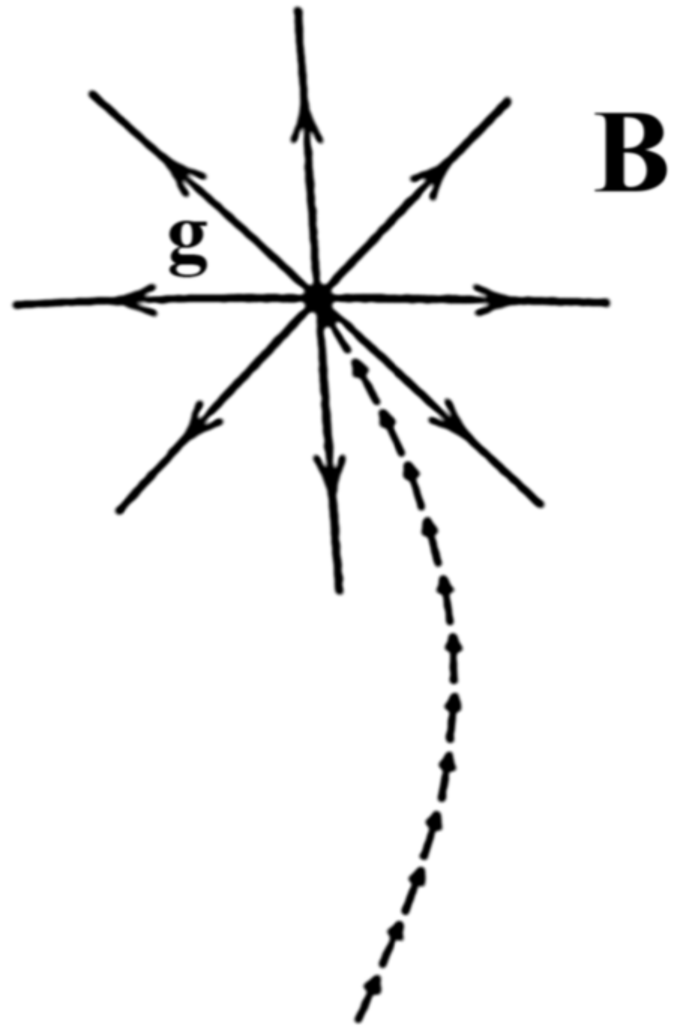
Quantum Magnetic Monopoles

arXiv:2411.18741

Dirac Strings



- Dirac (1948): electric charges + magnetic monopoles + EM field. Action + Quantum theory
- Strange non-locality
- Dirac string attached to each monopole
- Dirac quantisation condition $\Longrightarrow$ quantum theory well-defined and independent of positions of strings
- Dirac veto: position of strings arbitrary, so long as they do not intersect worldlines of electrically charged particles

Wu-Yang Monopoles

- Heavy monopole: treat as (semi-) classical background
- Quantise electrons and EM field in monopole background
- Remove location of monopole from spacetime, leaving space of non-trivial topology. EM field : U(1) bundle over this
- Gauge connection A_α in each patch U_α, no Dirac strings
- Magnetic charge is topological
- Use for `t Hooft lines in N=4 SYM

Kristjansen and Zarembo

Light monopoles

- SUSY theories can have light or massless monopoles
- If monopoles are light and dynamical, Wu-Yang approach problematic
- Quantum monopole doesn’t have definite location. Instead there is a wave function
- No clear choice of which points to exclude from spacetime for Wu-Yang bundle
- In path integral, sum over all monopole trajectories. Different bundle for each trajectory?

Dirac Approach Revisited

- Quantises system of dynamical electrons and monopoles plus EM field
- Dirac: field equations independent of positions of strings
- Independence of actions and path integral on positions of strings: generalised symmetries
- These generalised symmetries are anomalous. Restriction to configurations in which anomaly is zero = Dirac veto
- Anomaly cancellation?

