QUASINORMAL MODES FROM PENROSE LIMIT

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Overview

1. (Very brief) Introduction to QNMs.
2. The Penrose limit
3. Linear perturbations
4. Second-order equation for the Weyl scalar Ψ_0
5. Quadratic QNMs for the channel $(\ell \times \ell) \to 2\ell$
6. Conclusions

Motivation

- BH spectroscopy aims in the understanding of QNMs (the characteristic oscillations of a perturbed BH), which are triggered by BH interactions.
- The produced GWs carry information (mass, spin, enviroment and underlying theory).
- The QNM are usual in the linear regime and obey the linear Regge-Wheller-Zerilli equation.
- In high frequency dominance, (eikonal regime) the corresponding QNMs are linked to spherical harmonics with large angular quantum numbers ($\ell \gg 1$).
- This regime is tied to a collection of null geodesics focused around the so-called photon ring (the birth place of QNMs).
- QNMs in the eikonal regime can be treated as small perturbations around the Penrose limit of the background geometry centered on the photon ring.

- Full nonlinear numerical simulations have shown clear signs of non-linearities, the Quadratic-QNMs (QQNMs), due to nonlinear nature of General Relativity.
- At second order in perturbation theory, linear mode couplings lead to QQNMs.
- These modes probe GR deeper in the nonlinear regime, and therefore are of paramount importance.
- Our aim is see if QQNMs in the eikonal regime can be treated similarly to the linear QNMs, also at second-order in perturbation theory in the Penrose limit of the background Schwarzschild geometry.

Related publications:

1. M. H. Y. Cheung, V. Baibhav, E. Berti, V. Cardoso, G. Carullo, R. Cotesta, W. Del Pozzo, F. Duque, T. Helfer and E. Shukla, *et al.* "Nonlinear Effects in Black Hole Ringdown," Phys. Rev. Lett. **130** (2023) no.8, 8
2. K. Mitman, M. Lagos, L. C. Stein, S. Ma, L. Hui, Y. Chen, N. Deppe, F. Hébert, L. E. Kidder and J. Moxon, *et al.* "Nonlinearities in Black Hole Ringdowns," Phys. Rev. Lett. **130** (2023) no.8, 081402.
3. B. Bucciotti, L. Juliano, A. Kuntz and E. Trincherini, "Amplitudes and polarizations of quadratic quasi-normal modes for a Schwarzschild black hole," JHEP **09** (2024), 119; "Quadratic quasinormal modes of a Schwarzschild black hole," Phys. Rev. D **110** (2024) no.10, 104048
4. B. Bucciotti, V. Cardoso, A. Kuntz, D. Pereñiguez and J. Redondo-Yuste, "Ringdown nonlinearities in the eikonal regime," [arXiv:2501.17950 [gr-qc]].
5. A. Kehagias and A. Riotto, "Nonlinear effects in black hole ringdown made simple: Quasinormal modes as adiabatic modes," Phys. Rev. D **111** (2025) no.4, L041506
6. A. Kehagias, D. Perrone and A. Riotto, [arXiv:2503.09350 [gr-qc]].

1. (Very brief) Introduction to QNMs.

The Quasinormal modes (QNMs) describe the **damped oscillations** of perturbations around BH. They arise naturally when solving the **linearized Einstein equations** in a BH spacetime. They are solutions of the form

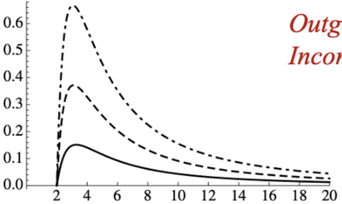
$$\Psi(t,r,\theta,\phi) = e^{-i\omega t}\psi(r)\ {}_{-2}Y_{\ell m}(\theta,\phi)$$

