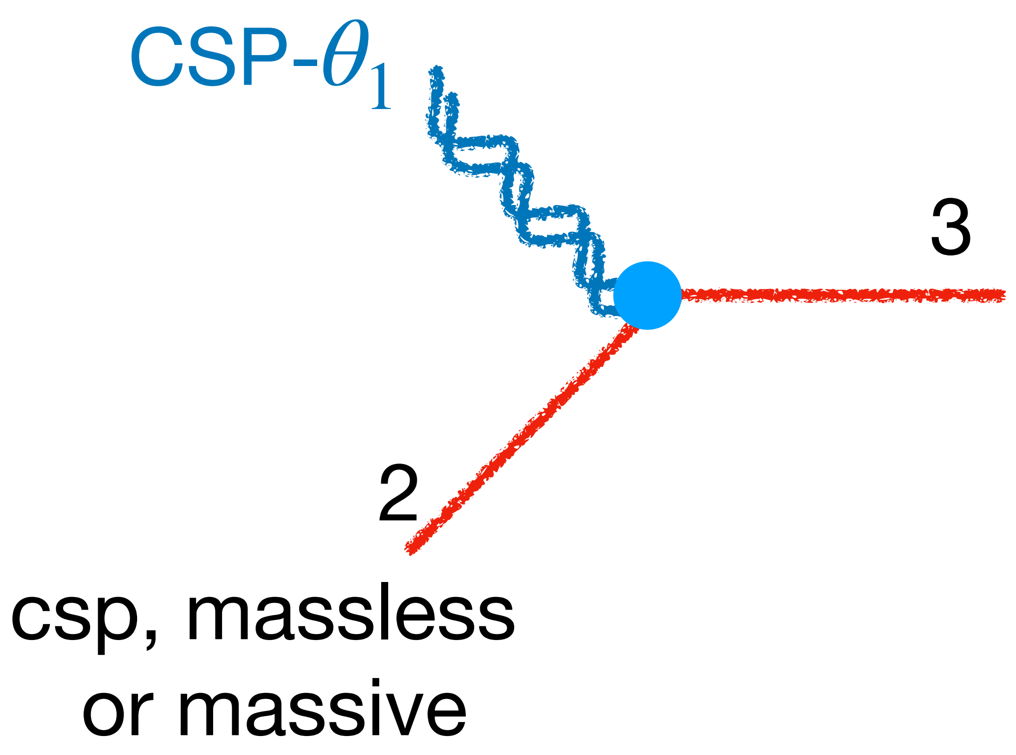
IR

infinitely many h_i's

Example 3pts: mostly photon-CSP

- mostly $\phi F_{\mu\nu}^2$, $aF_{\mu\nu}\widetilde{F}^{\mu\nu}$ ⟶ $\mathcal{M} = e^{\left(\mu e^{-i\theta_1}\frac{\langle 2\mathbb{1}\rangle}{\langle 21\rangle} - \mu e^{i\theta_1}\frac{[2\mathbb{1}]}{[21]}\right)} e^{(1\leftrightarrow 2)} \times \left(\frac{1}{f_1}\langle 12\rangle^2 e^{i\theta_1+i\theta_2} + \frac{1}{f_2}[12]^2 e^{-i\theta_1-i\theta_2}\right)$
- mostly $F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$ ⟶ $\mathcal{M} = e^{\left(\mu e^{-i\theta_1}\frac{\langle 2\mathbb{1}\rangle}{\langle 21\rangle} - \mu e^{i\theta_1}\frac{[2\mathbb{1}]}{[21]}\right)} e^{(1\leftrightarrow 2)} \times \frac{1}{M_{\rm Pl}}\left(\frac{\langle \mathbf{13}\rangle^4}{\langle 12\rangle^2} e^{-i\theta_1+i\theta_2} + \frac{[\mathbf{13}]^4}{[12]^2} e^{i\theta_1-i\theta_2}\right)$

Mass-splitting selection rule



no (on-shell) 3pt-solution for CSP coupling to same-mass particles (CSPs or not)

e.g. $\mathcal{M} \supset \mathrm{Exp}\left(e^{i\theta_1}\frac{\mu_1\langle 2\mathbb{1}\rangle}{\langle 1\mathbb{1}\rangle\langle 21\rangle} - e^{-i\theta_1}\frac{\mu_1[2\mathbb{1}]}{[1\mathbb{1}][21]}\right)$ $\quad \langle 12\rangle[21] = m_3^2 = \Delta m^2 \to 0$