CMH arXiv:2411.18741

Maxwell with Sources

$$d^\dagger F = j \qquad\qquad dF = *\tilde{j}$$

Electric 1-form current j, Magnetic 1-form current $\tilde{j}$

Conserved: $d^\dagger j = 0$ $\qquad d^\dagger \tilde{j} = 0$

4 dimensions, F is 2-form field strength

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Conserved: $d^\dagger j = 0$ $\qquad d^\dagger \tilde{j} = 0$

Dirac: introduce 2-form $\tilde{J}$ with

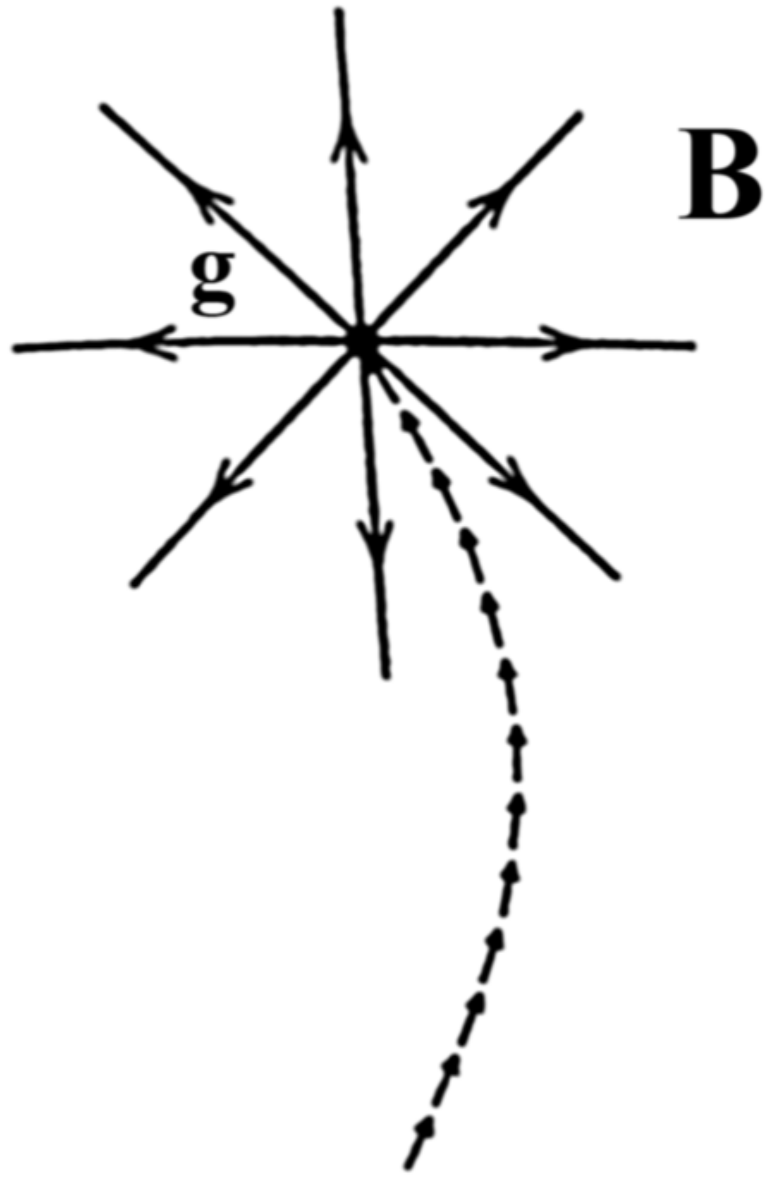
$$d^\dagger \tilde{J} = \tilde{j}$$

Dirac string is attached to each magnetic monopole, going to infinity.

2-form $\tilde{J}$ is the current density for these strings:

if $\tilde{j}$ localised on the world-line of a magnetic monopole,

then $\tilde{J}$ is localised on the world-sheet of the corresponding Dirac string.



