How well do we understand string spectra?

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Gravity, Strings & Supersymmetry Breaking Meeting

SCUOLA NORMALE SUPERIORE

Why (still) string theory?

- (closed) string spectrum:

  $\exists$ (single) massless spin–2 state, whose lowest–energy interactions *match* those of the graviton of General Relativity!

  $\Rightarrow$ gravity (after compactifications to $D = 4$)

- string scattering amplitudes:

  interaction vertices: "*zones*" (instead of points, like in field theory) $\Rightarrow$ amplitudes do not diverge @ high energies!

$\Rightarrow$ string theory: shares properties with what we expect quantum gravity to be

but how well do we understand its spectrum?

Generalities of the spectrum

- let's look at (a channel of) the Veneziano amplitude:

$$\mathcal{A}_4 \sim \frac{\Gamma\big(-\alpha(s)\big)\Gamma\big(-\alpha(t)\big)}{\Gamma\big(-\alpha(s)-\alpha(t)\big)}\,, \quad \alpha(s) = 1 + \alpha'\, s$$

$\Rightarrow$ *infinitely many simple poles* e.g. @ $s = -\frac{1}{\alpha'}, 0, \frac{1}{\alpha'}, \frac{2}{\alpha'}, \dots$

- it is actually a string amplitude $\Rightarrow$ theory with *infinitely many* (on–shell) *physical states*

$$M^2 = \text{integer} \times \frac{1}{\alpha'}$$

- they correspond to irreps of $\mathfrak{so}(D-1)$ or $\mathfrak{so}(D-2)$ $\Rightarrow$ TT

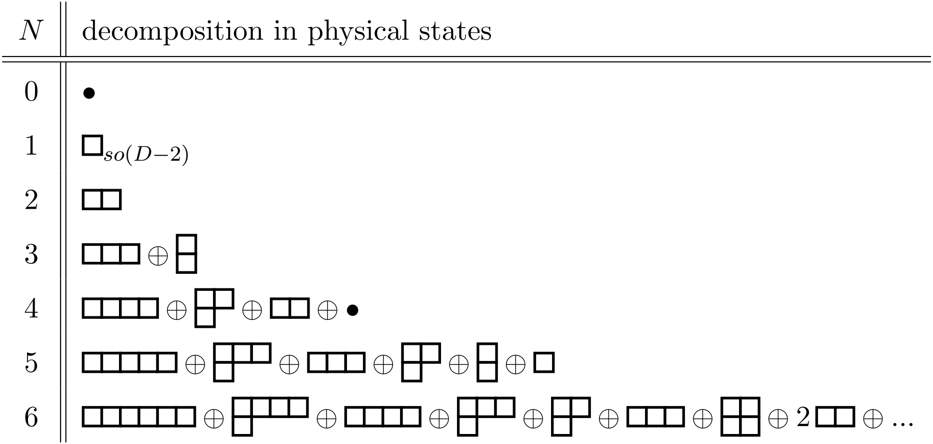
1–particle states à la Bargmann and Wigner ?

- among them $\exists$ ∞–many massive higher spins $\Rightarrow$ crucial to several properties of sting theory, e.g. UV softness

What does the spectrum look like?
Is there a bigger *symmetry*?

Let's start with the open bosonic string

(one box per spacetime index, symmetric rows, Young symmetry)



see e.g. Weinberg 1985, Mañes, Vozmediano 1989

How is this table constructed?

Traditional constructions

1. old covariant way: $|\text{phys}\rangle \sim F(\alpha^{\mu}_{-1}, \alpha^{\nu}_{-2}, \dots)\,|0;p\rangle$

   1. $(L_0 - 1)|\text{phys}\rangle = 0 \quad \Rightarrow \quad M^2 = (N-1)/\alpha'\,,$

      where $N := \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m \quad \Rightarrow \quad$ *choose "mass level" N*

   2. *write Ansatz* for F at chosen N and solve $L_1|\text{phys}\rangle = 0 = L_2|\text{phys}\rangle$

      Del Giudice, P. Di Vecchia 1970, Sasaki, Yamanaka 1985,...

