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Positivity Bounds on Electromagnetic Properties of Media

With O. Janssen, B. Salehian and L. Senatore, 2405.09614 (JHEP)

Riccardo's fest, Nov 9th 2024

Ode al Maestro[1]

(Festschrift)

Cari colleghi, ancora adunati[2]
   per festeggiare il Riccardo Barbieri,
qual schiera eletta tutt'accomunati,
   dalla sua Scienza, di oggi o di ieri,[3]

alla sua guisa, riposti gl'incensi,
   giunto è il momento del pàrlar verace,
per dire ora di quei motti densi
   che ai riti solenni ognuno tace...[4]

Andrea Gambassi

Tanti auguri Riccardo e grazie

Tanti auguri Riccardi e grazie

Positivity: LI case

Coefficients of EFT operators must satisfy inequalities
(if there is a "standard" UV completion)

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi 06

For example: $\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 + \frac{c}{\Lambda^4}(\partial\pi)^4$ $\qquad c \geq 0$

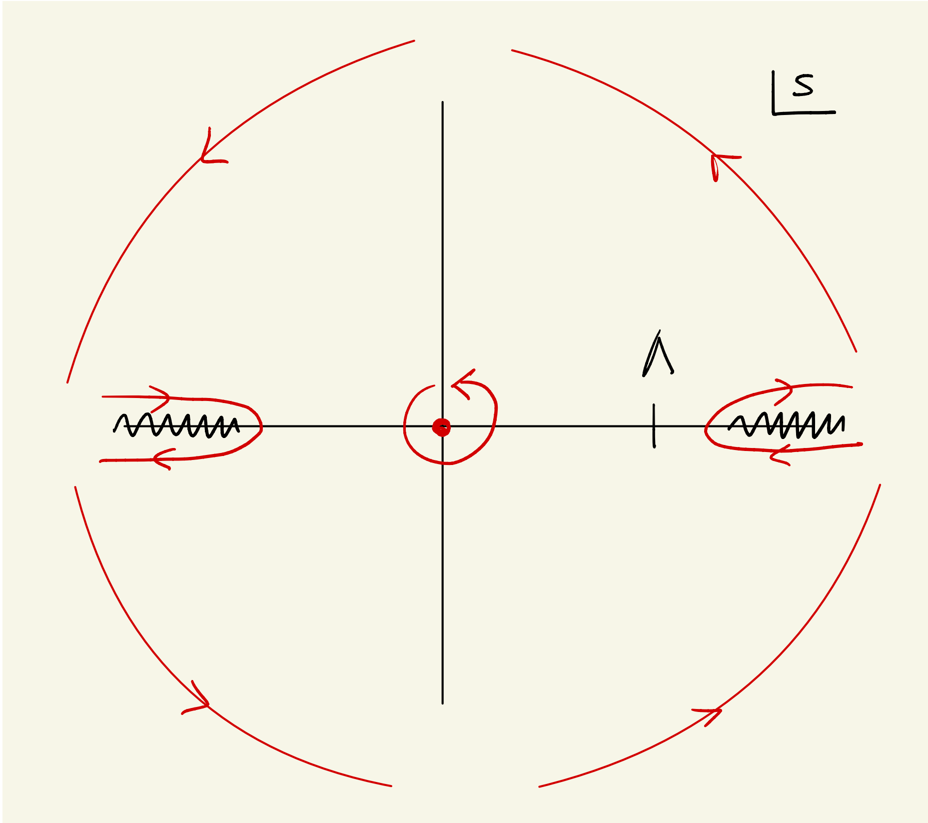
$\mathcal{A}(s) \equiv \mathcal{M}(s, t \to 0) \qquad \mathcal{A}(s) = c\frac{s^2}{\Lambda^4} + \ldots$

Crossing: $\mathcal{A}(s) = \mathcal{A}^*(-s^*)$

$$\oint \frac{ds}{2\pi i}\frac{\mathcal{A}(s)}{s^3} = \frac{c}{\Lambda^4}$$

Froissart bound: $|\mathcal{A}(s)| < s\log^2 s$

$$\frac{c}{\Lambda^4} = \frac{2}{\pi}\int ds\frac{s\sigma(s)}{s^3} \geq 0$$



Similar bounds for non-LI theories?

Motivation: in many interesting situations Lorentz is spontaneously broken

1. Cosmology. In particular Inflation and Dark Energy/Modifications of Gravity

   We are particularly interested in "peculiar" theories (Galileon, Ghost Condensate...): are they consistent?