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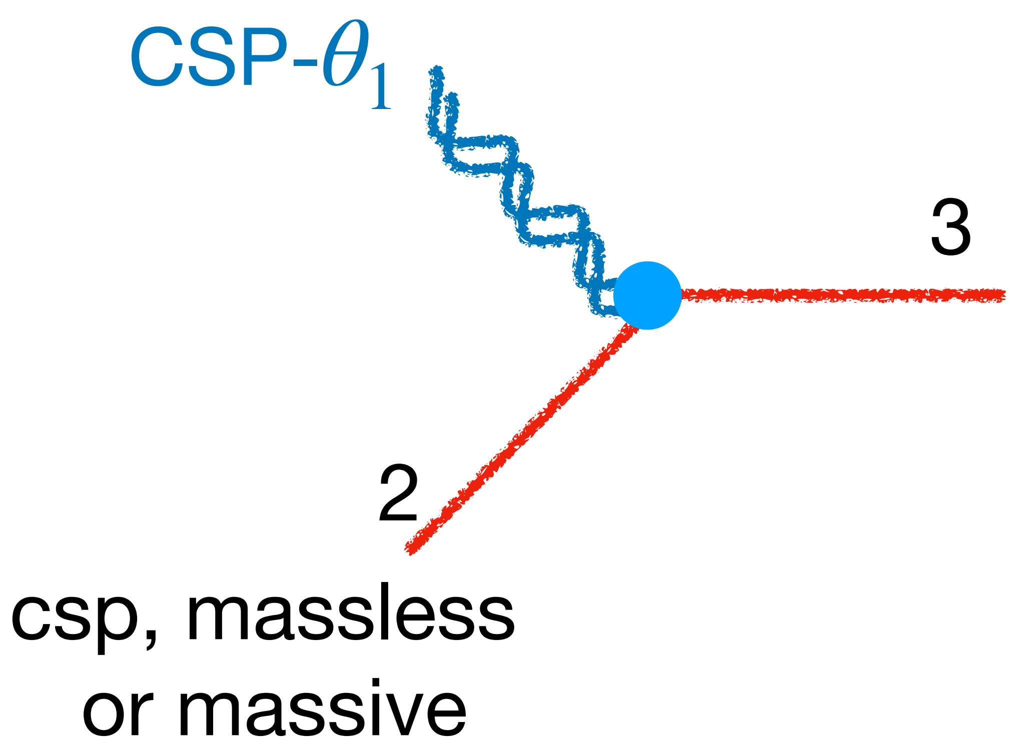

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is this
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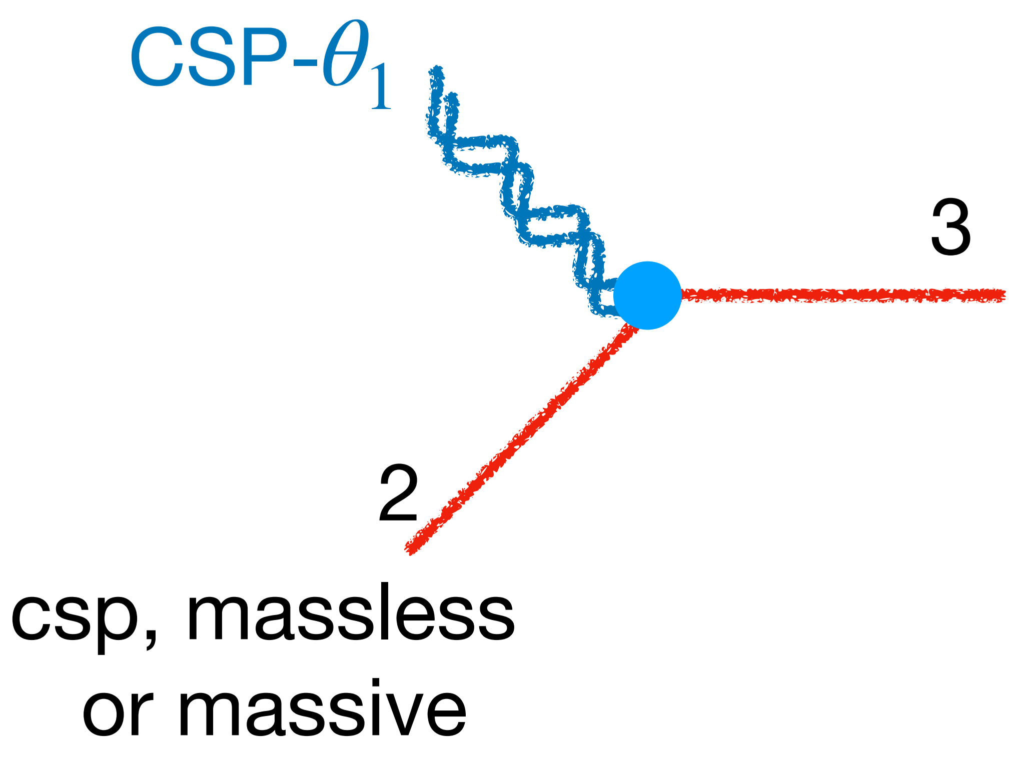

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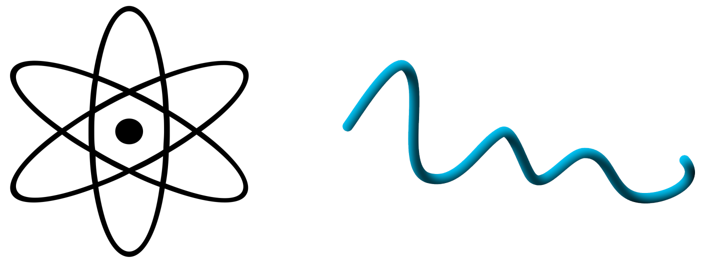
- 4pts trivial to write down: but $\mathscr{M}_{2\to 2} \to \mathscr{M}_{1\to 2} \times \mathscr{M}_{2\to 1}$ only if $\Delta m^2 \neq 0$
- No on-shell coupling to gravitons, to photons, to particle-antiparticles pairs,…
- Effective Δm^2 seen as off-shellness: good enough for some questions