$\psi(r)$ satisfies the Regge-Wheeler equation

$$\frac{d^2}{dr_*^2}\psi(r) + (\omega^2 - V_{RW})\psi(r) = 0.$$

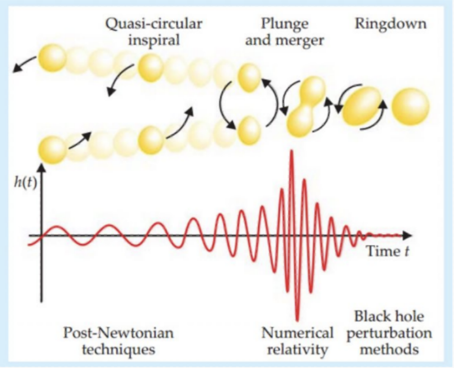
The Regee-Wheeler potential is

$$V_{RW} = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)$$



Outgoing BC at infinity

Incoming BC at horizon

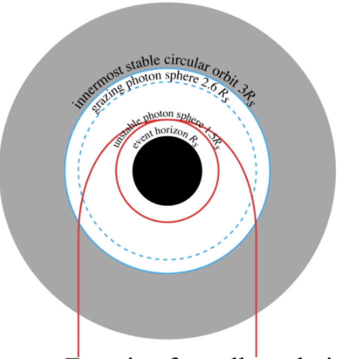


Baumgarte+Shapiro 2010

Black hole spectroscopy: reconstructing the remnant black hole mass/spin from measured quasinormal modes (Detweiler 1980, Dreyer+ 2004, Berti+ 2006, Carullo+ 2019, Isi+ 2021, ...)

ω are the QNM frequencies, which are (in the eikonal limit, large ℓ)

$$\boxed{\omega_{n\ell} = \frac{\ell}{3\sqrt{3}M} - i\left(n + \frac{1}{2}\right)\lambda, \qquad \lambda = \frac{1}{3\sqrt{3}M} \qquad \lambda \text{ is the Lyapunov exponent}}$$



QNMs are generated in **the strong gravity region near the black hole**, particularly in the vicinity of the **photon sphere.** For a Schwarzschild black hole of mass M, the **photon sphere** is the **unstable circular orbit** of photons at:

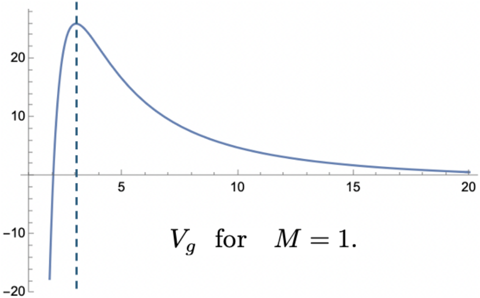
$$r = 3M$$

- Photons on this sphere **travel in perfect circular orbits** around the black hole.
- Any small perturbation causes the photon to either **fall into the black hole** or **escape to infinity**.
- The photon sphere defines the **edge of the black hole shadow** seen by distant observers.

Equation for null geodesics:

$$\dot{r}^2 = -V_g, \qquad V_g = -\left(1 - \frac{2M}{r}\right)\left(\frac{E^2}{1 - \frac{2M}{r}} - \frac{L^2}{r^2}\right)$$

Photon sphere: $\ddot{r} = \dot{r} = 0 \quad \Longrightarrow \quad r = 3M$



The Regge-Wheeler-Zerili equations are obtained by perturbations of the Einstein equations around the Schwarzschild BH $\bar{g}_{\mu\nu}$. We expand the metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \cdots$ so that Einstein tensor $G_{\mu\nu}$ is expanded as

$$G_{\mu\nu}[g] = G_{\mu\nu}[\bar{g}] + \epsilon G_{\mu\nu}[h^{(1)}] + \epsilon^2 \left(G_{\mu\nu}[h^{(2)}] + G_{\mu\nu}(h^{(1)^2} \right) + \cdots \tag{1}$$

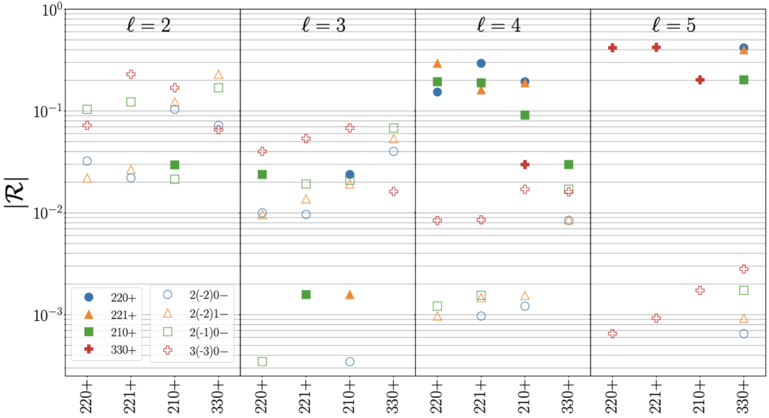
so that

$$G_{\mu\nu}(h^{(1)}) = 0, \qquad G_{\mu\nu}[h^{(2)}] = -G_{\mu\nu}(h^{(1)^2}) \tag{2}$$

$$h^{(1)} = \frac{M}{r} \sum_{n,\ell,m} \mathcal{A}^{(1)}_{n\ell m} e^{-i\omega_{n\ell m}(t-r_*)} {}_{-2}Y_{\ell m}(\theta, \phi) \qquad h^{(2)} = \frac{M}{r} \sum_{n,\ell,m} \mathcal{A}^{(2)}_{n\ell m} e^{-i\omega_{n\ell m}(t-r_*)} {}_{-2}Y_{\ell m}(\theta, \phi)$$

