Some questions about MMEs for surface diffeomorphisms with positive entropy

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June 19th, 2024

Outline

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2. Entropy and MMEs for surface diffeomorphisms
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Entropy and Variational Principle for continuous T on compact metric X

Definition (Bowen,Katok)

Topological entropy: $$h_{\mathrm{top}}(f) := \lim_{\epsilon\to 0} \underbrace{\limsup_{n\to\infty} \frac{1}{n} \log r_T(\epsilon, n, X)}_{h_{\mathrm{top}}(T,\epsilon)}$$

Kolmogorov-Sinai entropy of $\mu \in \mathbb{P}_{\mathrm{erg}}(T)$: $$h_\mu(f) := \lim_{\epsilon\to 0} \underbrace{\limsup_{n\to\infty} \frac{1}{n} \log r_T(\epsilon, n, \mu)}_{h_\mu(T,\epsilon)}$$

Misiurewicz tail entropy: $$h^*(T) := \lim_{\epsilon\to 0} \underbrace{\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \max_x r_T(\delta, n, B_T(x, \epsilon, n))}_{h^*(T,\epsilon)}$$

Theorem (Variational principle, Goodman, Dinaburg)

$$h_{\mathrm{top}}(T) = \sup_{\mu\in\mathbb{P}_{\mathrm{erg}}(T)} h_\mu(T)$$

Proof (Misiurewicz).

Let $E_n^\epsilon \subset X$ be (ϵ, n)-separated and $\mu_n^\epsilon := \frac{1}{n} \frac{1}{\# E_n^\epsilon} \sum_{x\in E_n^\epsilon, 0\le k<n} \delta_{T^k x}$

Any accumulation point μ^ϵ satisfies: $h_{\mu^\epsilon}(T) \geq h_{\mathrm{top}}(f, \epsilon)$. □

Entropy and MMEs in smooth dynamics

Is $\lim_{\epsilon\to 0}\mu^\epsilon$ above an MME?

Theorem (Newouse 1989)
If f C^∞ on compact manifold then $\exists$ *ergodic* Measure Maximizing the Entropy *(ergodic MME):*

$$\exists\mu\in\mathbb{P}_{\text{erg}}(T)\quad h_\mu(f)=\sup_{\mu\in\mathbb{P}_{\text{erg}}(T)} h_\mu(T)=h_{\text{top}}(T)$$

Proof.
$h^*(f)=0$ from Yomdin theory, then upper semicontinuity by Misiurewicz □

Remark Burguet-Liao-Yang (2015) established estimates on $h^*(f,\epsilon)\to 0$ (it can be bad)

Remark $h^*(f)=0$ holds beyond (pmm, expansive, hyperbolic, PH with $d_c=1$,...)

Remark $h^*(f)>0$ often in C^r topology

Corollary
$f\in\text{Diff}^\infty(M)\mapsto h_{\text{top}}(f)$ *is upper semicontinuous*

From now on: C^∞ smooth diffeomorphisms on a surface

Theorem (Katok 1980, Newhouse 1987)
$f \in \mathrm{Diff}^\infty(M) \mapsto h_{\mathrm{top}}(f) = \sup\{h_{\mathrm{top}}(f|H) : H \textit{ horseshoe}\}$ *is continuous*

Question Can one actually compute $h_{\mathrm{top}}(f)$ in this setting?

Consider χ-hyperbolic periodic orbits:

$$\mathrm{Per}_{f,\chi}(n) := \{x \in M : f^n(x) = x, \sigma(D_x f^n) \cap \{z \in \mathbb{C} : e^{-\chi n} \leq |z| \leq e^{+\chi n}\} = \emptyset\}$$

Theorem (Burguet 2020; Katok 1980, Sarig 2013)
There is $p \geq 1$ st for any $0 < \chi < h_{\mathrm{top}}(f)$

– [Periodic-expansivity] $\lim_{n\to\infty} \frac{1}{p\cdot n} \log \#\mathrm{Per}_{f,\chi}(p\cdot n) = h_{\mathrm{top}}(f)$

– [Equidistribution] *Any weak accumulation point of*

$$\frac{1}{\#\mathrm{Per}_{f,\chi}(p\cdot n)} \sum_{x\in \mathrm{Per}_{f,\chi}(p\cdot n)} \delta_x \textit{ is an MME}$$

Remark Kaloshin's examples (1997) show that $\chi = 0$ does not work

Question Can one use the above to compute the MME?
Find a rate of convergence?

