Explicit resolvent bounds for transfer operators

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General set-up

Set-up

$\mathcal{L} : X \to X$ transfer operator of hyperbolic dynamical system T

Want

Calculate spectral data of $\mathcal{L}$

How?

Take sequence of finite-rank discretisations $(\mathcal{L}_k)$ with $\mathcal{L}_k \to \mathcal{L}$

Hope

$$\text{spectral data of } \mathcal{L}_k \to \text{spectral data of } \mathcal{L}$$

Analytic scenario

Best case

Underlying system T is analytic: then possible to choose X such that $\mathcal{L} : X \to X$ is 'very' compact

Quantifying compactness

Let X be a Banach space, and $A : X \to X$ a bounded operator. Then for $n \in \mathbb{N}$

$$s_n(A) := \inf \{\|A - F\| \ : \ \operatorname{rank} F < n\} \quad (n \in \mathbb{N})$$

is called the n-th **approximation number** of A.

Properties

- $s_n(A) \to 0$ implies A compact
- if X is Hilbert, then
  - $s_n(A) \to 0$ iff A compact
  - $s_n(A) = \sqrt{\lambda_n(AA^*)}$ (= n-th singular value of A)

Analytic scenario ...

Assumption

Underlying system T is hyperbolic and analytic on a subset of $\mathbb{C}^d$

Fact I

Possible to choose X such that $\mathcal{L} : X \to X$ satisfies, for some $a > 0$

$$s_n(\mathcal{L}) = O(\exp(-an^{1/d}))$$

B-Jenkinson 08, Slipantschuk-B-Just 22, Jézéquel 22

Fact II

Often $\exists$ discretisation scheme $(\mathcal{L}_k)$ such that, for some $0 < a' \leq a$

$$\|\mathcal{L} - \mathcal{L}_k\| = O(\exp(-a'k^{1/d}))$$

Wormell 19, B-Slipantschuk, Wormell-Vytnova

Fact III

Often X can be chosen to be a Hilbert space

Analytic scenario......

Consequences

Since $\|\mathcal{L} - \mathcal{L}_k\| \to 0$ it follows

$$\text{spectral data of } \mathcal{L}_k \to \text{spectral data of } \mathcal{L}$$

Main problem

For a given $N \in \mathbb{N}$, how close is spectral data of $\mathcal{L}_N$ to that of $\mathcal{L}$?

Main problem quantified

Definition

Given $\sigma, \sigma' \subset \mathbb{C}$ closed, and $z \in \mathbb{C}$, write

$$\operatorname{dist}(z, \sigma) = \inf_{\lambda \in \sigma} |z - \lambda|$$
$$\widehat{\operatorname{dist}}(\sigma, \sigma') = \sup_{z \in \sigma} \operatorname{dist}(z, \sigma')$$

The **Hausdorff distance** of σ and σ' is defined as

$$\operatorname{Hdist}(\sigma, \sigma') = \max(\widehat{\operatorname{dist}}(\sigma, \sigma'), \widehat{\operatorname{dist}}(\sigma', \sigma))$$

Note

Hdist is a metric on the set of closed subsets of $\mathbb{C}$.

Main problem

If A and B are compact operators, find explicitly computable upper bounds for $\operatorname{Hdist}(\sigma(A), \sigma(B))$.

Finite dimensional prototype

Theorem (Ostrowski 57, Henrici 62, Elsner 85)

Let $n \in \mathbb{N}$. Then there is $C_n > 0$ such that for any $n \times n$ matrices A, B we have

$$\mathrm{Hdist}(\sigma(A), \sigma(B)) \leq C_n (2M)^{1-1/n} \|A - B\|^{1/n},$$

where $M := \max\{\|A\|, \|B\|\}$.

Remark

- Ostrowski, Henrici: $C_n \leq n$
- Elsner: $C_n = 1$, provided $\|\cdot\|$ is spectral norm
- $1 \leq C_n = O(1)$ for arbitrary matrix norms

Basic approach

Key ingredient

Need resolvent estimates of the form

$$\|(zI - A)^{-1}\| \leq g_A\left(\frac{1}{\operatorname{dist}(z, \sigma(A))}\right),$$

for some function $g_A : \mathbb{R}_0^+ \to \mathbb{R}_0^+$.