2. Condensed Matter. Can we deduce general inequalities for a system?

3. QFT at finite T or finite Q

4. Worldline EFT

In general the theory is <u>defined</u> with non-linearly realised Lorentz

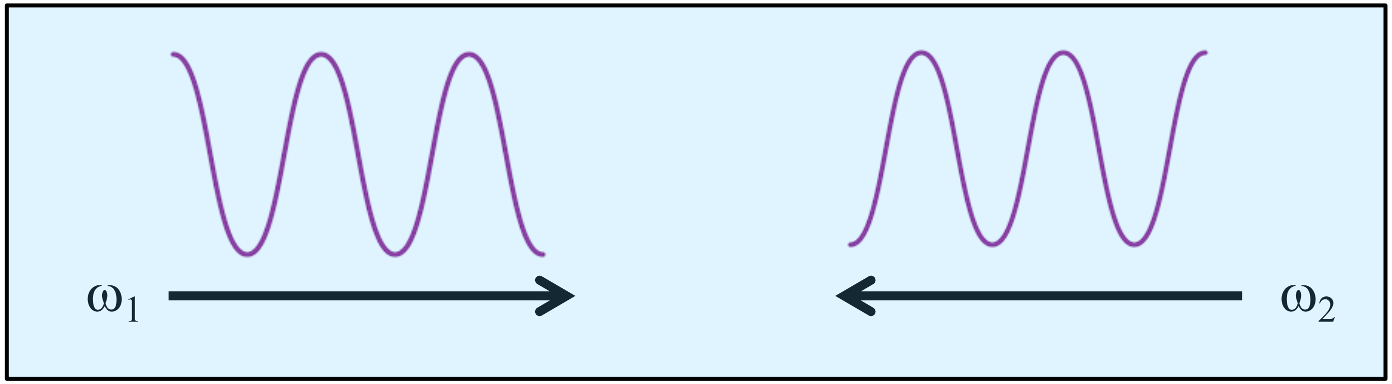
Cannot be "extrapolated" from a LI invariant theory: think about a fluid

Simply do the same?

Baumann, Green, Lee, Porto 15
Grall, Melville 21

$$\mathcal{L} = \frac{1}{2}\dot{\pi}^2 - \frac{1}{2}c_s^2(\partial_i\pi)^2 + \frac{\alpha_1}{\Lambda^2}\dot{\pi}^3 + \frac{\alpha_2}{\Lambda^2}\dot{\pi}(\partial_i\pi)^2 + \ldots$$

Look at π scattering



In a LI theory this is well-defined at arbitrary high energy
(calculable in EFT only at low energy)

If LI is broken, π is not a good asymptotic state at high energy:
scatter phonons at 10 TeV?

Even when asymptotic states can be followed,
S-matrix does not have required analytic properties

PC, Delladio, Janssen, Longo, Senatore 23 Hui, Kourkoulou, Nicolis, Podo, Zhou 23

Let us go back to 2-point function

- Quite rich object w/o LI: G(ω,k). Two variables like S-matrix with LI
- Electrodynamics of media, hydrodynamics, worldline EFT.

  Linear response theory
- The first use of analyticity are Kramers-Kronig relations. Non-LI!
- Constraints on conformal superfluids (CFTs at large Q) using $\langle J^\mu J^\nu \rangle$

PC, Janssen, Senatore 22

What are the implications of microcausality and positivity for EM in media?

Of course 80% is known (Russians!) maybe in an unconventional language…
(For us!)

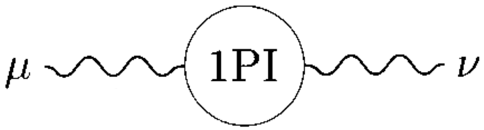
Positivity bounds for EM response of media

Fields are small (compared with the atomic ones) → linear optics

We want EOM for <E> and <B> after you integrate out the medium

IN-IN Effective action

$$\int \mathrm{d}^4x \; J^\mu(\psi) A_\mu$$



$$\Gamma_M[A_1, A_2] = \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y \left[A_{1\mu}(x) \;\; A_{2\mu}(x)\right] S^{\mu\nu}(x,y) \begin{bmatrix} A_{1\nu}(y) \\ A_{2\nu}(y) \end{bmatrix}$$

$$S^{\mu\nu}(x,y) = i \begin{bmatrix} \langle T J^\mu(x) J^\nu(y)\rangle & -\langle J^\nu(y) J^\mu(x)\rangle \\ -\langle J^\mu(x) J^\nu(y)\rangle & \left\langle \tilde{T} J^\mu(x) J^\nu(y)\right\rangle \end{bmatrix}_{1\mathrm{PI}}$$

Macroscopic Maxwell equations

$$\frac{1}{g^2}\partial_\nu F^{\nu\mu} + \int \mathrm{d}^4y\, \Pi^{\mu\nu}(x,y) A_\nu(y) = -J^\mu_{\mathrm{ext}}(x)$$

$$\Pi^{\mu\nu}(x,y) = i\theta(x^0 - y^0)\,\langle [J^\mu(x), J^\nu(y)]\rangle_{1\mathrm{PI}} + \langle N^{\mu\nu}\rangle_{1\mathrm{PI}}\,\delta(x-y)$$

$\Pi^{\mu\nu}$ and ε, μ

Conserved: $p_\mu \Pi^{\mu\nu} = 0$ $\qquad p^\mu = (\omega, \mathbf{k})$

Two tensor structures: $\Pi^{\mu\nu} = \pi_L(\omega,k) p^2 \mathcal{P}_L^{\mu\nu} + \pi_T(\omega,k) k^2 \mathcal{P}_T^{\mu\nu}$

$$\mathcal{P}_L^{00} = -\frac{k^2}{p^2}\,, \qquad \mathcal{P}_L^{0i} = -\frac{\omega k^i}{p^2}\,, \qquad \mathcal{P}_L^{ij} = -\frac{\omega^2}{p^2}\frac{k^i k^j}{k^2}\,,$$

$$\mathcal{P}_T^{00} = \mathcal{P}_T^{0i} = 0\,, \qquad \mathcal{P}_T^{ij} = \delta^{ij} - \frac{k^i k^j}{k^2}\,.$$

Macroscopic Maxwell equations

$$\frac{1}{g^2}\partial_\nu F^{\nu\mu} = -(J^\mu_{\rm in} + J^\mu_{\rm ext})\,, \qquad J^\mu_{\rm in}(x) \equiv \int {\rm d}^4 y\, \Pi^{\mu\nu}(x,y) A_\nu(y)\,,$$