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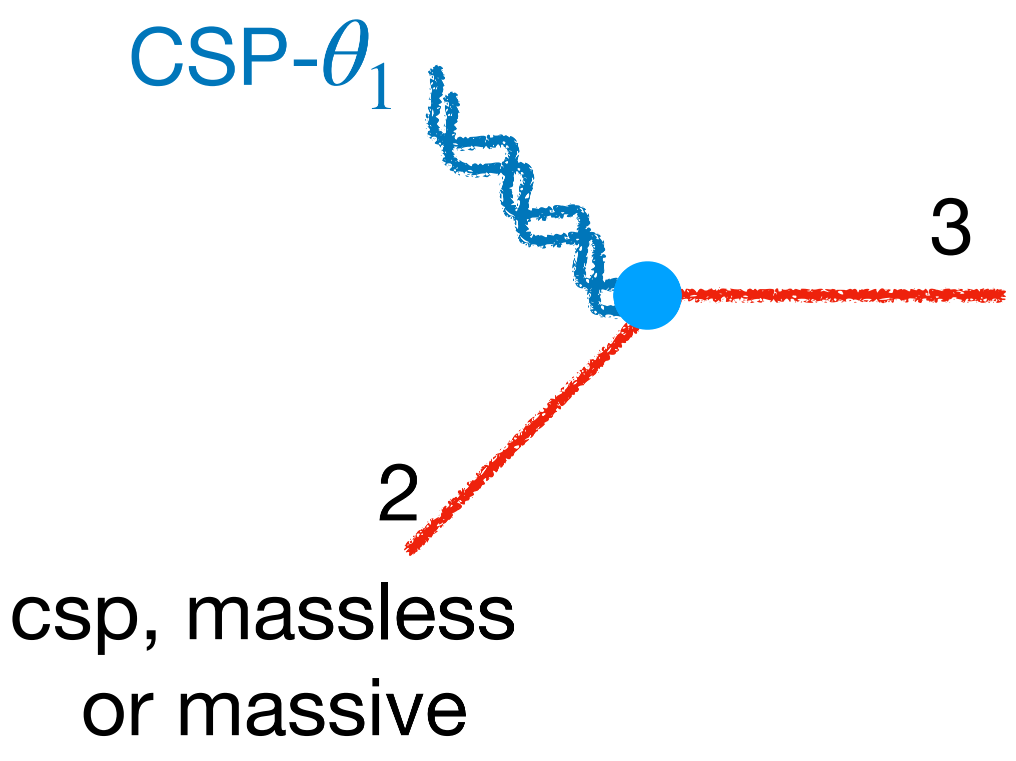

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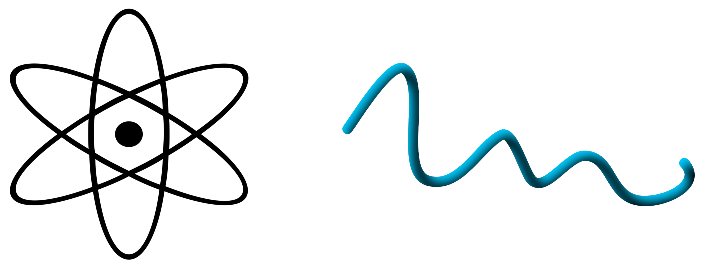

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- Effective Δm^2 seen as off-shellness: good enough for some questions

$$\cancel{\langle 0|T^{\mu\nu}(0)|1^{\theta_1}2^{\theta_2}\rangle} = \left(F_1(q^2)\langle 1|\sigma^\mu|2]\langle 1|\sigma^\nu|2]e^{i(\theta_1-\theta_2)} + \ldots\right)\exp\left(\frac{\mu_1^-}{\langle 1\mathbb{1}\rangle}\frac{\langle 2\mathbb{1}\rangle}{\langle 21\rangle} + \frac{\mu_1^+}{[1\mathbb{1}]}\frac{[2\mathbb{1}]}{[21]}\right)\exp\left(1 \leftrightarrow 2\right)$$

energy-momentum of 2 mostly-$|h| = 1$ CSPs

loophole in Weinberg-Witten

4pts: Examples

1. Lorentz invariant
2. ISO(2)-covariant
3. factorization/ 3pt-unitary

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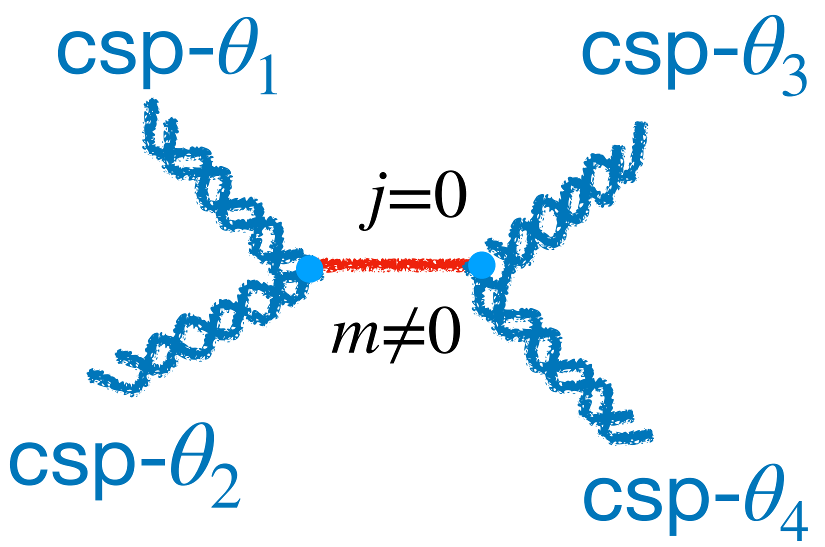
CSP-Euler-Heisenberg



$$\mathcal{M} = \frac{1}{s_{12}-m^2} e^{\left(\mu e^{-i\theta_1}\frac{\langle 2\mathbb{1}\rangle}{\langle 21\rangle} - \mu e^{i\theta_1}\frac{[2\mathbb{1}]}{[21]}\right)} e^{(1\leftrightarrow 2)} \times e^{\left(\mu e^{-i\theta_3}\frac{\langle 4\mathbb{3}\rangle}{\langle 43\rangle} - \mu e^{i\theta_3}\frac{[4\mathbb{3}]}{[43]}\right)} e^{(3\leftrightarrow 4)}$$

$$\times \left([12]^2[34]^2 e^{-i(\theta_1+\theta_2+\theta_3+\theta_4)} + [12]^2\langle 34\rangle^2 e^{-i(\theta_1+\theta_2-\theta_3-\theta_4)} + \ldots\right) + (2 \longleftrightarrow 3) + (2 \longleftrightarrow 4)$$