C^∞-smooth diffeomorphisms on surfaces with **positive** h_{top}

Theorem (B-Crovisier-Sarig 2022)

$f : M \to M$ be C^∞ smooth on compact surface with $h_{\mathrm{top}}(f) > 0$
If $h_{\mathrm{top}}(f) > 0$, there are finitely many ergodic MMEs and they depend upper semicontinuously on $f \in \mathrm{Diff}^\infty(M^2)$

If, additionally f is top transitive, there is a unique MME

Remark. The Markov partition is countable and *very* abstract

Theorem (B-Crovisier-Sarig 2022)

For any ergodic MME μ there is a hyperbolic periodic orbit $\mathcal{O}$ such that, setting:

$$W^s(\mathcal{O}) := \{x \in M : \lim_{n\to\infty} \frac{1}{n} \log d(f^{+n}x, \mathcal{O}) < 0\} \text{ and}$$
$$W^u(\mathcal{O}) := \{x \in M : \lim_{n\to\infty} \frac{1}{n} \log d(f^{-n}x, \mathcal{O}) < 0\}$$

then, $\mathrm{supp}(\mu) = H_{\mathcal{O}} := \overline{W^u(\mathcal{O}) \pitchfork W^s(\mathcal{O})}$ *(homoclinic class)*

and the χ-hyperbolic periodic points equidistribute toward μ

Question Can one use this to find $\mathrm{supp}(\mu) = H_{\mathcal{O}}$? Is there a rate of convergence?

Question Can one localize the MMEs? Structure them? Upper bound on their $\#$?

C^∞-smooth diffeomorphisms on surfaces with **positive** h_{top}

f is top mixing if for any nonempty open $U, V \subset M$, for all large n, $U \cap f^{-n}(V) \neq \emptyset$

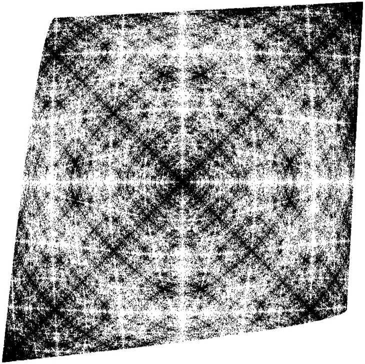
Theorem (B-Crovisier-Sarig ?2024?)

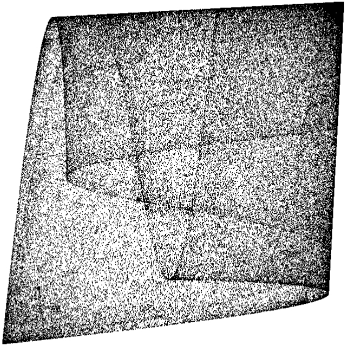
$f : M \to M$ be C^∞ smooth on compact surface with $h_{\text{top}}(f) > 0$
If $h_{\text{top}}(f) > 0$ and f is top mixing, there is a unique MME and it is exponentially mixing with CLT, ASIP,... wrt Hölder observables

Question Sure. But can one sample?Is there a somewhat tractable Banach anisotropic space like in the uniform case?

Question Can the transfer operator built by Gouezel-Liverani (2008) for Anosov diffeomorphisms be used to estimate the MME there?

Something completely different





$$f(x,y) = (1 - ax^2 - \epsilon y^2, 1 - ay^2 - \epsilon x^2) \text{ with } a = 1.9,\ \epsilon = 0.1$$

Conjecture The picture on the left, obtained by picking randomly (uniformly, independently) a preimage in $\Lambda := f([0,1]^2)$, is an equilibrium for $-\log \#(f^{-1}(x) \cap \Lambda)$

Conclusion

Some answers?

Thank you!