2. light–cone gauge (level–by–level, by construction non–covariant)

   Goddard, Thorn 1972, Goddard, Goldstone, Rebbi, Thorn 1973

3. DDF (level–by–level, either ∃ reference momentum or need to solve constraints)

   Del Giudice, Di Vecchia, Fubini 1972, Brower 1972, . . . , Skliros, Hindmarsh '11
   highly excited strings: Gross, Rosenhaus '21, Pesando, Biswas '24,
   Bianchi, Firrotta, Sonnenschein, Weissman '22–'now, Firrotta, Kiritsis, Niacrhos '24

4. obtain $SO(D-1)$ irreps from partition function (level–by–level, yields states' characters)

   Curtright, Thorn 1986, Forcella, Hanany, J. Troost '10

A bit on interactions

- open string spectrum:
  - massless vector ⇒ Yang–Mills

    Neveu, Scherk 1971, ...
  - lightest massive spin–2 ⇒ ~~bimetric~~ vertices

    Lüst, CM, Mazloumi, Stieberger '21
  - leading Regge:

$$|\text{leading}\rangle = \varepsilon_{\mu_1 \dots \mu_s}\, \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_s}\, |0; p\rangle \quad (\text{TT due to } L_1, L_2)$$

⇒ higher–spin couplings (at 3– and 4–points)

Sagnotti, Taronna '10
see also Angelantonj, Sagnotti '02

- closed string spectrum:
  - massless spin–2 ⇒ Einstein–Hilbert

    Scherk, Schwarz 1974, Yoneya 1974, ...
  - certain lightest massive spin–2 ⇒ cubic bimetric vertices

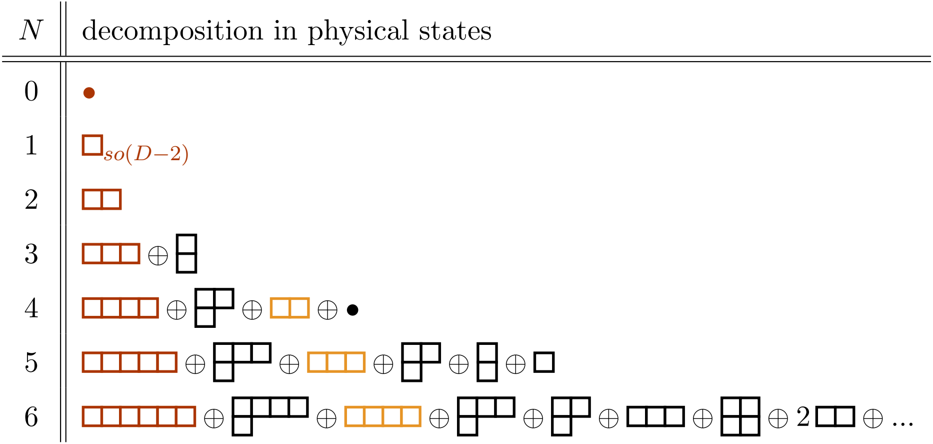
    Lüst, CM, Mazloumi, Stieberger '23

(bimetric theory ∼ interacting graviton and massive spin–2 in Minkowski)

de Rham, Gabadadze, Tolley '10, '11, Hassan, Rosen '11

All methods come with their own restrictions

Spectrum visualisation: Regge *trajectories*



beyond the leading Regge, the spectrum seems *repetitive*

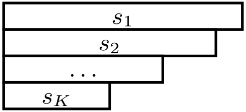
Main challenge: what do excited states look like?

Is there a certain *pattern*?

Is the spectrum concealing a *bigger organizing symmetry*?

What do states look like?

- *any* state has a polarization depicted by a Young diagram



$\Leftrightarrow \quad \varepsilon^{\mu(s_1),\,\lambda(s_2),\dots,\,\nu(s_K)}(p)$

  - convention: symmetric rows + additional relations
  - e.g. hook:  $\Leftrightarrow \ \varepsilon^{\mu_1\mu_2,\lambda}$ with $\varepsilon^{\mu_1\mu_2,\lambda} + \varepsilon^{\mu_2\lambda,\mu_1} + \varepsilon^{\lambda\mu_1,\mu_2} = 0$

- general string state:

  1. Young symmetry: $\varepsilon_{\dots,\mu_k(k),\dots,\mu_k\mu_\ell(\ell-1),\dots} = 0 \quad \Rightarrow \quad GL$ irreps
  2. tracelessness: $\eta^{\rho\sigma}\varepsilon_{\dots,\mu_k(k-1)\rho,\dots,\mu_\ell(\ell-1)\sigma,\dots} = 0 \quad \Rightarrow \quad SO$ irreps

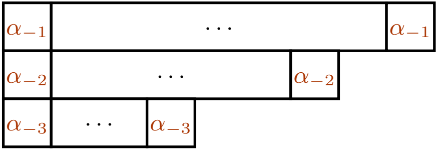
- render a given diagram physical: $\exists$ ∞–many physical polynomials

Where do states appear?