Internal (bound) current

$$g^2 \rho_{\rm in} \equiv (1-\varepsilon)\partial_i E^i\,,$$

$$g^2 J^i_{{\rm in},T} \equiv (\tilde\varepsilon - 1)\partial_t E^i_T + \left(1 - \frac{1}{\tilde\mu}\right)\varepsilon^{ijk}\partial_j B_k\,,$$

$$\varepsilon\, \boldsymbol{\nabla}\cdot \boldsymbol{E} = g^2 \rho_{\rm ext}\,, \qquad \frac{1}{\tilde\mu}\boldsymbol{\nabla}\times\boldsymbol{B} - \tilde\varepsilon\,\partial_t \boldsymbol{E}_T = g^2 \boldsymbol{J}_{{\rm ext},T}$$

$\Pi^{\mu\nu}$ and ε, μ

Ambiguity due to hom. Maxwell eq. $\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0$

Two useful choices:

- No magnetic response $\tilde{\mu} = 1$. Transverse/longitudinal electric permittivity.
- Single electric permittivity $\tilde{\varepsilon} = \varepsilon$ and magnetic permeability μ

$$\varepsilon\, \boldsymbol{\nabla} \cdot \boldsymbol{E} = g^2 \rho_{\text{ext}}\,, \qquad \frac{1}{\mu} \boldsymbol{\nabla} \times \boldsymbol{B} - \varepsilon\, \partial_t \boldsymbol{E} = g^2 \boldsymbol{J}_{\text{ext}}\,.$$

Textbook ε and μ now function of ω, k (I should write convolutions)

$$\varepsilon - 1 = -g^2 \pi_L\,, \qquad 1 - \frac{1}{\mu} = g^2 \left(\pi_T + \frac{\omega^2}{k^2} \pi_L \right)$$

(Let me not introduce ***H, D***...)

(We assumed parity invariance, otherwise one would have optical activity: sugar!)

Linear response

One should be careful about external vs total field

$$J^\mu_{\rm in}(x) = \int \mathrm{d}^4y\, G^{\mu\nu}_J(x,y) A_{{\rm ext},\nu}(y) \qquad G^{\mu\nu}_J(x,y) = i\theta(x^0-y^0)\,\langle[J^\mu(x),J^\nu(y)]\rangle + \langle N^{\mu\nu}\rangle\,\delta(x-y)$$

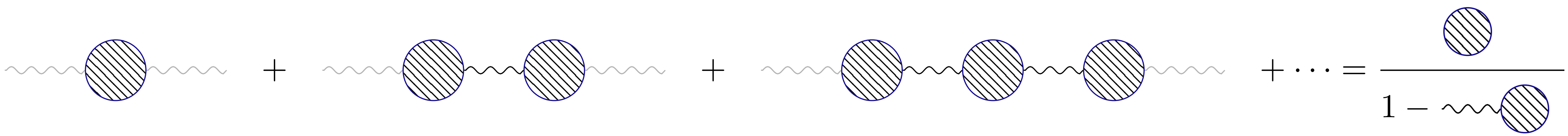
This is the microcausal object (commutator). It is not 1PI.

$\Pi^{\mu\nu}$ is the response to the total field

$$A^\mu(x) = A^\mu_{\rm ext}(x) + g^2 \int \mathrm{d}^4y\, \Delta^{\mu\nu}(x-y) J_{{\rm in},\nu}(y)$$

$$(\Pi^{-1})^{\mu\nu} = (G_J^{-1})^{\mu\nu} + g^2 \Delta^{\mu\nu}$$

$\Delta^{\mu\nu}$ is the free photon propagator



Positivity

We assume passive medium: it only absorbs energy from external EM

$$\Delta H = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \omega A_{\mathrm{ext},\mu}(-p)\,\mathrm{Im}\, G_J^{\mu\nu}(p) A_{\mathrm{ext},\nu}(p) \qquad > 0$$

$$\mathrm{Im}\, G_J^{\mu\nu}(p) = \frac{1}{2}\int \mathrm{d}^4 x\, e^{-ip\cdot x}\, \langle [J^\mu(x), J^\nu(0)] \rangle$$

$$2\,\mathrm{Im}\, G_J^{\mu\nu}(p) = \sum_{n,m} (2\pi)^4 \delta(p + p_n - p_m) \langle n| J^\mu(0) |m\rangle \langle m| J^\nu(0) |n\rangle (c_n - c_m)$$

$$\rho\,|n\rangle = c_n\,|n\rangle$$

Only absorbtion is c_n are monotonically decreasing, e.g. vacuum or thermal state

- In a laser there is population inversion and light is amplified
- Same property for $\Pi^{\mu\nu}$
- Not assuming a gap in Im G

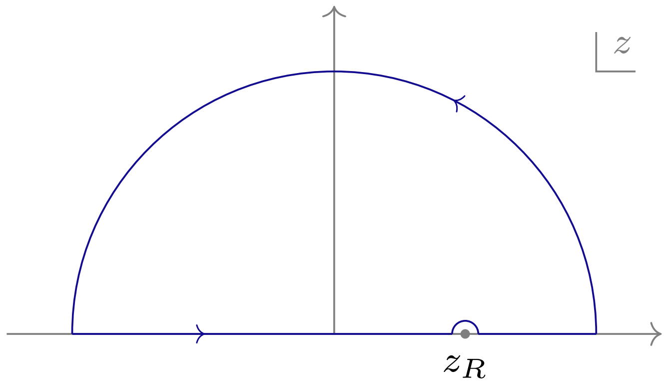
Microcausality and analyticity

$$G_J^{\mu\nu}(p) = \int \mathrm{d}^4x\, e^{-ip\cdot x} G_J^{\mu\nu}(x) \qquad G_J^{\mu\nu}(x,y) = i\theta(x^0 - y^0)\,\langle [J^\mu(x), J^\nu(y)]\rangle + \langle N^{\mu\nu}\rangle\,\delta(x-y)$$