We are interested to calculate the quadratic-to-linear amplitude ratio

$$\mathcal{R}_{\ell\times\ell} = \left| \frac{\mathcal{A}^{(2)}_{0\ell\ell}}{\left(\mathcal{A}^{(1)}_{0\ell\ell}\right)^2} \right|$$



Bucciotti, Juliano, Kuntz & Trincherini, 2024

$\mathcal{R}^{\rm num}_{\ell\times\ell} \sim 0.2 \qquad \ell$ large

2. The Penrose Limit

Theorem

Penrose: *Every spacetime has a pp-wave as a limit.*

The Penrose limit associates to every space-time metric g_{ab}, with line element $\mathrm{d}s^2$, and a null geodesic γ in that space-time, a (limiting) plane wave metric:
1) Write the metric in coordinates $V, Z, \bar{Z}$ "adapted" to γ, with the remaining coordinate U playing the role of the affine parameter and $\gamma(U)$ coinciding with the geodesic at $V = Z = \bar{Z} = 0$.
2) Change coordinates to

$$(U, V, Z, \bar{Z}) = (u, \lambda^2 v, \lambda z, \lambda \bar{z})$$

for some real λ and to take the limit

$$\lim_{\lambda \to 0} \lambda^{-2} \mathrm{d}s^2 = \mathrm{d}s^2_{\gamma}.$$

The resulting metric $\mathrm{d}s_\gamma^2$ is the so-called **Penrose limit** of the initial space-time, which, recast in Brinkmann coordinates, has the plane-parallel (pp) wave form

$$\mathrm{d}s_\gamma^2 = 2\mathrm{d}u\mathrm{d}v + H(u, z, \bar{z})\mathrm{d}u^2 - 2\mathrm{d}z\mathrm{d}\bar{z}. \tag{1}$$

The coordinate u plays the role of the affine parameter along the geodesic, and the function H controls the geodesic deviation properties along the transverse coordinates z and $\bar{z}$. For the Schwarzschild black hole of mass M and metric

$$\begin{aligned} \mathrm{d}s^2 &= f(r)\mathrm{d}t^2 - f^{-1}(r)\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \\ f(r) &= 1 - 2M/r, \end{aligned} \tag{2}$$

the Penrose limit around the circular photon ring located at $r_0 = 3M$ and $\theta = \pi/2$ has the metric (1) with

$$H(z, \bar{z}) = -\frac{1}{3M^2}(z^2 + \bar{z}^2), \tag{3}$$

with $u = -t/3$ and $v = -t + 3\sqrt{3}M\phi$ in terms of Schwarzschild coordinates.

Linear Perturbations

For a generic pp-wave space-time such as the one described by the metric (1), Einstein (vacuum) equations reduce to

$$H_{z\bar{z}} = 0. \tag{4}$$

For such spacetimes, metric perturbations $h_{\mu\nu}$ that satisfies the linearized Einstein vacuum equations can be expressed in terms of a complex scalar field Φ, the Hertz potential, which satisfies

$$\frac{1}{2}\Box\Phi = -\Phi_{z\bar{z}} - \frac{1}{2}H(z,\bar{z})\Phi_{vv} + \Phi_{uv} = 0. \tag{5}$$

Indeed, having a solution of the scalar wave equation (5), one can construct solutions of spin-two states by using the spin-raising operator. The latter is written as

$$\begin{aligned} R^{+} &= \mathrm{d}u\,\partial_{\bar{z}} + \mathrm{d}z\,\partial_{v}, \\ R^{-} &= \mathrm{d}u\,\partial_{z} + \mathrm{d}\bar{z}\,\partial_{v}. \end{aligned} \tag{6}$$

