Goldstones, Vectors and Positivity*

(* given by a former student of a group of former students of RB)

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About some future directions in fundamental physics

(organised by a group of former students of RB)

Amplitudes at large s

- Pre-LHC: $\mathcal{A}(WW \to WW) \sim s$: hint of new physics
- Post Higgs: still $\mathcal{A}(WW \to WW) \sim s$ (but slower)
- Not a conceptual problem: Froissart bound requires $A(s)/s^2 \to 0$

Dispersion relations and positivity

Regge behaviour (large s, fixed t) determines dispersion relations:

$$A(s)/s^2 \to 0 \qquad \Rightarrow \qquad \text{couplings of dimension 8 dispersive}$$

$$\frac{g_8}{\Lambda^4} \sim \int_{\Lambda^2}^{\infty} ds \frac{\sigma(s)}{s^2} > 0$$

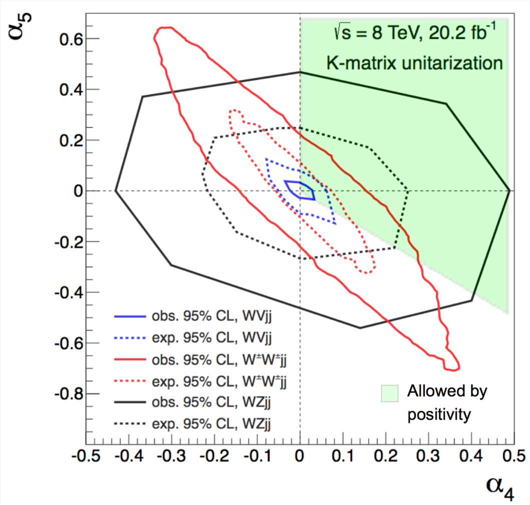
Sometimes can be interesting:

Light-by-light scattering

$$a_1(F_{\mu\nu}F^{\mu\nu})^2 + a_2(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2$$

$$a_1, a_2 \geq 0$$

WW scattering [Zhang and Zhou, '19]



Dispersion relations and positivity - 2

Regge behaviour (large s, fixed t) determines dispersion relations:

$$A(s)/s \to 0 \qquad \Rightarrow \qquad \text{couplings of dimension 6 dispersive}$$

$$1 - a^2 = \frac{v^2}{6\pi} \int_0^\infty \frac{ds}{s} \left(2\sigma^{\text{tot}}_{I=0}(s) + 3\sigma^{\text{tot}}_{I=1}(s) - 5\sigma^{\text{tot}}_{I=2}(s)\right)$$

[Falkowski and Rychkov, '12]

a coupling of h to WW (related to c_H in SILH notation)

Dispersion relations and positivity - 3

Regge behaviour (large s, fixed t) determines dispersion relations:

$$A(s) \to 0 \qquad \Rightarrow \qquad \text{couplings of dimension 4 dispersive}$$

Definitively not true for Goldstone bosons: inconsistent with shift symmetry

$$g_4 \sim \int_{\Lambda^2}^{\infty} ds\, \sigma(s) > 0$$

Pion scattering at large N

QCD at large N_c

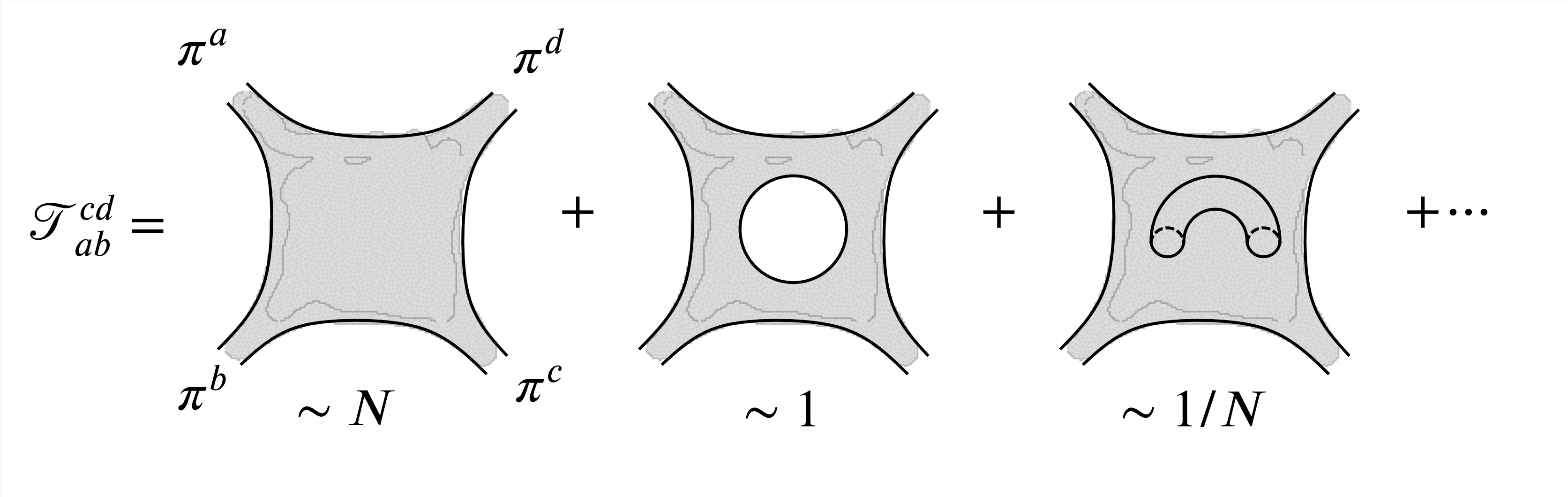
QCD with N_f massless quarks, in the 't Hooft limit: $N_c \to \infty$, fixed $\lambda = g^2 N_c$

Spectrum:

- Barions decouple: mass grows with N_c
- Mesons and Glueball marginally affected
- Confinement and chiral symmetry breaking $\Rightarrow$ Goldstone bosons π^a

Pion scattering at Large N

$$\pi^a \pi^b \longrightarrow \pi^c \pi^d$$



$q\bar{q}$-mesons loops & tetraquarks *glueballs*

Scattering amplitudes of mesons are meromorphic!