$p^\mu = (\omega, \boldsymbol{k}) \in \mathbb{C}^4$ $\qquad p^\mu = p_R^\mu + ip_I^\mu$ $\qquad$ Analytic $\; p_I^\mu \in \mathrm{FLC}$

"That approach really depressed me because I knew that I could never understand the theory of more than one complex variable. So I was pretty worried about how I could do research working in this mess." S.Weinberg

Reduce to one variable $\qquad p^\mu = (\omega, \boldsymbol{q} + \omega\boldsymbol{\xi}) \qquad \xi \equiv |\boldsymbol{\xi}| < 1$



$$\chi(z_R) = \frac{1}{i\pi}\mathrm{PV}\int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{z - z_R}\,\chi(z)$$

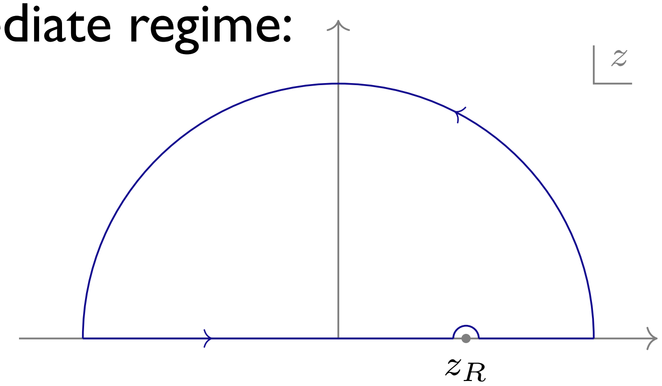
$$\chi(\omega, \boldsymbol{k}) = \frac{1}{i\pi}\mathrm{PV}\int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{z - \omega}\,\chi(z, \boldsymbol{k} + (z-\omega)\boldsymbol{\xi})$$

Leontovich (1961)

UV behaviour

For Leontovich, one needs functions to decay on the arc: medium negligible

In condensed matter one can close contour in intermediate regime:

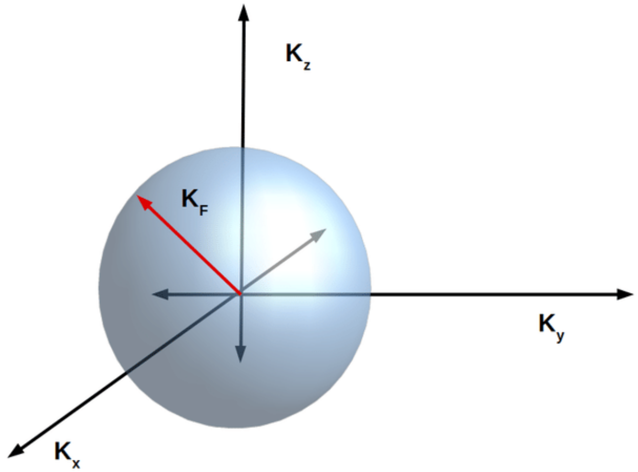


ω >> Collision frequency: free charge particles

ω << Electron mass: no QED loops

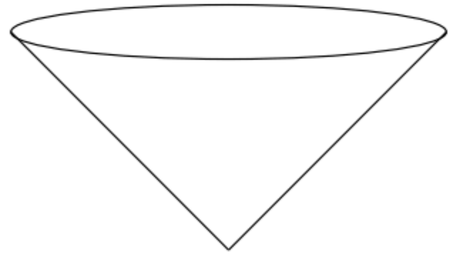
Plasma limit: $$g^2\pi_L \to -\frac{\omega_p^2}{\omega^2} + \ldots\,, \qquad g^2\pi_T \to -\frac{\omega_p^2}{k^2} + \ldots \qquad \omega_p^2 \equiv g^2 n/m$$

One can check this e.g. in response of a T=0 gas of fermions: Lindhardt function



Analyticity of $\Pi^{\mu\nu}$

Π is not obviously micro-causal being 1PI: it gives response to a localized total field. This requires external sources



Microcausal in perturbation theory

One can prove a partial result, but sufficient for our purposes

$$(G_\gamma^{-1})^{\mu\nu} = \frac{1}{g^2}(\Delta^{-1})^{\mu\nu} - \Pi^{\mu\nu}$$

Only possible singularities of Π are zeros of G_γ

Dropping k, Landau's argument for absence of zeros in ω UHP

$$\chi(\omega) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}z^2}{z^2 - (\omega + i\epsilon)^2}\ \mathrm{Im}\,\chi(z) \qquad \mathrm{Im}\,\chi(\omega) = \frac{\mathrm{Im}\,\omega^2}{\pi}\int_0^\infty \frac{\mathrm{d}z^2}{|z^2 - \omega^2|^2}\ \mathrm{Im}\,\chi(z)$$

Generalize $$\chi(\omega, \boldsymbol{q} + \omega\boldsymbol{\xi}) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}z^2}{z^2 - (\omega + i\epsilon)^2}\ \mathrm{Im}\,\chi(z, \boldsymbol{q} + z\boldsymbol{\xi})\,, \qquad (\boldsymbol{q}\cdot\boldsymbol{\xi} = 0)$$

$\rightarrow$ $\Pi^{\mu\nu}(\omega, \boldsymbol{q} + \omega\boldsymbol{\xi})$ analytic ω in UHP