The spin-two field reads

$$h_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = h^{++}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + h^{--}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} \tag{7}$$

where

$$\begin{aligned} h^{++}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} =& R^{+}\Big[R^{+}(\Phi)\Big], \\ h^{--}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} =& R^{-}\Big[R^{-}(\bar{\Phi})\Big]. \end{aligned} \tag{8}$$

Then in the radiation gauge (which also implies the transverse-free and traceless conditions)

$$\nabla^{\mu}h_{\mu\nu} = g^{\mu\nu}h_{\mu\nu} = 0, \quad h_{\mathrm{v}\mu} = 0, \tag{9}$$

the metric perturbed by the spin-two fields $h_{\mu\nu}$ is at first-order

$$\begin{aligned} \mathrm{d}s^2 &= 2\mathrm{d}u\mathrm{d}v + \left[H + \Phi_{\bar{z}\bar{z}} + \bar{\Phi}_{zz}\right]\mathrm{d}u^2 + 2\Phi_{v\bar{z}}\mathrm{d}u\mathrm{d}z + 2\bar{\Phi}_{vz}\mathrm{d}u\mathrm{d}\bar{z} \\ &+ \Phi_{vv}\mathrm{d}z^2 + \bar{\Phi}_{vv}\mathrm{d}\bar{z}^2 - 2\mathrm{d}z\mathrm{d}\bar{z}. \end{aligned} \tag{10}$$

The linearized Einstein equations are satisfied if Φ satisfies the equation

$$-\Phi_{z\bar{z}} - \frac{1}{2}H(z,\bar{z})\Phi_{vv} + \Phi_{uv} = 0.$$

For the QNMs, we first look for linear solutions to the above equation and demand an outgoing boundary condition for the unstable direction, and a decaying boundary condition for the stable direction. For the fundamental mode, we get (*Fransen, Giataganas et al.)*

$$\begin{aligned}\Phi(u,v,z,\bar{z}) =& Ae^{-iP_u u + iP_v v + \frac{3}{4}\ell\omega^2(1+i)(z^2+2iz\bar{z}+\bar{z}^2)} \\ &\times H_n(\sqrt{-3i(n-\ell)}\omega x_1)H_{n_2}(\sqrt{3(n_2-\ell)}\omega x_2),\end{aligned} \tag{11}$$

where

$$P_u = \frac{3}{2}\omega(i-1), \qquad P_v = \ell\omega \equiv \omega_\ell, \qquad \omega = \frac{1}{3\sqrt{3}M}. \tag{12}$$

The QNM frequencies are

$$\omega_{n\ell} = \left(\ell + \frac{1}{2}\right)\omega + i\omega\left(\frac{1}{2} + n\right). \tag{13}$$

The lesson

The pp-wave limit of the Schwarzschild BH captures the QNM frequencies at large ℓ.

and

The question

Is this an accident, or we can use it also at second order.

The answer

Penrose limit can go even beyond first-order perturbation theory.

We will examine in the sequence how second-order perturbations of the Schwarzschild BH, can be approximated by their pp-wave limit.

It is more convenient to work with the Weyl scalars. We first need to find the appropriate null tetrad. The latter in the coordinates $(u, v, z, \bar{z})$ is given up to first-order by

$$\begin{aligned}
\ell_\mu =& a(1,0,0,0), \\
n_\mu =& \frac{1}{a}\left[\frac{1}{2}(H+\Phi_{zz}+\bar{\Phi}_{\bar{z}\bar{z}}), 1, 0, 0\right] \\
m_\mu =& \left(-\frac{1}{2}\Phi_{v\bar{z}}, 0, -\frac{1}{2}\Phi_{vv}, 1\right), \\
\bar{m}_\mu =& \left(-\frac{1}{2}\bar{\Phi}_{vz}, 0, 1, -\frac{1}{2}\bar{\Phi}_{vv}\right), \qquad (14)
\end{aligned}$$

so that the first-order metric (10) can be also written as

$$\mathrm{d}s^2 = 2\ell_{(\mu}n_{\nu)} - 2m_{(\mu}\bar{m}_{\nu)}. \qquad (15)$$

A simple comparizon with perturbations of the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + (h_+ - ih_\times)\mathrm{d}z^2 + (h_+ + ih_\times)\mathrm{d}\bar{z}^2 - 2\mathrm{d}z\mathrm{d}\bar{z}. \tag{16}$$

reveals that

$$h_+ = \frac{1}{2}\left(\Phi_{vv} + \bar{\Phi}_{vv}\right), \qquad h_\times = \frac{1}{2i}\left(\bar{\Phi}_{vv} - \Phi_{vv}\right) \tag{17}$$