Bootstrapping scattering amplitudes

Mesons scattering amplitude: an infinite tower of tree level exchanges

Parameters space:

- masses, spin of resonances m_i^2, J_i
- on-shell couplings g_{ijk}

Constraints:

- Crossing
- Analiticity
- Unitarity (positivity)
- Regge behavior

Pion scattering at Large N - 2

$$\pi^a \pi^b \longrightarrow \pi^c \pi^d$$

- Crossing: $\mathcal{A}(s,t) = \mathcal{A}(t,s)$
- Analyticity: only poles on the real axis
- OZI rule: no poles on the negative real axis

Isospin: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

$q\bar{q}$ - mesons $\in 0, 1$

tetraquark - mesons $\in 2$ (suppressed at large N_c)

poles at negative $s \leftrightarrow$ poles in Isospin 2 channel

Pion scattering at Large N - 3

$$\pi^a \pi^b \longrightarrow \pi^c \pi^d$$

- Regge boundedness:

$$\lim_{|s|\to\infty} \mathcal{A}(s,t) \sim s^{\alpha(t)} \qquad (\text{fixed } t < 0)$$

  Eventually we want to expand for $t \simeq 0$ – important to know $\alpha(0)$:

  Finite N_c: $\alpha(0) \simeq 1.08$ (Pomeron)

  Large N_c: $\alpha(0) \simeq 0.5$ (Pomeron suppressed, ρ-trajectory)

- Unitarity

$$\mathcal{A}(s,t) = \sum_{J \text{ even}} f_J(s) \mathcal{P}_J \left(1 + \frac{2t}{s}\right),$$

  Unitarity guarantees positivity of the spectral density:

$$\rho_J(s) = Im\ f_J(s) \sim \sum_X g_\pi^2 \delta(s - m_X^2) \geq 0$$

Alternative prospective: Effective Field Theory

Integrating out all resonances at tree level produces an effective theory of pions

$$\mathcal{A}(s,t) = g_{0,1}(s+t) + g_{1,1}(s+t)(st) + g_{1,0}\, ts + g_{0,2}(s+t)^2 + \ldots$$

No states below cut off M: no poles.

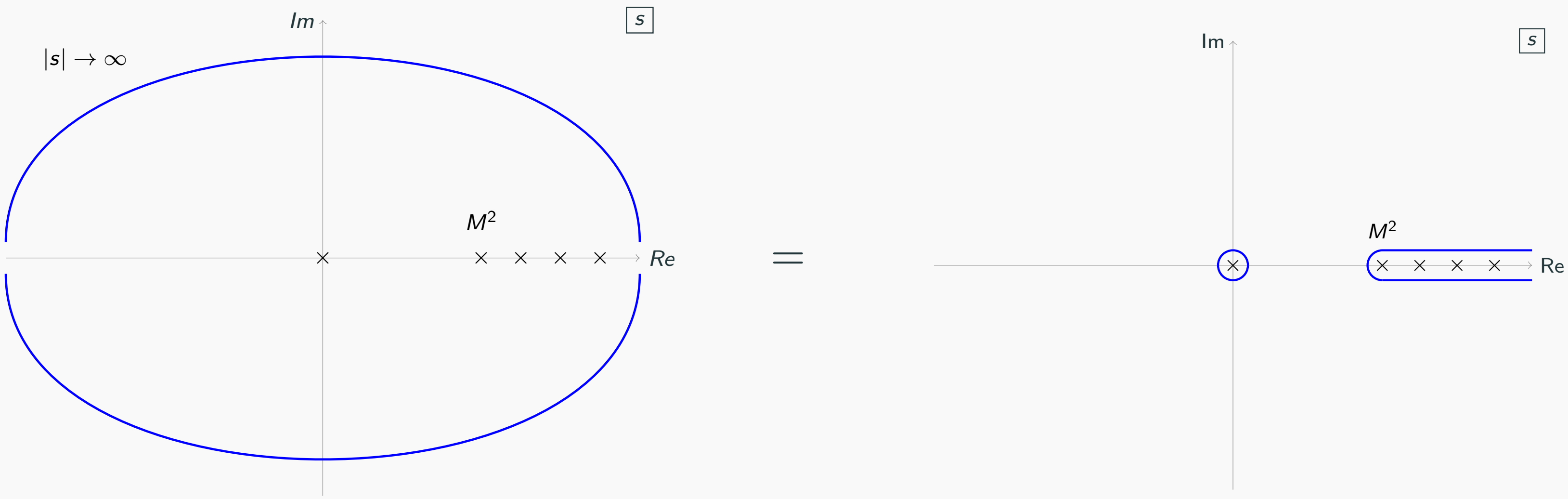
EFT also weakly coupled: all interactions $O(1/N)$: no logs

One can match the coefficients $g_{n,\ell}$ with the parameters of the Chiral Lagrangian

$$\mathscr{L} = -\frac{f_\pi^2}{4}\mathrm{Tr}[\partial^\mu U^\dagger \partial_\mu U] + \text{higher derivatives}, \qquad U = e^{i\pi^a\sigma^a}$$