Bounds on low-energy ε and μ

Focus on dieletrics: finite values of ε(0,0) and μ(0,0)

Conductors have $\varepsilon \sim i\,\sigma/\omega$, superconductors have $\varepsilon, \mu \sim 1/(\omega^2 - c_s^2 k^2)$

Longitudinal part

$$\varepsilon(\omega, \boldsymbol{q} + \omega \boldsymbol{\xi}) - 1 = \frac{1}{i\pi} \mathrm{PV} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{z - \omega}\; g^2 \pi_L(z, \boldsymbol{q} + z\boldsymbol{\xi})\,, \qquad (\boldsymbol{q} \cdot \boldsymbol{\xi} = 0)$$

$$\varepsilon(0,0) - 1 = \frac{2g^2}{\pi} \int_0^{+\infty} \frac{\mathrm{d}z}{z}\; \mathrm{Im}\, \pi_L(z, z\boldsymbol{\xi})$$

Transverse part

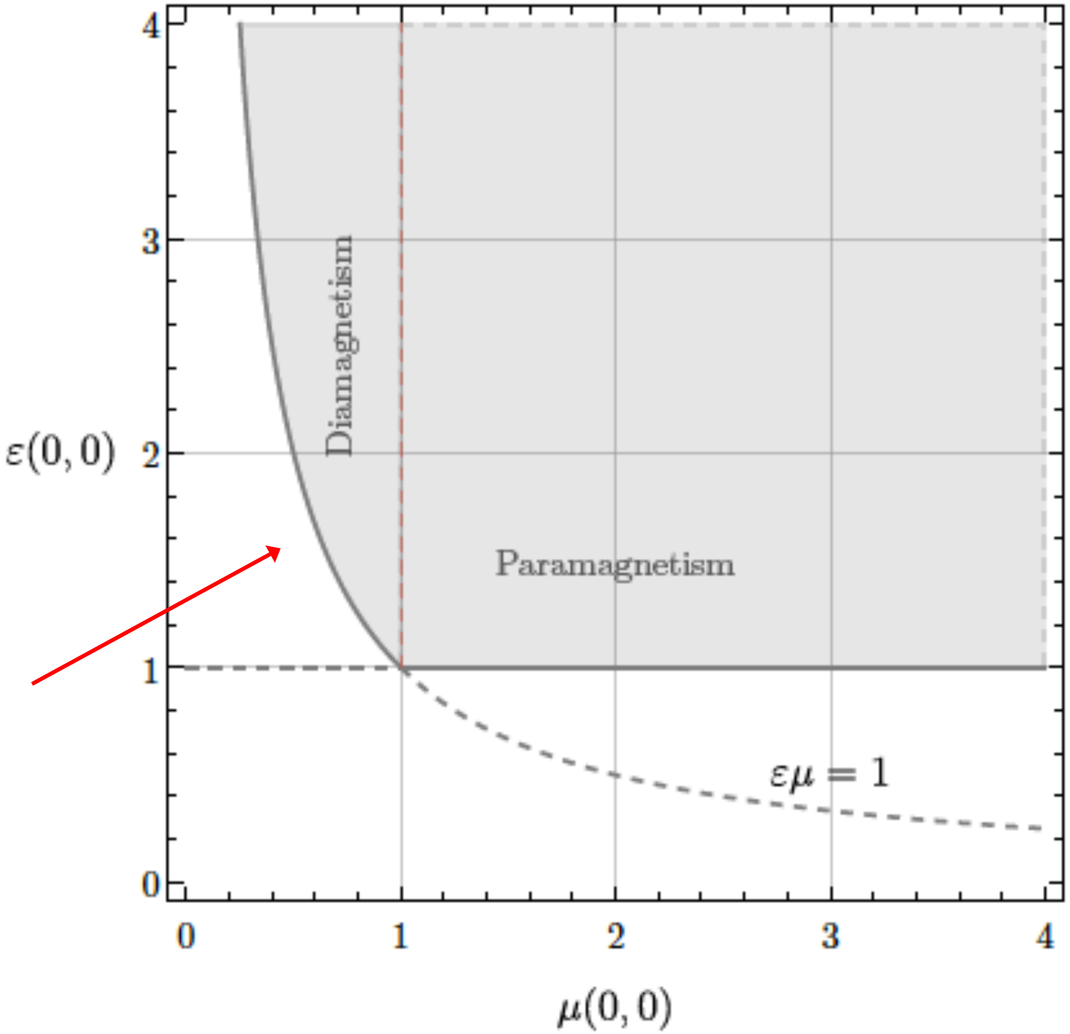
$$\frac{g^2 k^2 \pi_T}{\omega^2} = (\varepsilon - 1) + \frac{k^2}{\omega^2}\left(1 - \frac{1}{\mu}\right)$$

$$(\varepsilon(0,0) - 1) + \xi^2 \left(1 - \frac{1}{\mu(0,0)}\right) = \frac{2g^2\xi^2}{\pi} \int_0^{+\infty} \frac{\mathrm{d}z}{z}\; \mathrm{Im}\, \pi_T(z, z\boldsymbol{\xi})$$

$$\varepsilon(0,0) - \frac{1}{\mu(0,0)} = \frac{2g^2}{\pi} \int_0^{+\infty} \frac{\mathrm{d}z}{z}\; \mathrm{Im}\, \pi_T(z, z)$$