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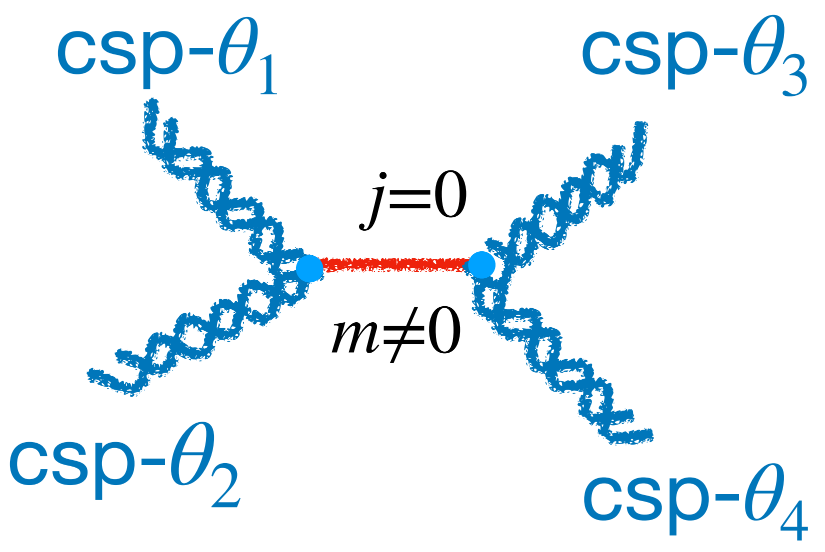
CSP-Euler-Heisenberg



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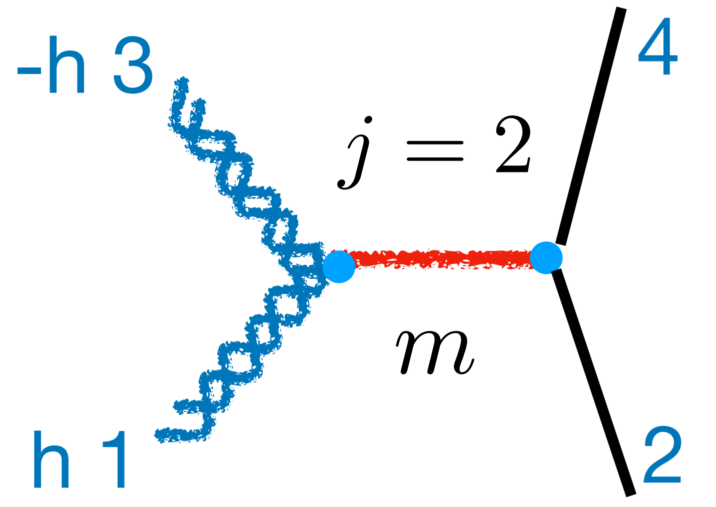
How CSP-photons gravitate?

bending of (CSP)-light



h-basis CSP's

$$\xrightarrow[m \to 0]{} \quad \text{F.T. } \langle 0|T^{\mu\nu}|1^{\theta_1} 3^{\theta_3}\rangle \frac{1}{t} \langle 0|T_{\mu\nu}|24\rangle$$

$$\uparrow$$

$$\sim \exp \frac{1}{t}$$

essential sing. from LG-phases → breaking equiv. principle @ $b\mu \gg 1$

4pts: Examples

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2. ISO(2)-covariant
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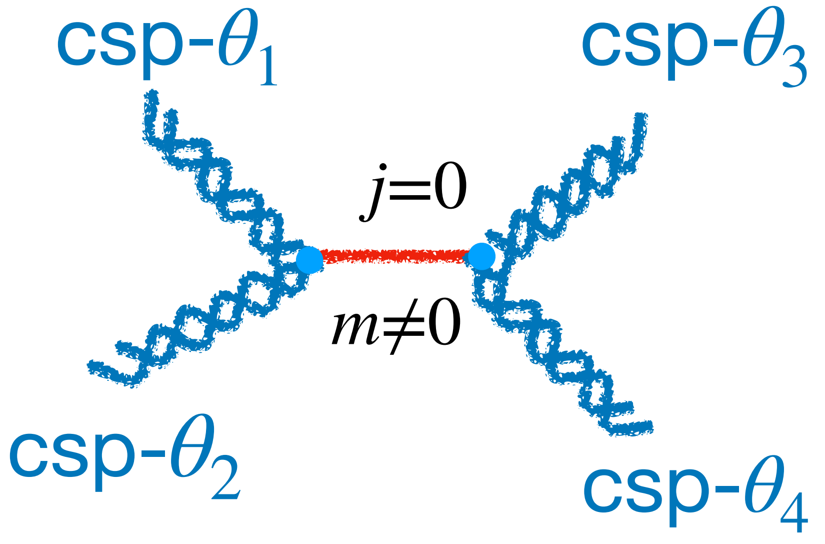
CSP-Euler-Heisenberg