Dispersion relations at large N_c

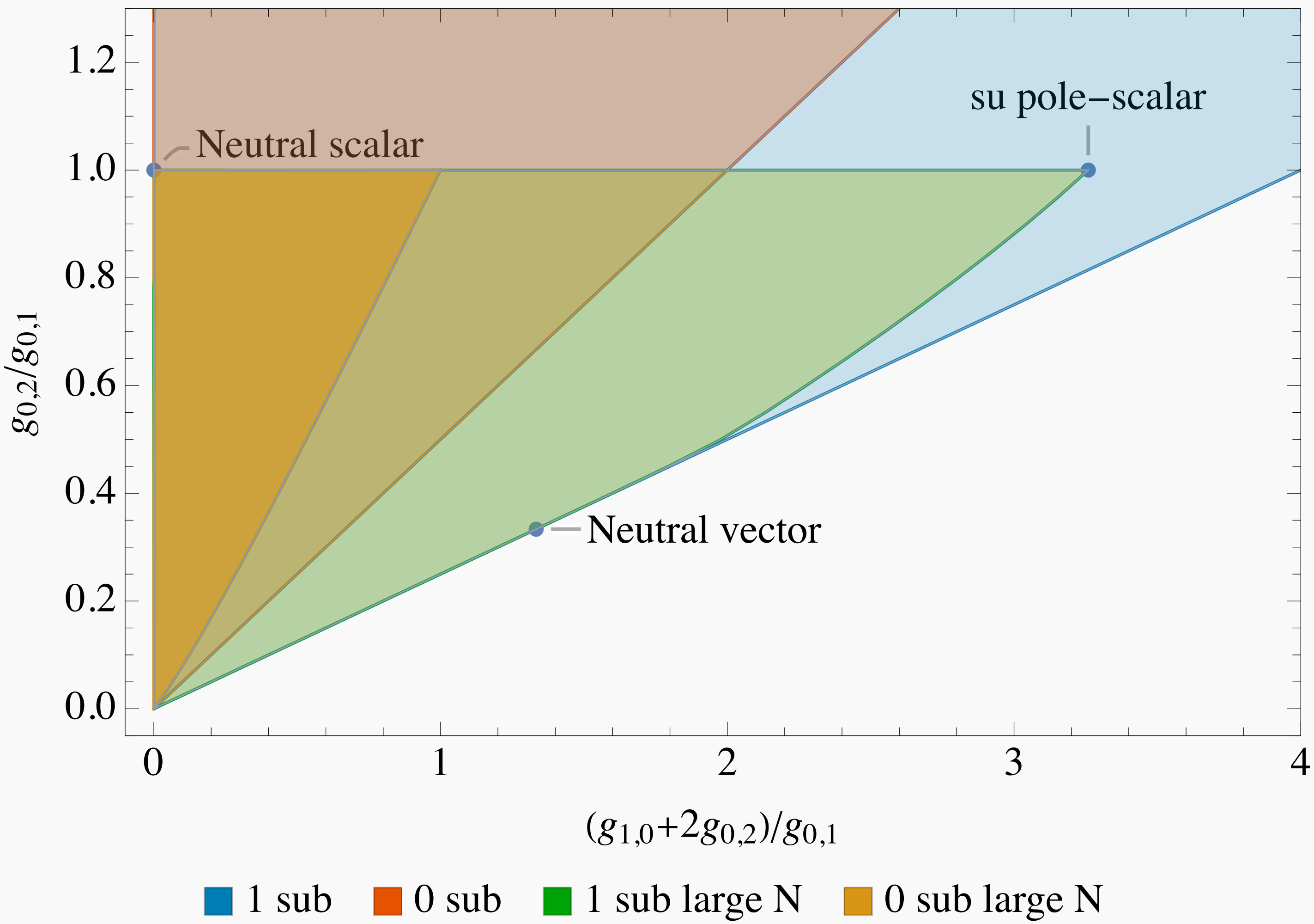
$$\oint_{\infty} \frac{ds}{2\pi i s} \frac{\mathcal{A}(s,t)}{s^k} = 0 \quad \text{(at large } N_c, k = 1, 2, \ldots)$$



low energy data $g_{n,\ell}$ $\longleftrightarrow$ high energy data $\rho_J(s)$

[Arkani-Hamed, Huang, Huang] , [Tolley, Wang, Zhou] ,
[Bellazzini, Mirò Rattazzi, Riembau,Riva] [Caron-Hout, Van Duong]

Allowed values of EFT coefficients



[Albert, Rastelli '22] [Fernandez, Pomarol, Riva, Sciotti '22]

[McPeak, Venuti, AV '23]

Simple (but uninteresting) amplitudes

- Scalar exchange:

$$\mathcal{A}(s,t) \sim \frac{g^2}{s-M^2} + \frac{g^2}{t-M^2}$$

- su -pole:

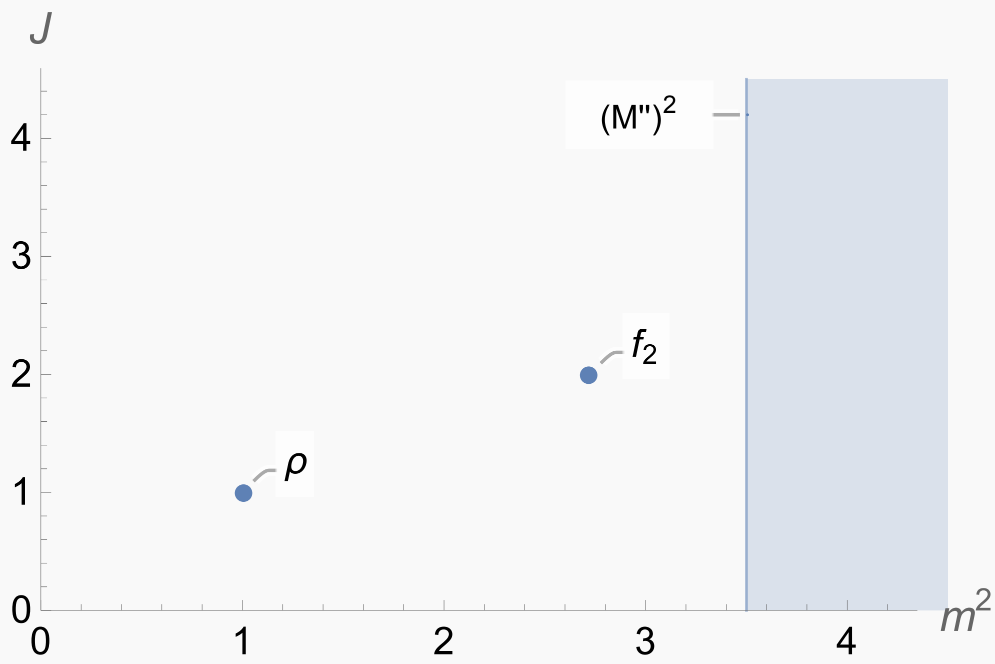
$$\mathcal{A}(s,t) \sim \frac{M^2 g^2}{(s-M^2)(t-M^2)} \longrightarrow \text{infinite tower of states at } M^2$$

- Vector exchange:

$$\mathcal{A}(s,t) \sim \frac{2t+M^2}{s-M^2} + \frac{2s+M^2}{t-M^2} \longrightarrow s^1 \quad (\text{ violates Regge?})$$

General lessons from Regge theory

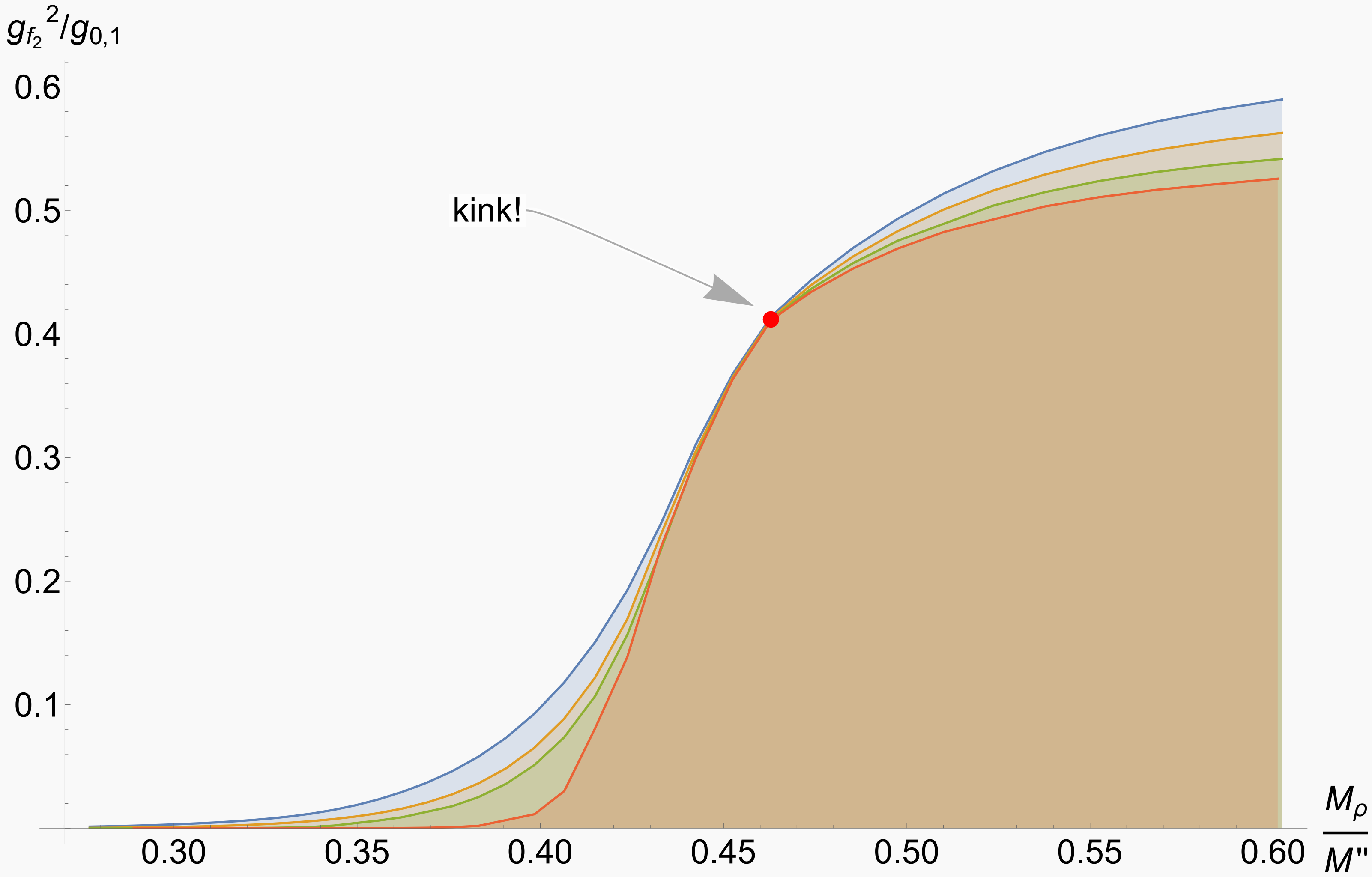
higher spins $\longrightarrow$ violation of Regge behavior $\longrightarrow$ need more states



f_2 cannot come alone: can't push cutoff to infinity

Forcing the f_2

Maximal value of spin-2 (aka f_2) coupling to pions as a function of cutoff M'' ?