Bounds on low-energy ε and μ

see also Dolgov, Kirzhnits, Losyakov 82



Low-energy speed <1

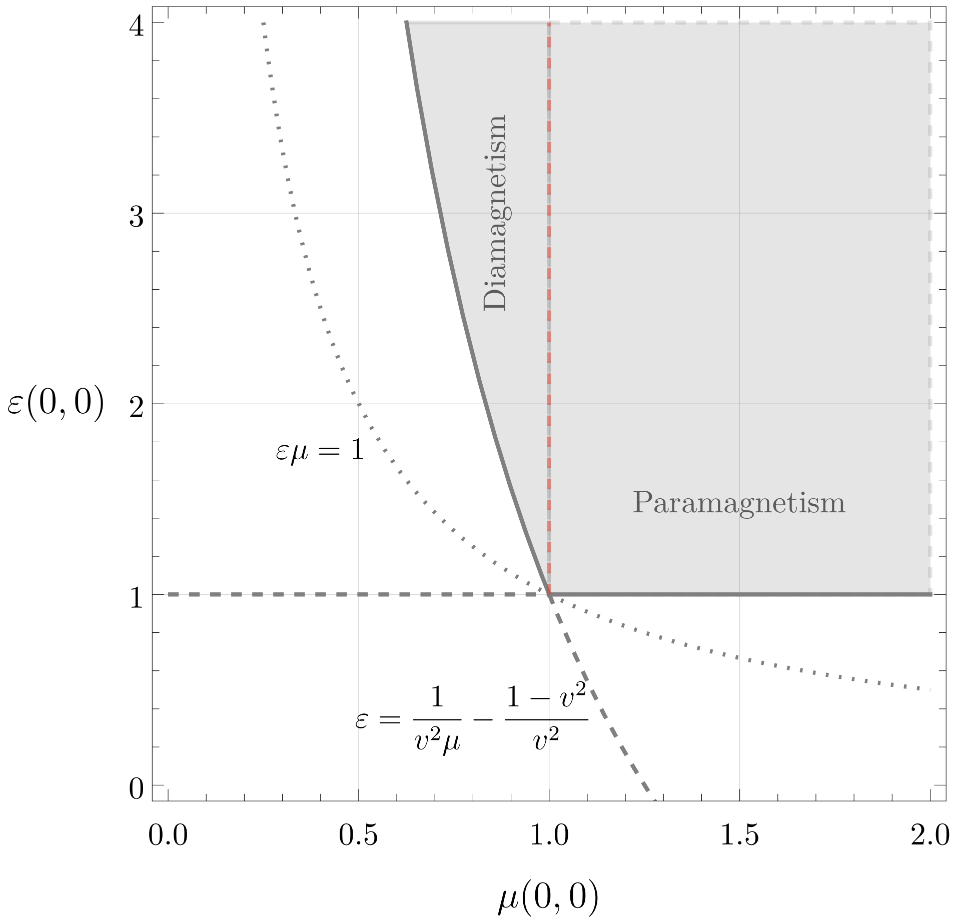
Microcausality imposes bounds on μ too

One can estimate part of the RHS integral for better bounds

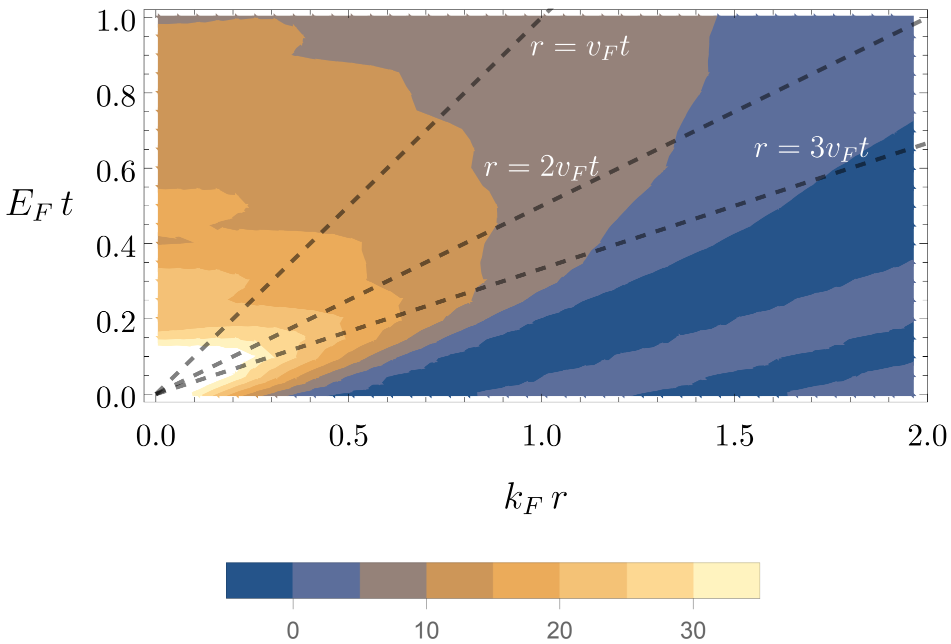
E.g. using plasma limit: $\varepsilon(0,0) - 1 \geq \frac{\omega_p^2}{\omega_{\rm UV}^2}$