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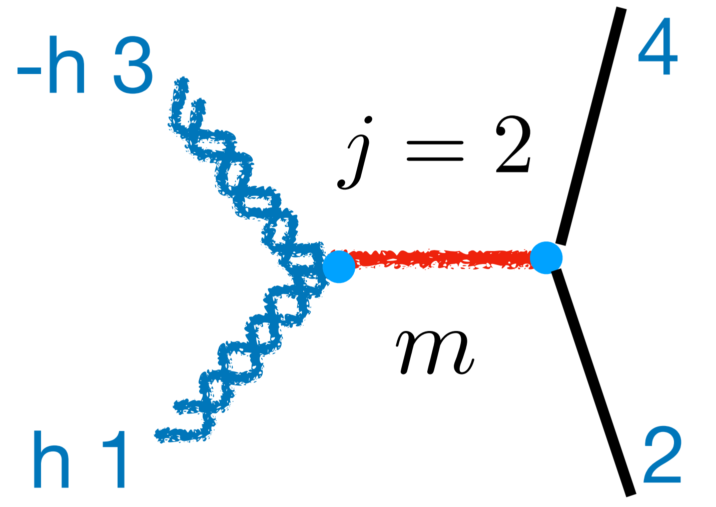

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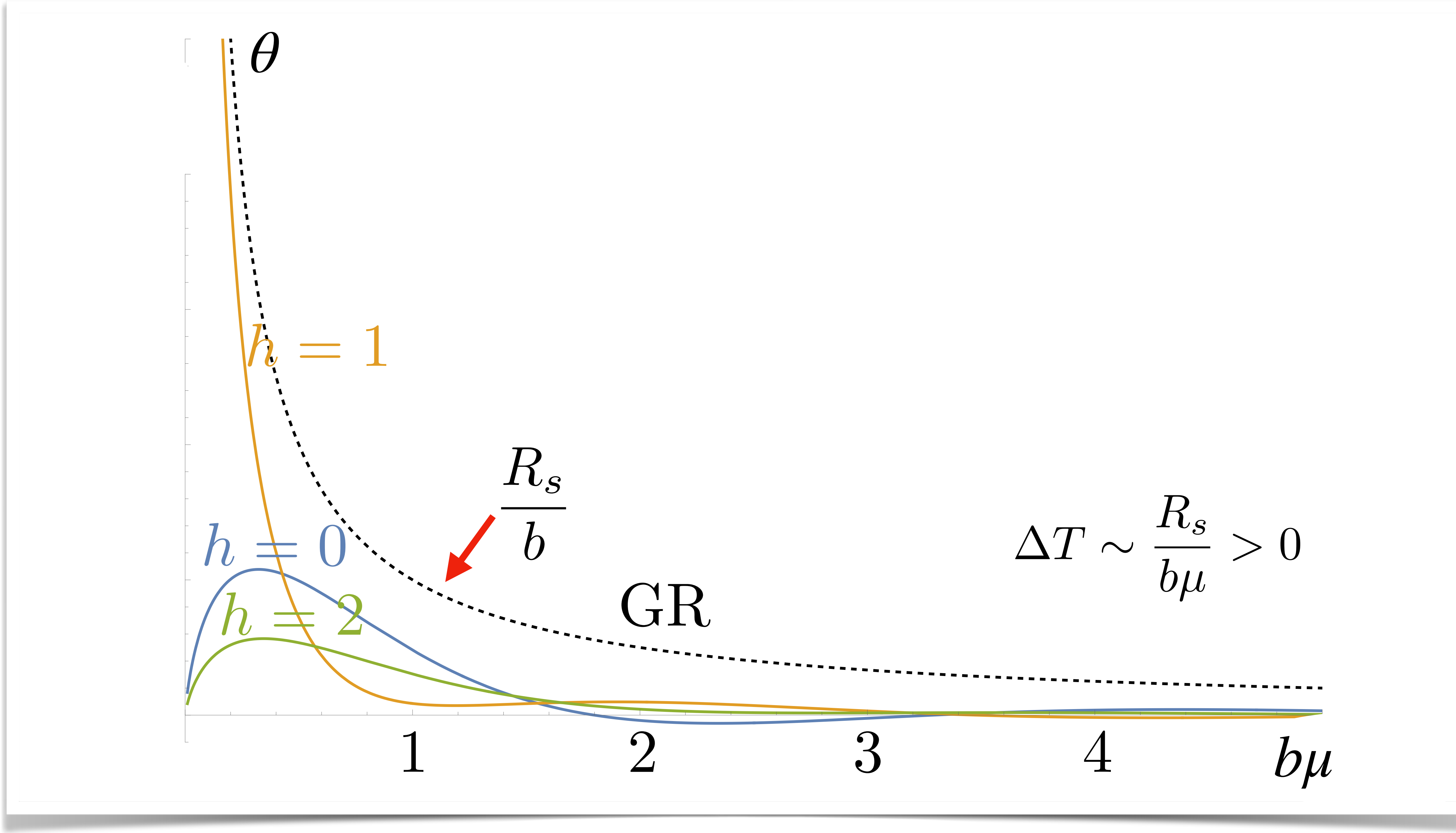
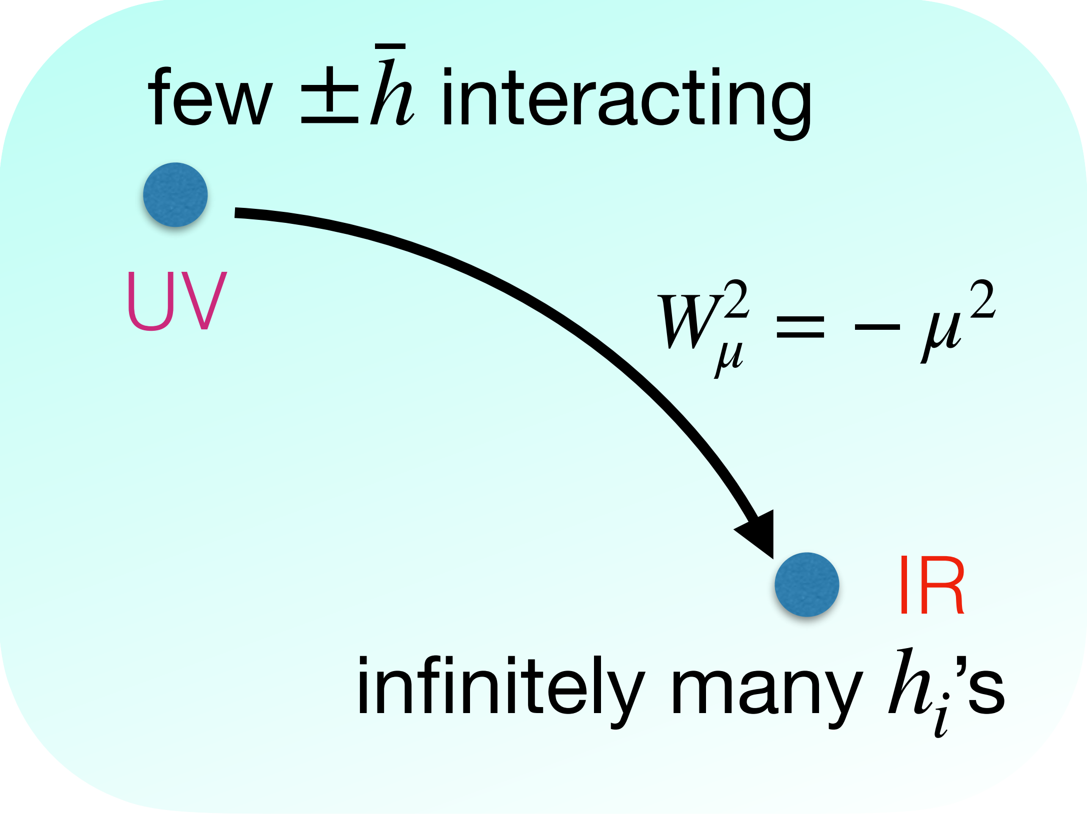