[Albert, Henriksson,Rastelli, AV '23]

Forcing the f_2

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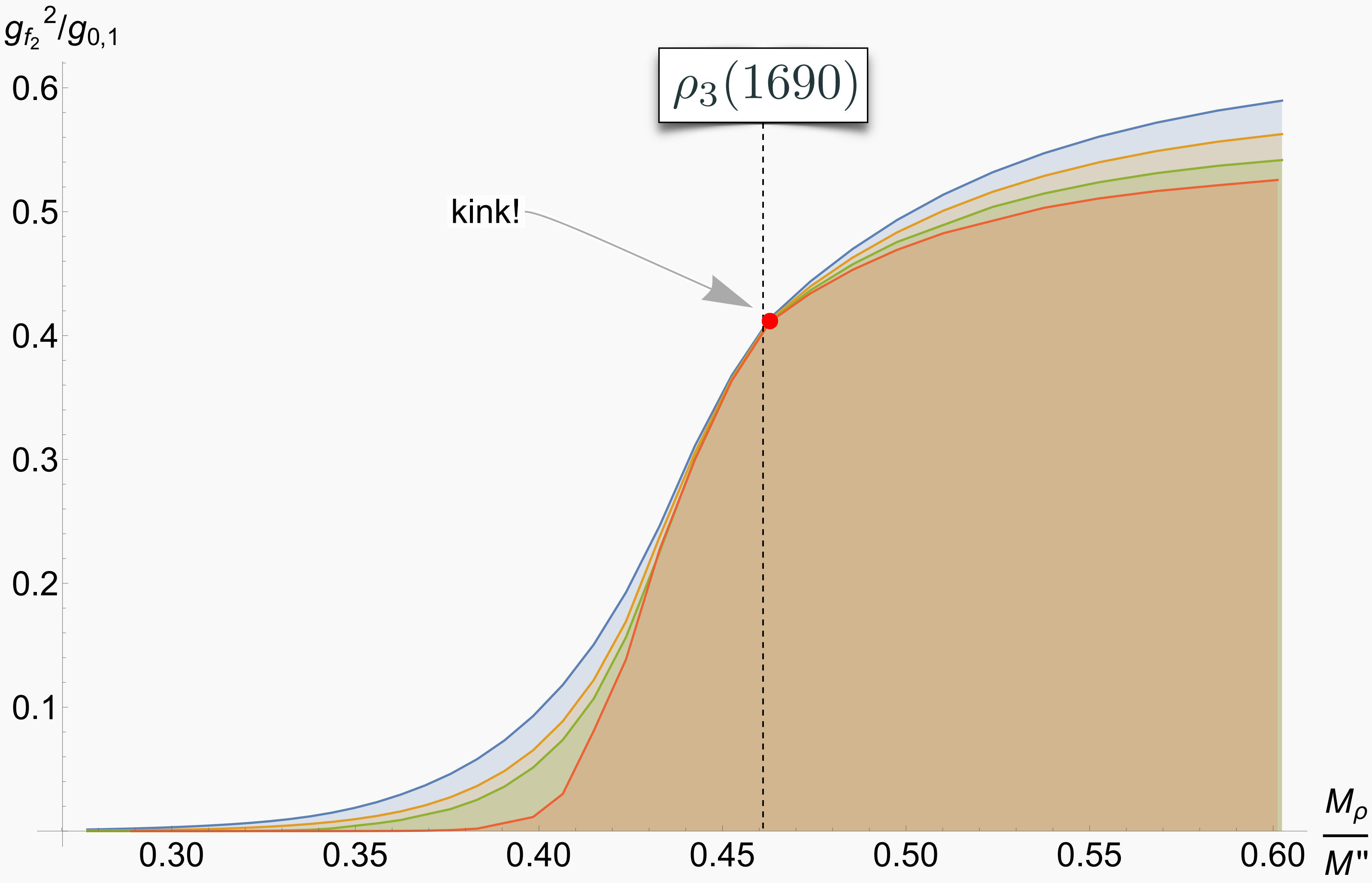
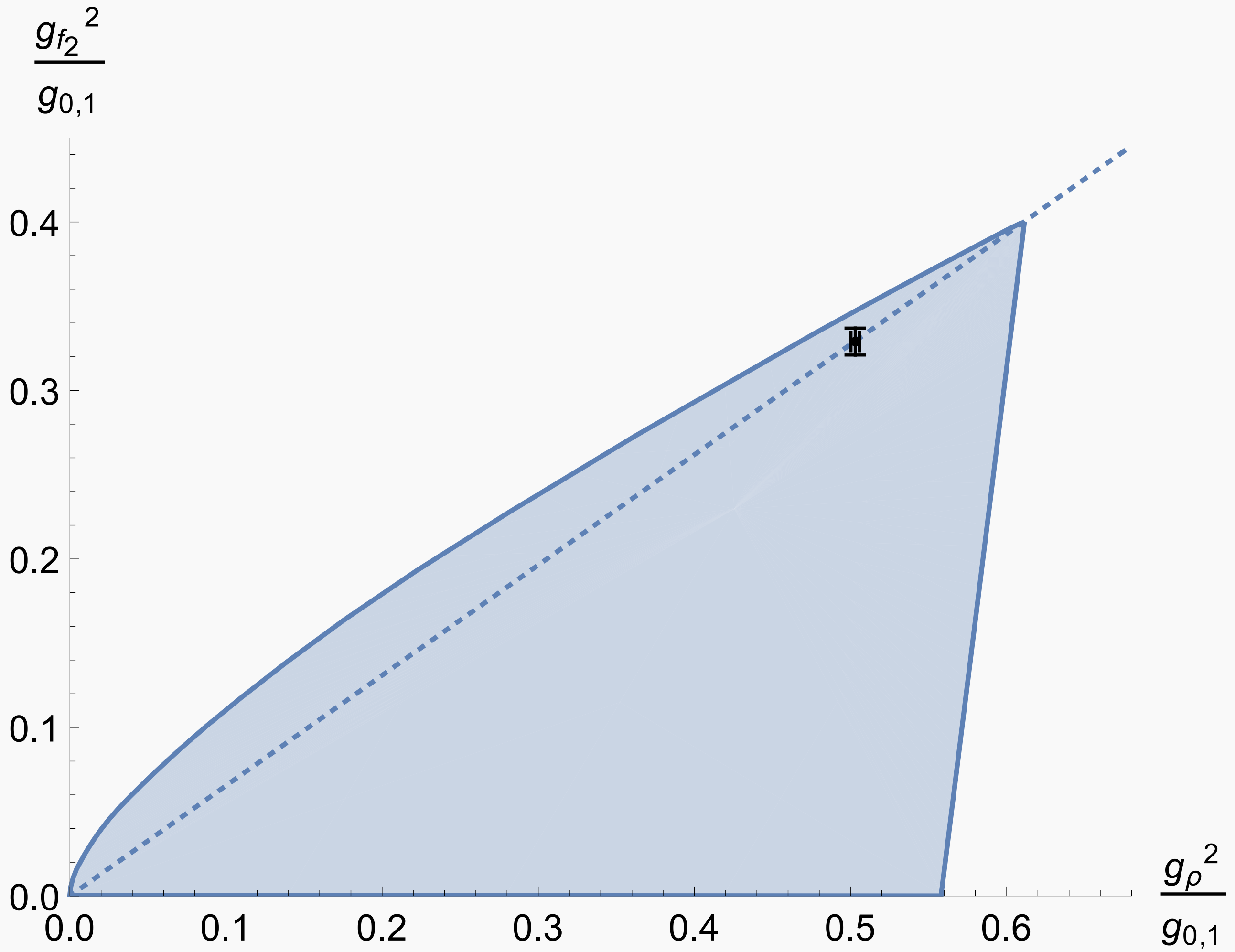