Then, the Weyl scalars are

$$\begin{aligned} &\Psi_0 = -\, C_{\mu\nu\rho\sigma}\,\ell^\mu m^\nu \ell^\rho m^\sigma, \qquad \Psi_1 = -C_{\mu\nu\rho\sigma}\,\ell^\mu n^\nu \ell^\rho m^\sigma, \\ &\Psi_2 = -\, C_{\mu\nu\rho\sigma}\,\ell^\mu m^\nu \bar{m}^\rho n^\sigma, \qquad \Psi_3 = -C_{\mu\nu\rho\sigma}\,\ell^\mu n^\nu \bar{m}^\rho n^\sigma, \qquad \Psi_4 = -C_{\mu\nu\rho\sigma}\,n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma, \end{aligned} \tag{18}$$

and the two Weyl scalars we will be interested below, Ψ_0 and Ψ_4, are

$$\Psi_0 = \frac{1}{2}a^2(h_+ - ih_\times)_{vv}, \tag{19}$$

$$\Psi_4 = \frac{1}{2a^2}(h_+ + ih_\times)_{vv}. \tag{20}$$

For completeness, the corresponding non-zero spin coefficients in the basis (14) up to first-order are

$$
\begin{aligned}
\tau^{(1)} &= -\frac{1}{2}\Phi_{vv\bar{z}}, & \alpha &= 0, \\
\rho &= 0, & \beta^{(1)} &= -\frac{1}{2}\Phi_{vv\bar{z}}, \\
\sigma^{(1)} &= \frac{1}{2}a\Phi_{vvv}, & \epsilon &= 0, \\
\kappa &= 0, & \nu^{(0)} &= -\frac{1}{2a^2}H_{\bar{z}} \\
\gamma^{(1)} &= \frac{1}{2a}\Phi_{v\bar{z}\bar{z}}, & \nu^{(1)} &= -\frac{1}{4a^2}\left(2\Phi_{\bar{z}\bar{z}\bar{z}} + 2\bar{\Phi}_{zz\bar{z}} + H_z\bar{\Phi}_{vv}\right) \\
\lambda^{(1)} &= \frac{1}{a}\left(\bar{\Phi}_{vz\bar{z}} + \frac{1}{4}H\bar{\Phi}_{vvv} - \frac{1}{2}\bar{\Phi}_{uvv}\right), & \pi^{(1)} &= -\frac{1}{2}\bar{\Phi}_{vvz}. \\
\mu^{(1)} &= \frac{1}{2a}\left(\Phi_{v\bar{z}\bar{z}} + \bar{\Phi}_{vzz}\right), & &
\end{aligned}
\tag{21}
$$

The non-zero Weyl scalars turn out to be

$$\Psi_0^{(1)} = \frac{1}{2}a^2\Phi_{vvvv} \qquad \Psi_1^{(1)} = -\frac{1}{2}a\Phi_{vvv\bar{z}}, \qquad \Psi_2^{(1)} = \frac{1}{2}\Phi_{vv\bar{z}\bar{z}},$$
$$\Psi_3^{(1)} = -\frac{1}{2a}\Phi_{v\bar{z}\bar{z}\bar{z}}, \qquad \Psi_4^{(0)} = \frac{1}{2a^2}H_{\bar{z}\bar{z}} \tag{22}$$

and

$$\Psi_4^{(1)} = \frac{1}{a^2}\left(\frac{1}{2}\Phi_{\bar{z}\bar{z}\bar{z}\bar{z}} + \frac{1}{2}\bar{\Phi}_{zz\bar{z}\bar{z}} + \frac{1}{4}H_z\bar{\Phi}_{vv\bar{z}} + \frac{1}{4}H_{\bar{z}}\bar{\Phi}_{vvz} + \frac{1}{2}H\bar{\Phi}_{vvz\bar{z}}\right.$$
$$\left. + \frac{1}{2}H_{z\bar{z}}\bar{\Phi}_{vv} + \frac{1}{8}H^2\bar{\Phi}_{vvvv} - \bar{\Phi}_{uvz\bar{z}} - \frac{1}{2}H\bar{\Phi}_{uvvv} + \frac{1}{2}\bar{\Phi}_{uuvv}\right). \tag{23}$$