Non-relativistic response

A response confined in a narrower cone, v << c, gives stronger bounds



Lindhardt response:
effectively confined to v_f



$$\frac{1}{\mu} - 1 < \frac{v^2}{c^2}(\varepsilon - 1)$$

Indeed normal diamagnetism has $\delta\mu \sim -10^{-5}$

Beyond condensed matter

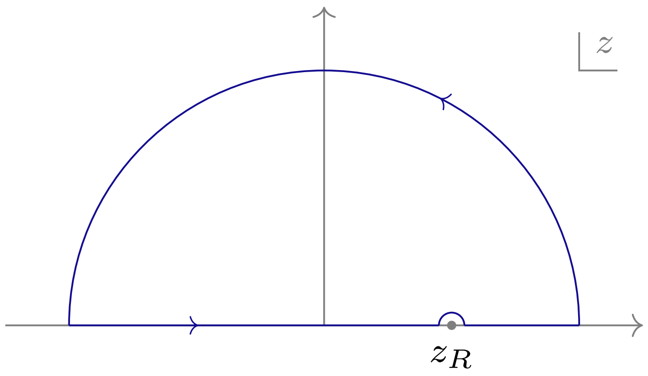
- Tiny diamagnetism in CM, because v << c

Cf. $$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\alpha}{\Lambda^2}\phi^2 F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu}F^{\mu\nu} \propto E^2 - B^2$$

Diamagnetic response saturates the bound. This happens for pions for instance

- For high-energy media (e.g. nuclear matter) one cannot close the contour below electron mass



Contour at infinity cannot be neglected, but it is known

$$\varepsilon(0,0) - \mathrm{Re}\,\varepsilon(\omega_{\mathrm{UV}},0) = \frac{2g^2}{\pi}\int_0^{\omega_{\mathrm{UV}}} \frac{\mathrm{d}z}{z}\ \mathrm{Im}\,\pi_L(z,0)$$

Future directions

- Other systems: superconductors, conductors, crystals...
- Derivatives of ε, μ
- Full analyticity of $\Pi^{\mu\nu}$
- Khallen-Lehman representation? Not every spectral density is ok:

$$\mathrm{PV}\int \mathrm{d}z\,\frac{\mathrm{Im}\,\chi(z,\boldsymbol{k}+(z-\omega)\boldsymbol{\xi}_1)}{z-\omega}=\mathrm{PV}\int \mathrm{d}z\,\frac{\mathrm{Im}\,\chi(z,\boldsymbol{k}+(z-\omega)\boldsymbol{\xi}_2)}{z-\omega}$$

Fluctuation dissipation theorem

We do not know anything without LI

- Induced dipole moments (and eventually Love numbers in gravity)
- Fluids using $\langle T^{\mu\nu}T^{\alpha\beta}\rangle$
- Inflation

Backup slides

Para/Dia magnetism and Electric response

Interaction with magnetic field

$$\Delta H = \mu_B(\vec{L} + g_0\vec{S}) \cdot H + \frac{e^2}{8mc^2} H^2 \sum_i (x_i^2 + y_i^2)$$

Paramagnetic Diamagnetic

Effect of E is second order

$$\Delta H_0 = 2E^2 e^2 \sum_n \frac{|\langle n|z|0\rangle|^2}{E_n - E_0}$$

Diamagnetic response is suppressed wrt electric one by $\Delta E/m \sim v^2$

Origin of analyticity

Consequence of microcausality: commutators vanish outside lightcone

See e.g. Itzykson Zuber's book

LSZ: $$S_{fi} = -\int d^4x\, d^4y\, e^{i(q_2\cdot y - q_1\cdot x)}(\Box_y + m_a^2)(\Box_x + m_a^2)\langle p_2|T\varphi^\dagger(y)\varphi(x)|p_1\rangle$$

Up to disconnected pieces: $T\varphi^\dagger(y)\varphi(x) \to \theta(y^0 - x^0)[\varphi^\dagger(y), \varphi(x)]$

$$S_{fi} = (2\pi)^4\delta^4(p_2 + q_2 - p_1 - q_1)i\mathscr{T}$$

$$\mathscr{T} = i\int d^4z\, e^{iq\cdot z}\langle p_2|\,\theta(z^0)\left[j^\dagger\left(\frac{z}{2}\right), j\left(-\frac{z}{2}\right)\right]|p_1\rangle \qquad (\Box + m_a^2)\varphi(x) = j(x)$$

$$q = \tfrac{1}{2}(q_1 + q_2)$$

Commutator vanishes outside FLC → $\mathcal{T}(q^\mu)$ analytic for Im q^μ in FLC

S - Matrix

with Delladio, Janssen, Longo, Senatore 23

Also Hui, Kourkoulou, Nicolis, Podo, Zhou 23

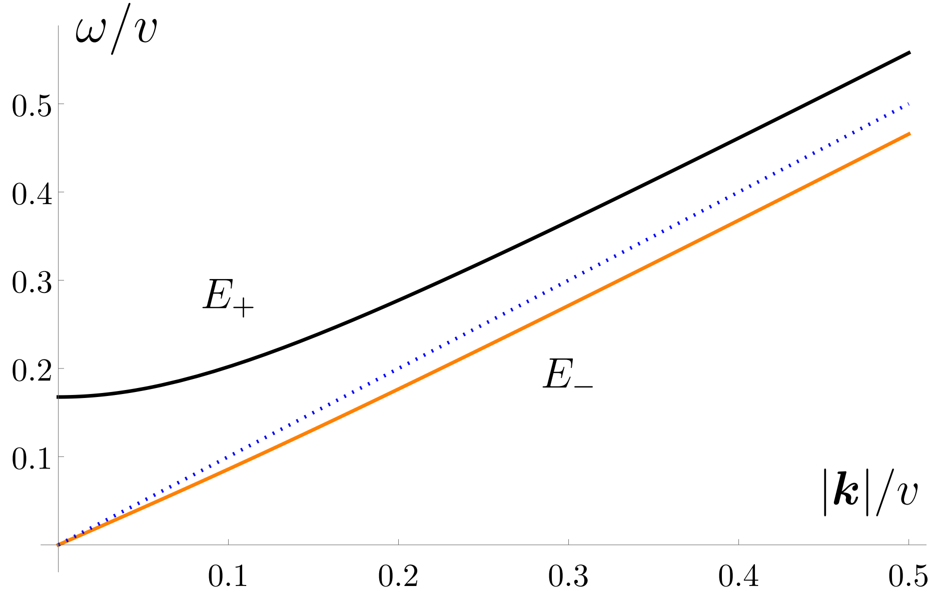
What if the low energy states *do* exist at high energy?