[Albert, Henriksson,Rastelli, AV '23]

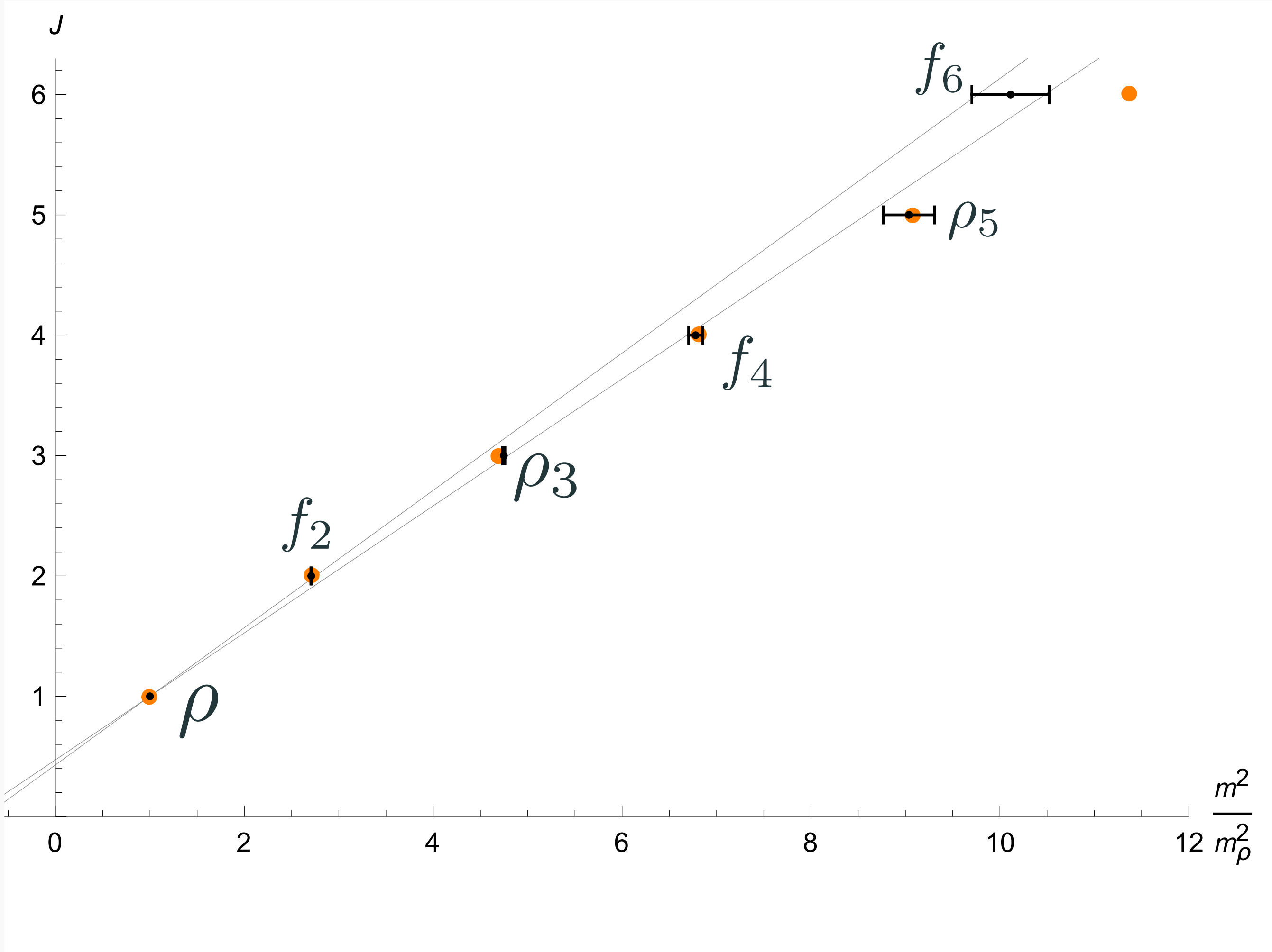
couplings @ kink VS real world

Fix the cut-off $M'' = M_{\text{kink}}$: $\rho-$coupling vs f_2-coupling



Ratio $\frac{g_{f_2}^2}{g_\rho^2}$ close to real word QCD!