- How can we *minimize* the level? contract i–th row with α^{μ}_{-i}

see e.g. Weinberg 1985

e.g. 3–row diagrams:



e.g. hook: $F^{\boxplus}_{\text{simpl}} = \varepsilon_{\mu_1\mu_2,\,\nu}\,\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}\alpha^{\nu}_{-2}$ (TT) $\Rightarrow$ $N_{\text{min}} = 4$

$\Rightarrow$ *any* diagram appears at the respective N_{min} for the “first” time

- a given diagram also appears at higher levels $N = N_{\text{min}} + w$

$\Rightarrow$ let’s call w “depth”

- let’s call a trajectory the set of diagrams with a *fixed* number of rows at *fixed* w

CM, Skvortsov ’23

Re–organizing the spectrum

the spectrum consists of $w = 0$ trajectories and their ∞–many clones!

N	decomposition in physical states $\qquad w = {\color{brown}0}, {\color{violet}1}, {\color{orange}2}, {\color{purple}3}, {\color{teal}4}, \dots$
0	${\color{brown}\bullet}$
1	${\color{brown}\ydiagram{1}_{so(D-2)}}$
2	${\color{brown}\ydiagram{2}}$
3	${\color{brown}\ydiagram{3}} \oplus {\color{brown}\ydiagram{1,1}}$
4	${\color{brown}\ydiagram{4}} \oplus {\color{brown}\ydiagram{2,1}} \oplus {\color{orange}\ydiagram{2}} \oplus {\color{teal}\bullet}$
5	${\color{brown}\ydiagram{5}} \oplus {\color{brown}\ydiagram{3,1}} \oplus {\color{violet}\ydiagram{2,1}} \oplus {\color{orange}\ydiagram{3}} \oplus {\color{orange}\ydiagram{1,1}} \oplus {\color{teal}\ydiagram{1}}$
6	${\color{brown}\ydiagram{6}} \oplus {\color{brown}\ydiagram{4,1}} \oplus {\color{brown}\ydiagram{2,2}} \oplus {\color{violet}\ydiagram{3,1}} \oplus {\color{orange}\ydiagram{4}} \oplus {\color{orange}\ydiagram{2,1}} \oplus {\color{purple}\ydiagram{3}} \oplus 2\,{\color{teal}\ydiagram{2}} \oplus \dots$

$\Rightarrow$ polynomial *complexity*: measured by w (and number of rows)

CM, Skvortsov '23

All trajectories at once

- idea: let's consider the *most general* polynomial $F = F(\alpha^{\mu}_{-1}, \alpha^{\nu}_{-2}, \dots)$ and impose the Virasoro constraints:

$$\Rightarrow \quad 2L^{\perp}_{n>0}F = \left[\sum_{m=1}^{n-1} \alpha_{n-m}\cdot\alpha_m + 2\sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_{n+m}\right]F = 0$$

- let's define the operators $(k, \ell = 1, 2, \dots, N)$

$$T^{k}{}_{\ell} := \frac{1}{k}\,\alpha_{-k}\cdot\alpha_{\ell}\,, \quad T_{k\ell} := \alpha_k\cdot\alpha_\ell\,, \quad T^{k\ell} = \frac{1}{k\ell}\,\alpha_{-k}\cdot\alpha_{-\ell}$$

operator	$T^{k}{}_{\ell}$	$T_{k\ell}$	$T^{k\ell}$
energy units	$k-\ell$	$-k-\ell$	$k+\ell$

- can now rewrite the Virasoro constraints on a general F as:

$$L^{\perp}_{n>0}F = \left[\sum_{m=1}^{n-1} T^{\perp}_{m,n-m} + 2\sum_{m=1}^{\infty} mT^{m}{}_{n+m}\right]F = 0$$

CM, Skvortsov '23

A bigger organizing symmetry

- use $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu} \;\Rightarrow\; \{T^k{}_\ell\,, T_{k\ell}, T^{k\ell}\}$ generate $\mathfrak{sp}(2N)$!
- can choose lowering & raising operators
  - lowering: $T_{k\ell}, \quad T^{k<\ell}{}_\ell$
  - raising: $T^{k\ell}, \quad T^{k>\ell}{}_\ell$

$\Rightarrow$ lowest weight states: $T^k{}_k\,F = s_k\,F\,, \quad T_{k\ell}F = 0\,, \quad T^{k<\ell}{}_\ell\,F = 0$