Example

If A is normal (that is $A^*A = AA^*$), then

$$\|(zI - A)^{-1}\| = \frac{1}{\operatorname{dist}(z, \sigma(A))}$$

Basic tool: Bauer-Fike Lemma

Lemma (Bauer and Fike 60)

Let $A : X \to X$ be bounded. Suppose there is an increasing surjection $g_A : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\|(zI - A)^{-1}\| \leq g_A\left(\frac{1}{\operatorname{dist}(z, \sigma(A))}\right).$$

Then, for any bounded $B : X \to X$ we have

$$\widehat{\operatorname{dist}}(\sigma(B), \sigma(A)) \leq h_A(\|A - B\|),$$

where

$$h_A(x) = \frac{1}{g_A^{-1}(1/x)}.$$

Proof of Bauer-Fike Lemma

Write $E = B - A$.

Claim

$$z \in \sigma(B) \setminus \sigma(A) \implies \frac{1}{\|E\|} \leq \|(zI - A)^{-1}\|$$

Proof of claim

Let $z \in \sigma(B) \setminus \sigma(A)$ and suppose to the contrary that $\|E\| \|(zI - A)^{-1}\| < 1$. Then

$$zI - B = (zI - A)\left(I - (zI - A)^{-1}E\right)$$

But $\left(I - (zI - A)^{-1}E\right)$ is invertible, so $zI - B$ is invertible, so $z \notin \sigma(B)$.

Wrapping up

If $z \in \sigma(B) \setminus \sigma(A)$ then, by the Claim

$$\|E\|^{-1} \leq \|(zI - A)^{-1}\| \leq g_A\left(\frac{1}{\operatorname{dist}(z, \sigma(A))}\right)$$

$$\implies g_A^{-1}(\|E\|^{-1}) \leq \frac{1}{\operatorname{dist}(z, \sigma(A))}$$

$$\implies \operatorname{dist}(z, \sigma(A)) \leq \frac{1}{g_A^{-1}(\|E\|^{-1})} = h_A(\|A - B\|)$$

QED

Corollary

If A and B are normal, then

$$\operatorname{Hdist}(\sigma(A), \sigma(B)) \leq \|A - B\| .$$

Resolvent bounds for trace class operators

Theorem (B& Güven 15)

Let $A : X \to X$ be a trace class operator on a Hilbert space X, that is, $\sum_{n=1}^{\infty} s_n(A) < \infty$. Then

$$\|(zI - A)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, \sigma(A))} \prod_{n=1}^{\infty} \left(1 + \frac{s_n(A)}{\operatorname{dist}(z, \sigma(A))}\right)^2$$

Proof relies on following classic bound for trace class operators A

$$\|(I + A)^{-1}\| \leq \frac{\prod_{n=1}^{\infty}(1 + s_n(A))}{\prod_{n=1}^{\infty}|1 + \lambda_n(A)|}$$

which in turn follows from the following bound for an $N \times N$ matrix

$$\|A^{-1}\| = \frac{\prod_{n=1}^{N-1} s_n(A)}{|\det(A)|}$$

Resolvent bounds for operators with summable approximation numbers

Theorem (B 24)

Let $A : X \to X$, where X is an arbitrary Banach space, have summable approximation numbers, that is, $\sum_{n=1}^{\infty} s_n(A) < \infty$. Then

$$\|(zI - A)^{-1}\| \leq \frac{c}{\operatorname{dist}(z, \sigma(A))} \prod_{n=1}^{\infty} \left(1 + \frac{c\, s_n(A)}{\operatorname{dist}(z, \sigma(A))}\right)^4$$

where c is a constant not depending on A or X with $c \leq \sqrt{2\mathrm{e}}$.

Proof relies on Banach space Weyl inequality due to Pietsch 80.

Applications to transfer operators of analytic systems

Corollary

Suppose $(\mathcal{L}_k)$ *is a discretisation of* $\mathcal{L}$ *with* $\|\mathcal{L} - \mathcal{L}_k\| = O(\exp(-a' k^{1/d}))$. *Moreover suppose that there is* $M > 0$ *with*

$$s_n(\mathcal{L}) \leq M \exp(-a n^{1/d})$$

$$s_n(\mathcal{L}_k) \leq M \exp(-a n^{1/d})$$

Then there is an explicitly computable function $H_{a,d} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *with*

$$\mathrm{Hdist}(\sigma(\mathcal{L}), \sigma(\mathcal{L}_k)) \leq M H_{a,d}\left(\frac{\|\mathcal{L} - \mathcal{L}_k\|}{M}\right) = O(\exp(-c k^{1/d(d+1)}))$$

where $c > 0$.