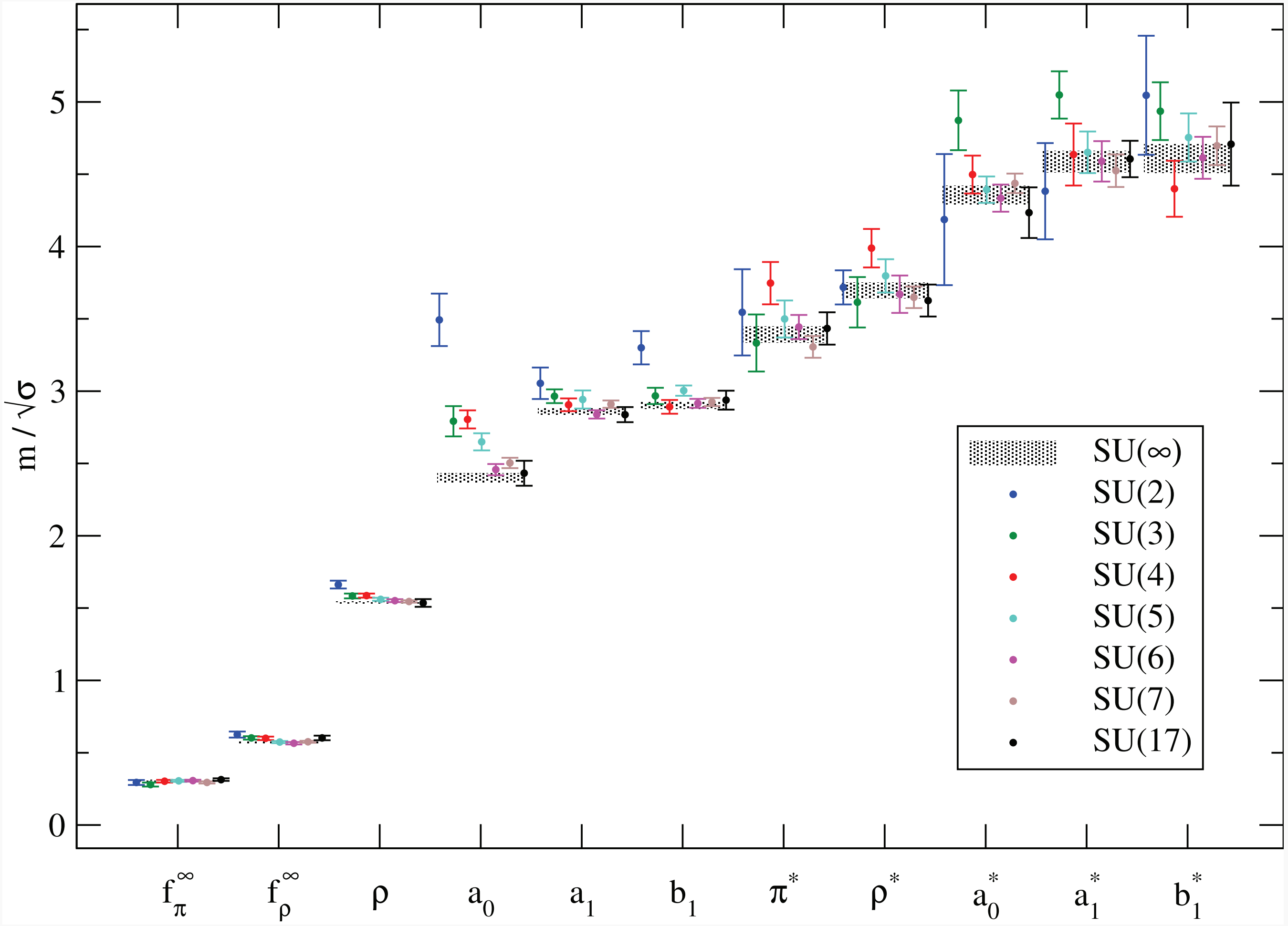
spectrum @ kink VS real world



Impressive agreement with real world QCD...

Is $3 \gg 1$?

Is it surprising that we agree with real world data?



[Bali, Bursa, Castagnini, Collins, Del Debbio, Lucini,Panero '13]

...perhaps not so much!