3. Second-order equation for the Weyl scalar Ψ_0

For **Petrov-type D** spacetimes like the Kerr black hole, the first-order $\Psi_4^{(1)}$ is completely determined by the zeroth-order background (Teukolsky equation) and similarly, the second-order $\Psi_4^{(2)}$ is determined purely by first-order quantities. For **Petrov-type N**, like the pp-wave background we are considering here, this is not true any longer. In fact, the first-order $\Psi_4^{(1)}$ is determined now by the other first-order Weyl scalars and similarly the second-order $\Psi_4^{(2)}$ is determined by the second-order quantities. In other words, to determine $\Psi_4^{(2)}$, we need all the second-order perturbations of the metric. Therefore, for Petrov-type N spacetimes, it is more convenient to consider Ψ_0 instead. In order to determine the equation for Ψ_0, we will need the following two Bianchi identities for Ricci-flat spacetimes

$$-(\bar{\delta}+\pi-4\alpha)\Psi_0+(D-4\rho-2\epsilon)\Psi_1+3\kappa\Psi_2=0, \tag{24}$$

$$-(\Delta+\mu-4\gamma)\Psi_0+(\delta-4\tau-2\beta)\Psi_1+3\sigma\Psi_2=0, \tag{25}$$

where

$$D=\ell^\mu\nabla_\mu, \qquad \Delta=n^\mu\nabla_\mu, \qquad \delta=m^\mu\nabla_\mu, \qquad \bar{\delta}=\bar{m}^\mu\nabla_\mu. \tag{26}$$

Let us now act on Eq. (24) with $\delta^{(0)}$ and on Eq. (25) with $D^{(0)}$ and substract them. $\Longrightarrow$

$$-\left[\delta^{(0)}(\bar{\delta}+\pi-4\alpha)-D^{(0)}(\Delta+\mu-4\gamma)\right]\Psi_0+\left[\delta^{(0)}(D-4\rho-2\epsilon)-D^{(0)}(\delta-4\tau-2\beta)\right]\Psi_1$$
$$+3\left(\delta^{(0)}\kappa-D^{(0)}\sigma\right)\Psi_2=0 \qquad \Longrightarrow \textbf{up to second order} \tag{27}$$

$$\begin{aligned}
&-\left[\delta^{(0)}(\bar{\delta}+\pi-4\alpha)^{(0)}-D^{(0)}(\Delta+\mu-4\gamma)^{(0)}\right]\Psi_0^{(1)}\\
&-\left[\delta^{(0)}(\bar{\delta}+\pi-4\alpha)^{(1)}-D^{(0)}(\Delta+\mu-4\gamma)^{(1)}\right]\Psi_0^{(1)}\\
&-\left[\delta^{(0)}(\bar{\delta}+\pi-4\alpha)^{(0)}-D^{(0)}(\Delta+\mu-4\gamma)^{(0)}\right]\Psi_0^{(2)}\\
&+\left[\delta^{(0)}(D-4\rho-2\epsilon)^{(0)}-D^{(0)}(\delta-4\tau-2\beta)^{(0)}\right]\Psi_1^{(1)}\\
&+\left[\delta^{(0)}(D-4\rho-2\epsilon)^{(1)}-D^{(0)}(\delta-4\tau-2\beta)^{(1)}\right]\Psi_1^{(1)}\\
&-3D^{(0)}\left(\sigma^{(1)}\Psi_2^{(1)}\right)=0.
\end{aligned} \tag{28}$$

Note that

$$\left[\delta^{(0)}(D-4\rho-2\epsilon)^{(0)}-D^{(0)}(\delta-4\tau-2\beta)^{(0)}\right]\Psi_1^{(2)}$$
$$=\left(\delta^{(0)}D^{(0)}-D^{(0)}\delta^{(0)}\right)\Psi_1^{(2)}=0, \tag{29}$$

due to the commutation relation

$$[\delta, D]=(\bar{\alpha}+\beta-\bar{\pi})D+\kappa\Delta-(\bar{\rho}+\epsilon-\bar{\epsilon})\delta-\sigma\bar{\delta}, \tag{30}$$

so that

$$\left[\delta^{(0)}, D^{(0)}\right]=0. \tag{31}$$

Eq. (28) can explicitly be written by using Eqs. (21), (22) and (23) and the following explicit form of the differential operators δ, D and Δ

$$\delta^{(0)}=-\partial_z, \qquad \delta^{(1)}=-\frac{1}{2}\Phi_{uu}\partial_{\bar{z}}, \qquad D^{(0)}=a\partial_v, \qquad \Delta^{(0)}=\frac{1}{a}\left(\partial_u-\frac{1}{2}H\partial_v\right),$$
$$\Delta^{(1)}=-\frac{1}{2a}(\Phi_{\bar{z}\bar{z}}+\bar{\Phi}_{zz})\partial_v+\frac{1}{a}(\Phi_{v\bar{z}}\partial_{\bar{z}}+\bar{\Phi}_{vz}\partial_z). \tag{32}$$