CONCLUSIONS

Conclusions & Outlook

(to our surprise)
CSP-amplitudes exist
&
easy to get

- Lorentz + ISO(2)-covariance ✓
- good high-energy behavior ✓
  IR-deformation of massless amp.
  (e..g. in gravity!)
- factorisation $4pts \to 3pts^2$ ✗ ✓
  mass selection rule
- (linear) coupling to gravity ✗ ✓



Future Stress Test:

- Classical Limit?
- Thermodynamics?
- Loops, running, RG-evolution?
- Off-shell correlators? (e.g. Compton $\langle 0 | J^\mu J^\nu | 1^{\theta_1} 2^{\theta_2} \rangle$, $\langle 0 | T^{\mu\nu} T^{\rho\sigma} | 1^{\theta_1} 2^{\theta_2} \rangle$)

THANK YOU!

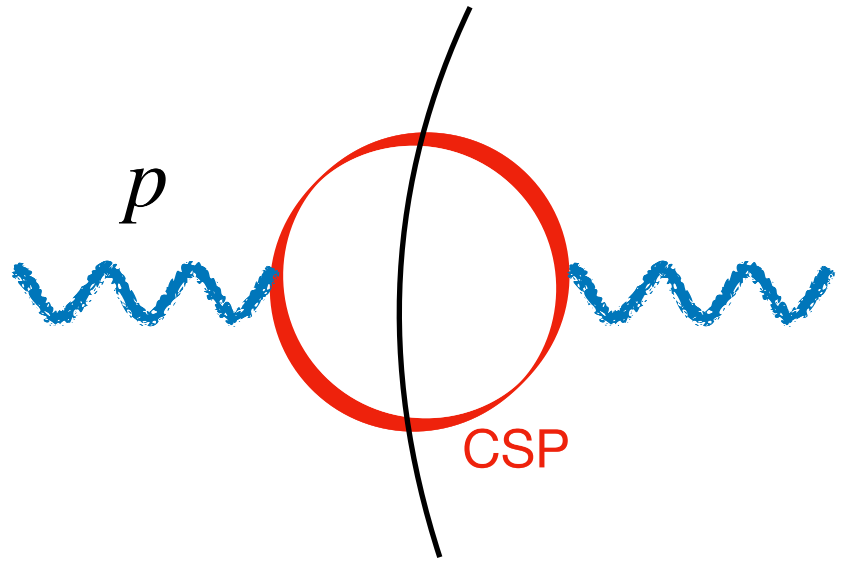
&

HAPPY BIRTHDAY RICCARDO!