ρ-meson scattering at large N

ρ-meson scattering

$$\rho^a \rho^b \longrightarrow \rho^c \rho^d$$

- amplitude parametrised by 17 functions $M_i(s,u)$
- Pion exchange $\longrightarrow \frac{1}{s}$ pole
- ρ exchange parametrized by two self couplings $\lambda^{(1)}_{\rho\rho\rho}$, $\lambda^{(2)}_{\rho\rho\rho}$

Parametrization of pion-rho interactions:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi_a - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{2} m^2_\rho \rho^a_\mu \rho^\mu_a \\ &- \lambda_{\pi\pi\rho} f_{abc} \pi^a \partial^\mu \pi^b \rho^c_\mu - \lambda_{\pi\rho\rho} d_{abc} \epsilon^{\mu\nu\sigma\tau} F^a_{\mu\nu} F^b_{\sigma\tau} \pi^c \\ &- \lambda^{(1)}_{\rho\rho\rho} f_{abc}\, \rho^a_\mu \rho^b_\nu F^{c,\mu\nu} - \lambda^{(2)}_{\rho\rho\rho} f_{abc} F^{a,\nu}_\mu F^{b,\sigma}_\nu F^{c,\mu}_\sigma + \cdots ,\end{aligned}$$

with $F^a_{\mu\nu} \equiv \partial_\mu \rho^a_\nu - \partial_\nu \rho^a_\mu$

[Bertucci, Henriksson, McPeak, Ricossa, Riva, AV '24]
[Albert, Rastelli, Henriksson, AV : in progress]

Target 1: $\lambda^{(1)}_{\rho\rho\rho}$, $\lambda^{(2)}_{\rho\rho\rho}$

$$\mathcal{L} \supset -\lambda^{(1)}_{\rho\rho\rho} \underbrace{f_{abc}\, \rho^a_\mu \rho^b_\nu F^{c,\mu\nu}}_{dim\ 4} - \lambda^{(2)}_{\rho\rho\rho} \underbrace{f_{abc} F^{a,\nu}_{\mu} F^{b,\sigma}_{\nu} F^{c,\mu}_{\sigma}}_{dim\ 6},$$

- $\lambda^{(1)}_{\rho\rho\rho}$: naturally present, ex from non-abelian kinetic term
- $\lambda^{(2)}_{\rho\rho\rho}$: needs higher spins or loops. Also, grows faster with s. Probe of cutoff?

Target 2: Hidden Local Symmetry

Can the ρ meson be a gauge boson of a spontaneously broken local symmetry?

[Bando, Kugo, Uehara, Yamawaki, Yanagida '84]

HLS: $\mathcal{G}/\mathcal{H} \longrightarrow (G \times \mathcal{H}_{\text{local}})/\mathcal{H}$

Interesting predictions:

- KSRF relation:

$$m_\rho^2 = 2\lambda_{\pi\pi\rho}^2 f_\pi^2$$

- Universality of ρ-couplings:

$$\lambda_{\rho\rho X} = g \text{ (i.e the gauge coupling)}$$

- ρ-dominance in photon coupling to pions

These relations can be tested considering mixed π, ρ scattering

Conclusions

Cornering large-N QCD?

	spectrum VS real-word	Asymptotically Linear Regge trajectories	daughter trajectories	degenerate ρ, f trajectories
Large-N QCD	✓	expected	suppressed(?)	✓
Kink solution	✓	X	not seen	✓

π, ρ-scattering will definitively help...

Happy Birthday Riccardo!!

BACKUP SLIDES

Sum rules and null constraints

$$\left.\begin{aligned} g_{0,1} &= \langle \tfrac{1}{m^2} \rangle \\ g_{1,1} &= \ldots \end{aligned}\right\} \leftarrow \text{sum rules} \qquad g_{0,1} > 0 \ (\sim 1/f_\pi^2)$$

$$\left.\begin{aligned} 0 &= \langle \tfrac{(J-2)J(J+1)(J+3)}{m^6} \rangle \\ 0 &= \ldots \end{aligned}\right\} \leftarrow \text{null constraints } \mathcal{X}_{k,\ell}$$

Notation:

$$\langle F(m^2, J) \rangle \equiv \sum_{J \text{ even}} n_J^{(4)} \int_{M^2}^{\infty} \frac{dm^2}{\pi m^2} \rho_J(m^2) \left[F(m^2, J) \right] .$$

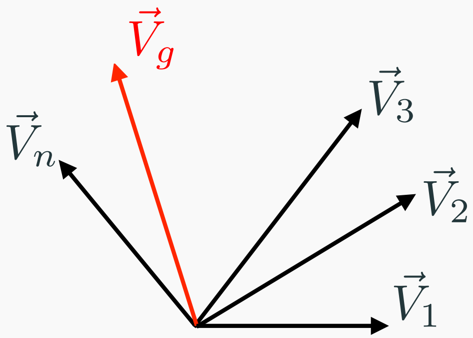
Unitarity $\Rightarrow \rho_J(s) \geq 0$

Bootstrap equations

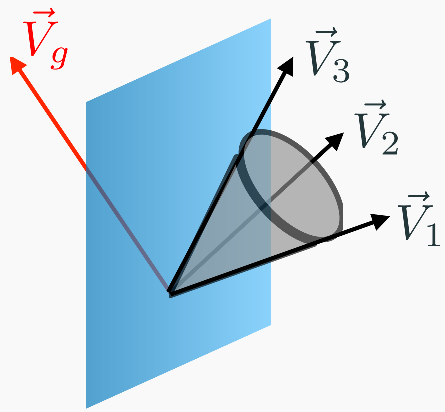
Schematic form of equations:

$$\underbrace{\begin{pmatrix} g_{0,1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{V}_g} = \sum_X g^2_{\pi\pi X} \underbrace{\begin{pmatrix} \cdots \\ \mathcal{X}_{3,1} \\ \cdots \\ \mathcal{X}_{4,1} \\ \cdots \end{pmatrix}}_{\vec{V}_X}$$

($X \equiv$ quantum numbers of states exchanged in $\pi\pi \to \pi\pi$)



VS



Feasibility can be recast in a semi-definite positive problem and tested numerically