- the $\mathfrak{sp}$ lowering operators check Young symmetry & tracelessness on $w = 0$ diagrams!

e.g. hook  at $w = 0$: $F_{\text{simpl}} = \varepsilon_{\mu_1\mu_2,\nu}\,\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}\alpha^{\nu}_{-2}$

- $T^1{}_2\,F_{\text{simpl}} = 0 \quad\Leftrightarrow\quad \varepsilon_{\mu_1\mu_2,\nu} + \text{cyclic} = 0$
- $T_{12}F_{\text{simpl}} = 0 \quad\Leftrightarrow\quad \eta^{\rho\sigma}\varepsilon_{\mu\rho,\sigma} = 0$ etc

$\Rightarrow$ the lowest weight states of $\mathfrak{sp}(2N)$ *are* the $w = 0$ trajectories!

CM, Skvortsov '23

Building the spectrum: a new technology

- $\mathfrak{sp}(2N)$ commutes with $\mathfrak{so}(D-1,1)$ $\Rightarrow$ employ *Howe duality* !

CM, Skvortsov '23

$$\text{irreps of} \quad \mathfrak{sp}(2N) \quad \overset{\text{“1–1”}}{\longleftrightarrow} \quad \text{irreps of} \quad \mathfrak{so}(D-1,1)$$

Howe 1989, ...

- implementation: use raising operators of $\mathfrak{sp}(2N)$ to reach $w > 0$

$$F^{f}_{w>0} = f_{w>0}(T^{mn}_{\perp}, T^{k>\ell}{}_{\ell}\,)\, F_{w=0} \qquad \text{and solve} \quad L_1, L_2$$

$\Rightarrow$ *can reach the* ∞*–many appearances of a given Young diagram from its first!*

- e.g. construct the first clone of the leading Regge

$$F^{f}_{w=2} = \left[\beta_1 T^{11}_{\perp} + \beta_2 T^{3}{}_{1} + \beta_3 (T^{2}{}_{1}\,)^2\right] F_{w=0}$$

$$\Rightarrow \quad \beta_2 = -\beta_1 \frac{D+2s-1}{s}\,, \quad \beta_3 = \beta_1 \frac{D+2s-1}{s(s-1)}$$

spin is not fixed $\Rightarrow$ full trajectory is known!

CM, Skvortsov '23

The superstring, super quickly

- add *fermionic* modes b_r^μ, with $\{b_m^\mu, b_n^\nu\} = \delta_{m+n}\,\eta^{\mu\nu}$

  $\Rightarrow$ favour columns, but how do we *glue rows to columns*?

1. Neveu–Schwarz sector: $\mathfrak{osp}(2M|2N)$

$w = 0$:

$b_{-\frac{1}{2}}$	α_{-1}	$\cdots$	α_{-1}		
$b_{-\frac{1}{2}}$	$b_{-\frac{3}{2}}$	α_{-2}	$\cdots$	α_{-2}	
$b_{-\frac{1}{2}}$	$b_{-\frac{3}{2}}$	$b_{-\frac{5}{2}}$	α_{-3}	$\cdots$	α_{-3}

2. Ramond sector: $\mathfrak{osp}(2M+1|2N)$ (K: diagonal length)

$w = 0 \;\Rightarrow\; 2^K$ configurations :

	α_{-1}	$\cdots$	α_{-1}		
b_{-1}		α_{-2}	$\cdots$	α_{-2}	
b_{-1}	b_{-2}		α_{-3}	$\cdots$	α_{-3}

ambiguity due to $Q^n{}_n := \frac{1}{n}\,\alpha_{-n}\cdot b_n \Rightarrow$ non–trivial multiplicity!

- construct $w > 0$: dress with raising operators of the $\mathfrak{osp}$, apply super–Virasoro constraints

Basile, CM '24

Conclusions & further directions

- we have developed a **new technology** that excavates entire string trajectories deeper in the (super)string spectrum
  - it is **covariant** and **efficient**
  - amplitudes for entire trajectories available, we have several examples

    CM, Skvortsov '23, Basile, CM, '24
- further developments:
  - technology for the *closed* string spectrum?

    Basile, CM *to appear*
  - technology around curved bkg? for $D = 4$?
  - can we probe deeper into black–holes?
  - can we decode chaos in the string S–matrix?

Is the *complete* spectrum within reach?

How well do we understand the underlying principles of string theory?

The key to decoding the deep mysteries surrounding its geometry may lie in its **massive spectrum**