Then it turns out from Eq.(28) that the second-order $\Psi_0^{(2)}$ satisfies the equation

$$\mathcal{T}_2 \Psi_0^{(2)} = S_2^{(2)} \tag{33}$$

where

$$\mathcal{T}_2 = -\delta^{(0)}(\bar{\delta} + \pi - 4\alpha)^{(0)} + D^{(0)}(\Delta + \mu - 4\gamma)^{(0)}, \tag{34}$$

and the source $S_{\Psi_0}^{(2)}$ is

$$\begin{aligned} S_{\Psi_0}^{(2)} =& \left[\delta^{(0)}(\bar{\delta} + \pi - 4\alpha)^{(1)} - D^{(0)}(\Delta + \mu - 4\gamma)^{(1)}\right]\Psi_0^{(1)} \\ &- \left[\delta^{(0)}(D - 4\rho - 2\epsilon)^{(1)} - D^{(0)}(\delta - 4\tau - 2\beta)^{(1)}\right]\Psi_1^{(1)} \\ &+ 3D^{(0)}\left(\sigma^{(1)}\Psi_2^{(1)}\right) \end{aligned} \tag{35}$$

We can now use Eqs. (21), (22) and (23) in (34) from where we find that

$$\mathcal{T}_2 \Psi_0^{(2)} = \frac{1}{2}\Box \Psi_0^{(2)}. \tag{36}$$

The source is:

$$
\begin{aligned}
S^{(2)}_{\Psi_0} =& a^2 \Bigg[-\frac{3}{2}\Phi^2_{vvv\bar{z}} + \Phi_{vvv}\Phi_{vvv\bar{z}\bar{z}} + \frac{3}{2}\Phi_{vv\bar{z}\bar{z}}\Phi_{vvvv} \\
& -2\Phi_{vv\bar{z}}\Phi_{vvvv\bar{z}} + \frac{1}{4}\Phi_{vv}\Phi_{vvvv\bar{z}\bar{z}} + \frac{1}{4}\bar{\Phi}_{vv}\Phi_{vvvvzz} \\
& +\Phi_{v\bar{z}\bar{z}}\Phi_{vvvvv} - \frac{1}{2}\Phi_{v\bar{z}}\Phi_{vvvvv\bar{z}} - \frac{1}{2}\bar{\Phi}_{vz}\Phi_{vvvvvz} \\
& +\frac{1}{4}\Phi_{\bar{z}\bar{z}}\Phi_{vvvvvv} + \frac{1}{4}\bar{\Phi}_{zz}\Phi_{vvvvvv} \Bigg].
\end{aligned}
\tag{37}
$$

When substituting the solution of the linear problem for Φ, the source in Eq. (37) becomes

$$
S^{(2)}_{\Psi_0} = -6(1+i)a^2\ell^7\omega^8\Phi^2 - \frac{3}{4}a^2\ell^7\omega^8\left[1 + 3\ell\omega^2(z-\bar{z})^2\right]\Phi\bar{\Phi}. \tag{38}
$$

Then, it is straightforward to verify that the solution to the equation

$$\frac{1}{2}\Box\Psi_0^{(2)} = S_{\Psi_0}^{(2)}, \tag{39}$$

is the sum of two functions, one oscillating and decaying, the other only decaying. The oscillating part satisfies

$$\frac{1}{2}\Box\Psi_{0\,\mathrm{osc}}^{(2)} = -6(1+i)a^2\ell^7\omega^8\Phi^2, \tag{40}$$

and its solution is

$$\Psi_{0\,\mathrm{osc}}^{(2)} = 2ia^2P_v^6\Phi^2. \tag{41}$$

Similarly, the the non-oscillating (decaying) part satisfies

$$\frac{1}{2}\Box\Psi_{0\,\mathrm{dec}}^{(2)} = -\frac{3}{4}a^2\ell^7\omega^8\left[1+3\ell\omega^2(z-\bar{z})^2\right], \tag{42}$$

and it is given by

$$\Psi_{0\,\mathrm{dec}}^{(2)} = -\frac{1}{4}a^2\ell^6\omega^6\Phi\bar{\Phi}. \tag{43}$$