The Dirac Formalism

$$dF = *\tilde{j} \qquad\qquad d^{\dagger}\tilde{J} = \tilde{j}$$

$$\Longrightarrow \quad d(F - *\tilde{J}) = 0$$

Solve: $F = *\tilde{J} + dA$ 1-form potential A

The Dirac Formalism

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$$\implies \quad d(F - *\tilde{J}) = 0$$

$$F = *\,\tilde{J} + dA$$

1-form potential A

$$S[A] = S_{particles} + \int \frac{1}{2} F \wedge *F - A \wedge *j$$

$$S_{particles} = \sum (mass) \times (length\ of\ worldline)$$

Dual formulation

Also: Dirac strings for electric charges, 2-form current J

$$d^\dagger J = j$$

Solve $$d^\dagger F = j$$

$$* F = * J + d\tilde{A}$$

Dual potential $\tilde{A}$ $$\tilde{F} = * F$$

$$S[\tilde{A}] = \int \frac{1}{2}\tilde{F} \wedge *\tilde{F} - \tilde{A} \wedge *\tilde{j}$$

Currents

Particle of electric charge q, world-line is a curve $\mathscr{C}$, $x^\mu = X^\mu(\tau)$

$$q\int_{\mathscr{C}} A = q\int d\tau\ A_\mu(X(\tau))\frac{dX^\mu}{d\tau} = \int_{\mathscr{M}} A\wedge *j$$

Multivalued. Dirac quantisation:

$W(\mathscr{C}) = e^{\frac{i}{\hbar}q\int_{\mathscr{D}} F}$ well-defined

$$j^\mu(x) = q\int d\tau\frac{dX^\mu}{d\tau}\delta(x - X(\tau))$$

$$j = q\delta_{\mathscr{C}}$$

$\delta_{\mathscr{C}}$ 1-form delta-function

$d^\dagger J = j$ $\qquad J = q\delta_{\mathscr{D}}$ $\qquad \mathscr{D}$ is a 2-surface with boundary $\mathscr{C}$

Wilson Loops

$\mathscr{C}$ a closed curve bounding 2-surface $\mathscr{D}$

$$q\int_{\mathscr{C}} A = q\int_{\mathscr{D}} F \qquad \Longrightarrow$$

$$\int_{\mathscr{M}} A \wedge *j = \int_{\mathscr{M}} F \wedge *J \qquad J = q\delta_{\mathscr{D}} \qquad d^{\dagger}J = j$$

For two surfaces $\mathscr{D}, \mathscr{D}'$ with same boundary $\mathscr{C}$

$$q\int_{\mathscr{D}'} F - q\int_{\mathscr{D}} F = q\int_{\mathscr{D}'-\mathscr{D}} F = 2\pi pq$$

Magnetic charge p

Then $\qquad W(\mathscr{C}) = e^{\frac{i}{\hbar}q\int_{\mathscr{D}} F}$

well-defined if $\qquad pq = 2\pi n, \quad n \in \mathbb{Z}$

Dirac quantisation

N particles, $i = 1,\ldots,N$

Masses m_i, electric charges q_i and magnetic charges p_i moving

Worldlines $\mathscr{C}_i$ given by $x^\mu = X_i^\mu(\tau_i)$

$$j(x) = \sum_{i=1}^{N} q_i \delta_{\mathscr{C}_i}(x), \quad \tilde{j}(x) = \sum_{i=1}^{N} p_i \delta_{\mathscr{C}_i}(x) \qquad \tilde{J} = \sum_i p_i \delta_{\mathscr{D}_i}$$

$$S_{particles} = \sum_i m_i \int d\tau_i \sqrt{\dot{X}_i^2}$$

Dirac's Action

$$S[A] = S_{particles} + \int \frac{1}{2} F \wedge *F - A \wedge *j \qquad F = *\tilde{J} + dA$$

- Veto $\Longrightarrow$ field equations independent of positions of strings

3-form symmetries

$d^\dagger J = j \qquad d^\dagger \tilde{J} = \tilde{j}$ Don't determine $J, \tilde{J}$ completely

$\delta J = d^\dagger \rho \qquad \delta \tilde{J} = d^\dagger \tilde{\rho}$ SYMMETRY