Back-up slides

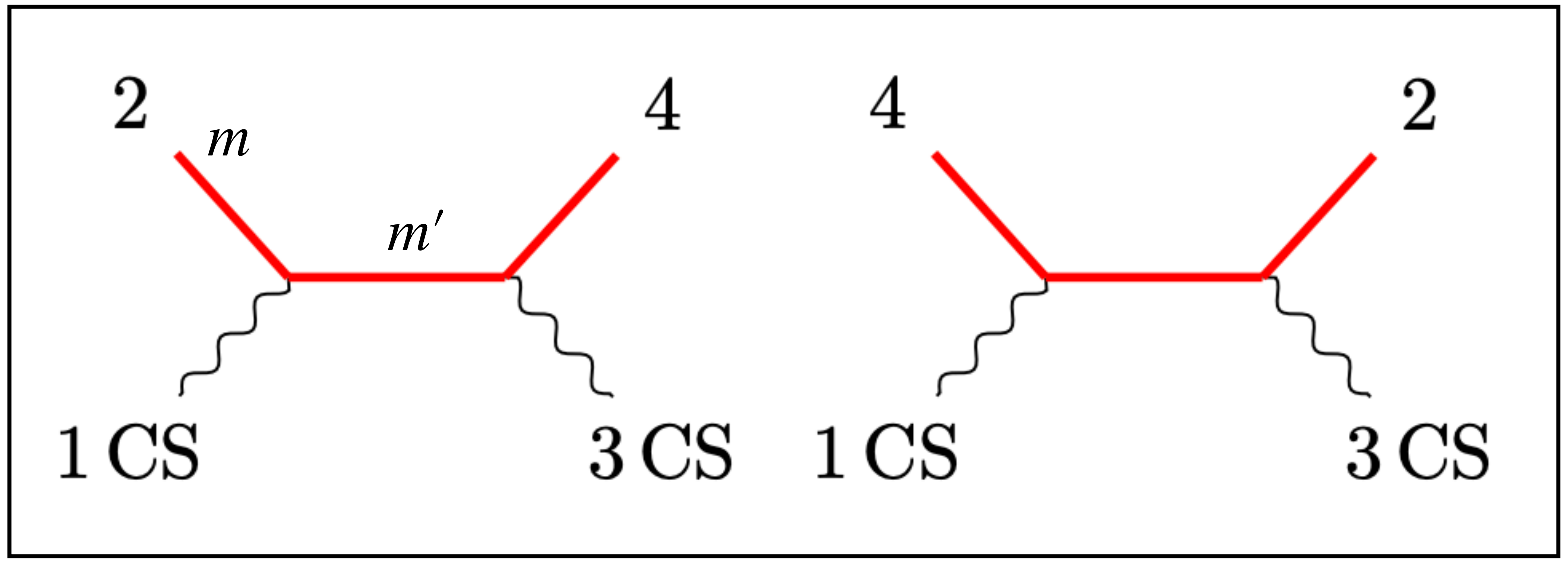
2pts and Running

$\text{Im}\ \langle J^{\mu}(p) J^{\nu}(-p) \rangle$

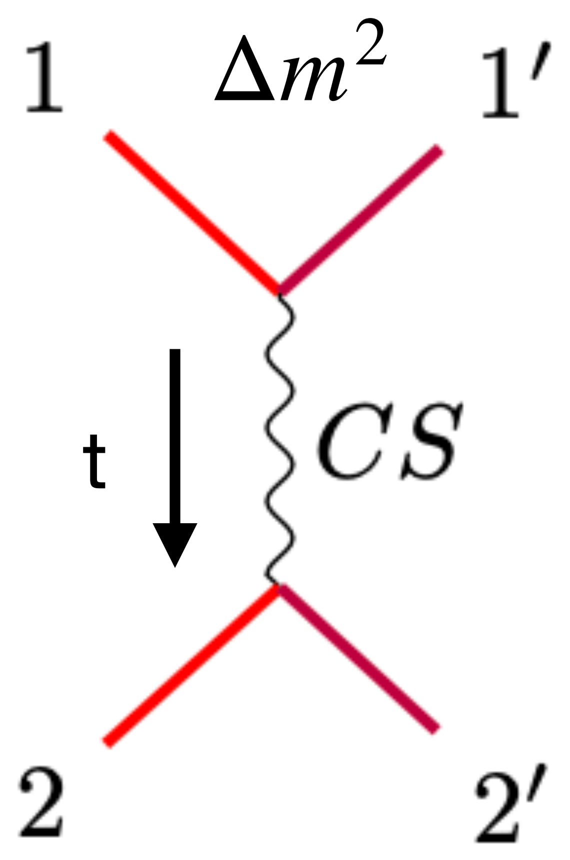


Scalar Compton

$$\mathcal{M}(1^{\theta_1} 2\, 3^{\theta_3} 4) = -e^{2i\mu\frac{\epsilon_1\cdot p_2+\epsilon_3\cdot p_4}{s-m^2}}\frac{\Lambda^2}{s-m'^2} - e^{2i\mu\frac{\epsilon_1\cdot p_4+\epsilon_3\cdot p_2}{u-m^2}}\frac{\Lambda^2}{u-m'^2}$$