4.Quadratic QNMs for the channel $(\ell \times \ell) \to 2\ell$

We are now in the position to calculate the non-linear QNMs of the gravitational waves on the photon ring. This is not yet the full answer as we will need to propagate it away from the photon ring at large distances. Using Eqs. (17) and (19) we find

$$\Psi_0 = \frac{1}{2} h_{vv} = -\frac{1}{2} P_v^2 h, \tag{44}$$

so that

$$h = \frac{2\Psi_0}{P_v^2}, \tag{45}$$

From Eqs. (41) and (22) we obtain

$$h^{(2)} = \frac{2\Psi^{(2)}_{0\,\mathrm{osc}}}{(2P_v)^2} = P_v^4 \Phi^2, \tag{46}$$

$$h^{(1)} = \frac{2\Psi^{(1)}_0}{P_v^2} = P_v^2 \Phi. \tag{47}$$

We finally obtain

$$\boxed{\frac{h^{(2)}}{\left(h^{(1)}\right)^2} = 1.} \tag{48}$$

Notice that we have calculated this ratio in the radiation gauge augmented with the traceless and transverse-free condition, which is the same gauge where the non-linearities are extracted numerically (*Loutrel et al, Ripley et al*). In this way, the issues about the non gauge-invariance of the Weyl scalars at second-order are avoided.

Summarizing: The Penrose limit of large multipoles the non-linear QNM ratio does not depend on the multipole itself for the process $(\ell \times \ell) \to 2\ell$. There is also symmetry argument to explain why $h^{(2)}/(h^{(1)})^2$ is independent from multipoles.

The last step to calculate the quadratic-to-linear amplitude ratio $\mathcal{R}_{\ell\times\ell}$ is to connect the result in the Penrose limit to the asymptotic solutions. This can be done by matching the Hertz potential Φ at the photon ring to large distances using the **WKB approximation**. The result is:

$$\mathcal{R}_{\ell\times\ell} = \frac{1}{2}\frac{1}{\sqrt{\gamma_\ell}} \cdot \frac{c_\ell^2}{c_{2\ell}} \frac{h_{2\ell}^{(2)}}{\left(h_\ell^{(1)}\right)^2} = \frac{1}{2}\frac{1}{\sqrt{\gamma_\ell}} \cdot \frac{c_\ell^2}{c_{2\ell}}, \tag{49}$$

where

$$c_\ell = {}_{-2}Y_{\ell\ell}\left(\frac{\pi}{2}, 0\right) = \frac{(-1)^\ell}{2^\ell}\sqrt{\frac{2\ell+1}{4\pi}\frac{(2\ell)!}{(\ell+2)!(\ell-2)!}} \tag{50}$$

and

$$\gamma_\ell = e^{-i\pi/4}\left(2Q_0''\right)^{1/4}, \tag{51}$$

with

$$Q_0'' = -\left(1 - \frac{r_s}{r_0}\right)\frac{\ell^2}{r_0^4}\left(6 - 20\frac{r_s}{r} + 15\frac{r_s^2}{r^2}\right), r_0 = 3r_s/2. \tag{52}$$

For large ℓ we have

$$c_\ell \approx \frac{e^{-2/\ell}\ell^{1/4}}{\sqrt{2}\pi^{3/4}}, \qquad \gamma_\ell \approx \frac{\sqrt{2}\ell^{1/2}}{3\sqrt{3}} e^{-i\pi/4}, \tag{53}$$

and therefore, $|\mathcal{R}_{\ell\times\ell}|$ asymptotes to

$$|\mathcal{R}_{\ell\times\ell}| \approx \frac{1}{4}\left(\frac{3}{\pi}\right)^{3/4} \approx 0.24. \tag{54}$$

This value should be compared to the value

$$|\mathcal{R}^{\text{num}}_{\ell\times\ell}| \approx 0.2 \tag{55}$$

seems to indivate the numerical results for $\ell = 10$.

Conclusions

- The Pensore limit is an interesting approximation to understand Schwarzschild BH perturbations.
- It allows to zoom-in onto the photon ring where the QNMs are generated.
- It can capture both QNM’s frequencies and amplitudes.
- It can be extended similarly to the Kerr BH.
- For Kerr BHs a question to be answered is the scaling of the non-linearities with the BH spin.