$F = * \tilde{J} + dA \qquad * F = * J + d\tilde{A}$ invariant if

$\delta A = * \tilde{\rho} \qquad \delta \tilde{A} = * \rho$

Symmetry of action with 3-form parameter ρ

Interpretation of Symmetry

Smooth deformation of Dirac string world-sheet $\mathscr{D}$ to $\mathscr{D}'$:

Family of Dirac string world-sheets $\mathscr{D}(\xi)$ parameterised by ξ with

$\mathscr{D}(0) = \mathscr{D}$ and $\mathscr{D}(1) = \mathscr{D}'$

Family of 2-dimensional world-sheets sweeps out a 3-dimensional surface $\mathscr{E}$

$$\tilde{J}' - \tilde{J} = p d^\dagger \delta_{\mathscr{E}}$$

Change in $\tilde{J}$ for infinitesimal deformation of $\mathscr{D}$, of the form $\delta\tilde{J} = d^\dagger\tilde{\rho}$ where $\tilde{\rho}$ is a 3-form current localised on $\mathscr{E}$

Generalised Symmetries in Maxwell Theory

Gaiotto, Kapustin, Seiberg, Willett

$$dF = 0 \qquad\qquad d * F = 0$$

$F, * F$ conserved 2-form currents

$$F = dA$$

0-form gauge symmetry $\delta A = d\sigma$

1-form symmetry $\delta A = \lambda$ $\qquad d\lambda = 0$

Constrained parameter: regard as global symmetry

1-form symmetries modulo 0-form symmetries: Cohomology class $[\lambda]$

Gauging 1-form Symmetry

Seek theory with symmetry for *unconstrained* λ

Introduce 2-form gauge field B

$$\delta B = d\lambda, \quad \delta A = \lambda$$

$$F = dA - B$$ invariant

$$S = \int \frac{1}{2} F \wedge *F$$

Dual Symmetry

$$* F = d\tilde{A}$$

$$\delta\tilde{A} = \tilde{\lambda}, \quad \delta\tilde{B} = d\tilde{\lambda}$$

$$* F = d\tilde{A} - \tilde{B}$$

Dual Gauge Symmetry in Original Theory

$$F = dA$$

Couple $\tilde{B}$ to Noether current F

$$S = \frac{1}{2}\int F \wedge *F + \tilde{B} \wedge F$$

Invariant under $\delta\tilde{B} = d\tilde{\lambda}$

Anomaly

Obstruction to gauging both symmetries

$$S = \int \frac{1}{2} F \wedge {*F} + \tilde{B} \wedge dA \qquad F = dA - B$$

Invariant under $\delta\tilde{B} = d\tilde{\lambda}$

Under $\delta B = d\lambda, \ \ \delta A = \lambda$ $\qquad \delta S = \int d\lambda \wedge \tilde{B}$

Alternative action $\qquad S = \int \frac{1}{2} F \wedge {*F} + \tilde{B} \wedge F$

Invariant under $\delta B = d\lambda, \ \ \delta A = \lambda$, but $\qquad \delta S = -\int d\tilde{\lambda} \wedge B$

Obstruction to simultaneous gauge of $\lambda, \tilde{\lambda}$ 1-form symmetries:

“Mixed Gauge Anomaly”

Ungauged theory has “mixed ’t Hooft anomaly” in $\lambda, \tilde{\lambda}$ 1-form symmetries

Gaiotto, Kapustin, Seiberg, Willett

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Anomaly can be cancelled by adding a 5-d action on a 5-d space with boundary the 4-d spacetime

$$S_5 = \int B \wedge d\tilde{B}$$

Dirac Action

Write $\rho = *\tilde{\lambda}, \;\; \tilde{\rho} = *\lambda$ $\quad B = -*\tilde{J}, \;\; \tilde{B} = -*J$

$$F = *\tilde{J} + dA \quad \rightarrow \quad F = dA - B$$

Dirac action becomes gauged Maxwell theory, with Dirac string currents $J, \tilde{J}$ reinterpreted as gauge fields $\tilde{B}, B$. 1-form symmetries with 1-form parameters $\lambda, \tilde{\lambda}$