(mostly h=0) Internal CSP

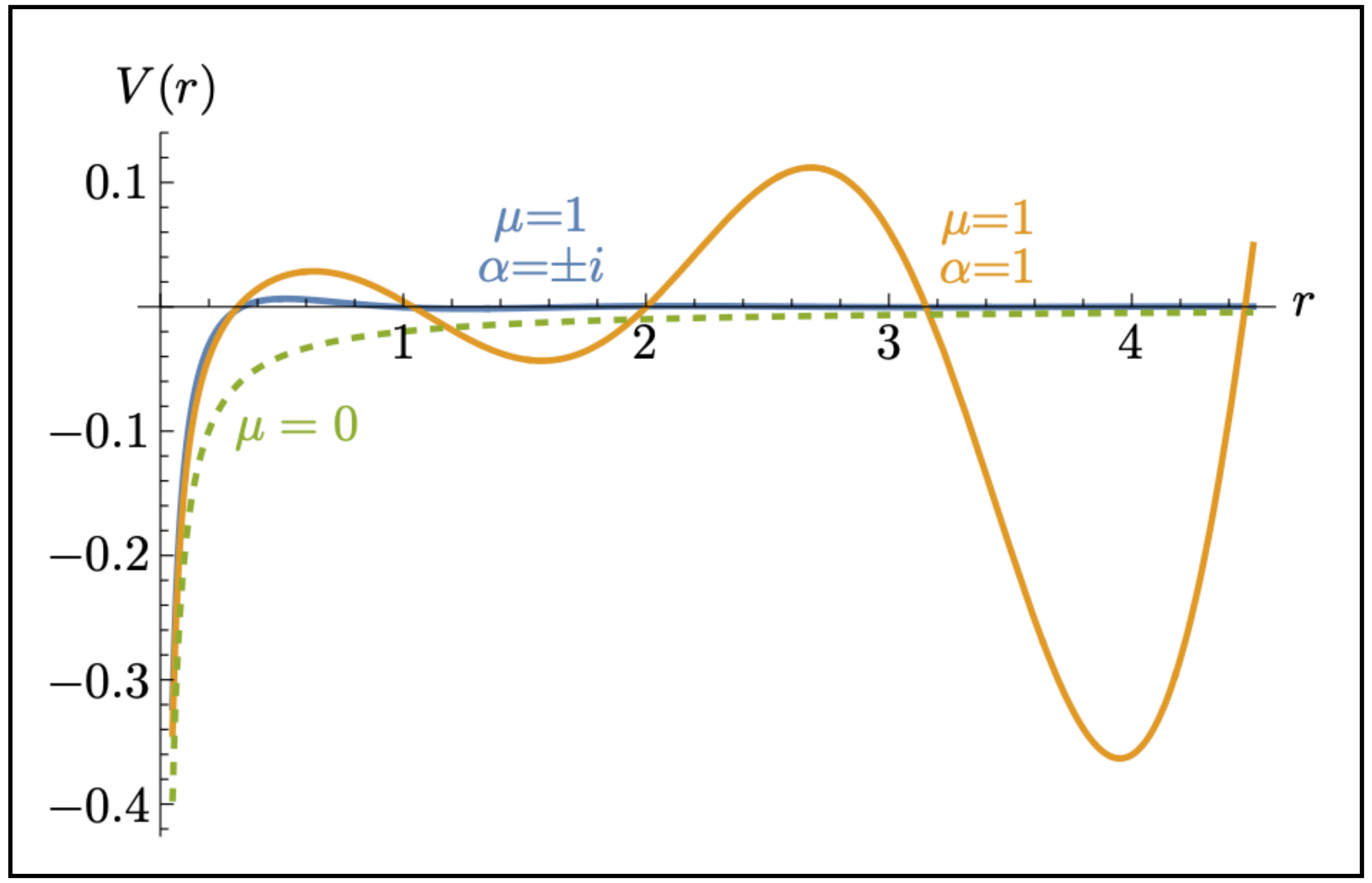


$$\mathcal{M}(1\,1'\,2\,2') \underset{t\to 0}{=} -\frac{\Lambda^2}{t} J_0\left(\frac{2\mu\sqrt{-s}}{\Delta m^2}\right) + \mathcal{O}(t)$$

Residue

$$\mathcal{M}(1\,1'\,2\,2') = -\frac{\Lambda^2}{t} J_0\left(\frac{2\mu\sqrt{-s}}{\alpha t + \Delta m^2}\right) - \frac{\Lambda^2}{u} J_0\left(\frac{2\mu\sqrt{-s}}{\alpha u + \Delta m^2}\right)$$

Family of solutions factorise well



potential at $\Delta m^2 \to 0\,, \quad \alpha \neq 0$

CSP-light bending

momentum kick
via KMOC-formalism

$$ {}_{\text{in}}\langle\psi|P_1^\mu|\psi\rangle_{\text{in}} - {}_{\text{in}}\langle\psi|S^\dagger P_1^\mu S|\psi\rangle_{\text{in}} = \Delta p^\mu_{\text{CS}} = {}_{\text{in}}\langle\psi|S^\dagger[P_1^\mu, S]|\psi\rangle_{\text{in}} $$

eikonal limit:

$$ \Delta p^\mu_{\text{CS}} = \left\langle -i\int \frac{\mathrm{d}^4 q}{(2\pi)^2}\,\delta(2p_1\cdot q)\delta(2p_2\cdot q)e^{ib\cdot q}q^\mu\,\mathcal{M}_4(1^{\theta_1}\phi\,\phi\,4^{\theta_4})\Big|_{\substack{p_3=-p_2-q\\ p_4=-p_1+q}} + \ldots \right\rangle $$

generic wave-packet ψ

$$ \boxed{\begin{aligned} \Delta p^\mu_{\text{CS}} = \frac{\kappa^2}{64\pi}(s-m^2)\sum_{h_1,h_4} c^*_{h_4}c_{h_1}\int_0^{+\infty}\mathrm{d}q\,[J_{h_1-1}(4\mu/q)\,J_{h_4-1}(4\mu/q) + J_{h_1+1}(4\mu/q)\,J_{h_4+1}(4\mu/q)] \\ \times(-1)^{\Delta h}\Big\{\Big[J_{\Delta h+1}\Big(\sqrt{-b^2}q\Big) - J_{\Delta h-1}\Big(\sqrt{-b^2}q\Big)\Big]\frac{b^\mu}{\sqrt{-b^2}} \\ +i\Big[J_{\Delta h+1}\Big(\sqrt{-b^2}q\Big) + J_{\Delta h-1}\Big(\sqrt{-b^2}q\Big)\Big]\frac{v^\mu}{\sqrt{-v^2}}\Big\}\,. \end{aligned}} $$

definite-θ wave-packet

$$ \boxed{\Delta p^\mu_{\text{CS}} \overset{\delta\theta=0}{=} -\frac{\kappa^2}{16\pi}(s-m^2)\frac{b^\mu}{b^2} = 2\Delta p^\mu_{\text{GR}}} $$

definite-h wave-packet
time-delay at large-b

$$ \boxed{\Delta T_h(\sqrt{-b^2}\mu \gg 1) = \left(\frac{\kappa^2\sqrt{s}}{32\pi^2}\right)\frac{2-(-1)^h\sqrt{2}\sin\Big(4\sqrt{2}\sqrt{-b^2}\mu\Big)}{\sqrt{-b^2}\mu} > 0} $$