$$\mathcal{L} = \partial\Phi^\dagger \cdot \partial\Phi + m^2\,\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 \qquad \Phi = \frac{\rho}{\sqrt{2}}e^{i\theta/v} \qquad \theta = \mu^2 t/2 + \pi \qquad \rho = v+h$$

$$\mathcal{L} = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\pi)^2 + \frac{1}{2v^2}\left(\mu^2\dot{\pi} + (\partial\pi)^2\right)\left(h^2+2vh\right) - \frac{\lambda}{4}(h^2+2vh)^2$$

Integrating out h one gets low energy EFT for Goldstone π

$$\frac{1}{2}\begin{pmatrix} \tilde{\pi}_{-k} & \tilde{h}_{-k} \end{pmatrix}\begin{pmatrix} k^2 & i\mu^2\omega/v \\ -i\mu^2\omega/v & k^2 - M^2 \end{pmatrix}\begin{pmatrix} \tilde{\pi}_k \\ \tilde{h}_k \end{pmatrix}$$



$$E_\pm(\boldsymbol{k})^2 \equiv \boldsymbol{k}^2 + \frac{1}{2}\left(M^2 + \frac{\mu^4}{v^2}\right) \pm \sqrt{\frac{\mu^4}{v^2}\boldsymbol{k}^2 + \frac{1}{4}\left(M^2+\frac{\mu^4}{v^2}\right)^2}$$

LSZ reduction

$$\phi^a(t,\boldsymbol{x}) \equiv \sum_{l=\pm} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2E_l(\boldsymbol{k})} \left(Z_l^a(\boldsymbol{k}) a_l(\boldsymbol{k}) e^{-i(E_l(\boldsymbol{k})t - \boldsymbol{k}\cdot\boldsymbol{x})} + \mathrm{h.c.} \right) \;,\; a \in \{\pi, h\}$$

Imposing EOM and CCR one gets e.g. $Z_-^\pi(\boldsymbol{k}) = \sqrt{\dfrac{M^2 + \boldsymbol{k}^2 - E_-(\boldsymbol{k})^2}{E_+(\boldsymbol{k})^2 - E_-(\boldsymbol{k})^2}}$

LSZ formula, using polology

$$\prod_i^n \int \mathrm{d}^4 y_i \, e^{ip_i\cdot y_i} \prod_j^m \int \mathrm{d}^4 x_j \, e^{-ik_j\cdot x_j} \langle 0|T(\pi(y_1)\ldots\pi(y_n)\pi(x_1)\ldots\pi(x_m))|0\rangle \sim$$

$$\prod_i^n \frac{iZ_-^\pi(\boldsymbol{p}_i)}{p_i^{0\,2} - E_-^2(\boldsymbol{p}_i) + i\varepsilon} \prod_j^m \frac{i\bar{Z}_-^\pi(\boldsymbol{k}_j)}{k_j^{0\,2} - E_-^2(\boldsymbol{k}_j) + i\varepsilon} \langle \boldsymbol{p}_1 \ldots \boldsymbol{p}_n|S|\boldsymbol{k}_1 \ldots \boldsymbol{k}_m\rangle$$

$$Z_\pm^h(\boldsymbol{k}) \equiv \langle \Omega|h(0)|\boldsymbol{k}, \pm\rangle$$

(Another procedure is to write creation/annihilation operators in terms of fields: different LSZ expression, but same conclusions)

Lack of analyticity

The usual arguments of S-matrix analyticity breaks down

$$S = -\int \mathrm{d}^4x \mathrm{d}^4y \, e^{i(q_2\cdot y - q_1\cdot x)} \frac{-\partial^2_{y^0} - E^2_-(-i\partial_{y_i})}{Z^\pi_-(-i\partial_{y_i})} \frac{-\partial^2_{x^0} - E^2_-(-i\partial_{x_i})}{\bar{Z}^\pi_-(-i\partial_{x_i})} \langle \boldsymbol{p}_2 | T(\pi(y)\pi(x)) | \boldsymbol{p}_1 \rangle$$

$$S = (2\pi)^4 \delta^{(4)}(p_2 + q_2 - p_1 - q_1) i\mathcal{T}$$

$$\mathcal{T} = i \int \mathrm{d}^4z \, e^{iqz} \frac{-\partial^2_{z^0/2} - E^2_-(-i\partial_{z_i/2})}{Z^\pi_-(-i\partial_{z_i/2})} \frac{-\partial^2_{z^0/2} - E^2_-(-i\partial_{z_i/2})}{\bar{Z}^\pi_-(-i\partial_{z_i/2})} \langle \boldsymbol{p}_2 | \theta(z^0)[\pi(\frac{z}{2})\pi(-\frac{z}{2}))] | \boldsymbol{p}_1 \rangle$$

Vanishes outside FLC in z

$\mathcal{T}(q^\mu)$ analytic for Im q^μ in FLC

Without Lorentz invariance Z(k) and E(k) introduce non-analyticities