The anomaly and the veto

Under gauge transformation $\delta B = d\lambda, \quad \delta A = \lambda$

$$\delta S = \int_{\mathscr{M}} \lambda \wedge {*j} = \sum_i q_i \int_{\mathscr{M}} \lambda \wedge {*\delta_{\mathscr{C}_i}}$$

For a λ arising from change of $\tilde{J} = \sum_i p_i \delta_{\mathscr{D}_i}$ that results from

moving the Dirac strings:

$$\lambda = \sum_i p_i * \delta_{\mathscr{E}_i},$$

$$\delta S = \sum_{i,j} q_i p_j \int_{\mathscr{M}} \delta_{\mathscr{E}_j} \wedge \delta_{\mathscr{C}_i}$$

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Family of Dirac string world-sheets $\mathscr{D}_j(\xi)$ parameterised by ξ sweeps out a 3-dimensional surface $\mathscr{E}_j$

Veto: None of these must cross any of the electric worldlines $\mathscr{C}_i$

For $i \neq j$ Dirac veto $\Longrightarrow \delta_{\mathscr{E}_j} \wedge \delta_{\mathscr{C}_i} = 0$

For $i = j$, $\mathscr{C}_i$ is tangent to $\mathscr{E}_i \Longrightarrow \delta_{\mathscr{E}_j} \wedge \delta_{\mathscr{C}_i} = 0$

Dirac Veto $\Longrightarrow \delta S = 0$

Dirac Veto: restricts to configurations of background gauge fields $B, \tilde{B}$ for which anomaly vanishes

Dodging the veto: Anomaly inflow

Anomaly can be cancelled by adding a 5-d action on a 5-d space with boundary the 4-d spacetime

$$S_5 = \int B \wedge d\tilde{B}$$

- 4-d Spacetime M boundary of 5-d space X
- Gauge fields on M
- Charged particles on M are ends of charged strings on X
- Dirac strings on M are boundary of Dirac membranes on X

3-forms $J_3 = *\tilde{B}_2, \quad \tilde{J}_3 = *B_2$

Dodging the veto: D3-brane

Electric charges on D3: Fundamental strings ending on D3

Magnetic charges on D3: D-strings ending on D3

F strings and D strings mutually local, so no veto?

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4d D3 brane action $$S_{Dirac} + \int C_4 + \ldots$$

Anomalous variation of S_{Dirac} can be cancelled by δC_4

$$F = dA - B, \quad B = b_{NSNS} + {}^*J$$

Conclusion

- Dirac’s action has generalised 1-form symmetries
- Veto is restriction to configurations in which anomaly vanishes
- Treats case in which electric and magnetic particles both light and dynamical
- Generalisations: p-form gauge fields coupling to branes: higher form symmetries. Born-Infeld and Chern-Simons interactions. Self-dual.
- Embed in string theory: D3 brane action
- Issue: Requires $\tilde{j}$ to be conserved off-shell to write $d^\dagger \tilde{J} = \tilde{j}$
- Field theory for electric and magnetic particles? [Zwanziger, Schwinger]

p-form delta-function

For p-dimensional submanifold $\mathscr{N} \subset \mathscr{M}$ define **p-form delta-function:** $\delta_{\mathscr{N}}$

de Rham current

For any p form ω on $\mathscr{M}$

$$\int_{\mathscr{N}} \omega = \int_{\mathscr{M}} \omega \wedge {}^*\delta_{\mathscr{N}}$$

If $\mathscr{N} \subset \mathscr{M}$ specified by $x^\mu = X^\mu(\sigma^a)$

$$\delta_{\mathscr{N}}(x) = \int d^p\sigma \quad \varepsilon^{a_1 a_2 \ldots a_p} \frac{\partial X^{\mu_1}}{\partial \sigma^{a_1}} \frac{\partial X^{\mu_2}}{\partial \sigma^{a_2}} \ldots \frac{\partial X^{\mu_p}}{\partial \sigma^{a_p}} \delta(x - X(\sigma))$$

For submanifolds with boundary:

$$\delta_{\partial \mathscr{N}} = d^\dagger \delta_{\mathscr{N}}$$