Democratic formulations and manifest duality

Karapet Mkrtchyan

Imperial College London

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Motivation

Some problems to solve during our lifetime

- S-duality in gauge theory: Montonen-Olive duality and its extensions. Can we make it manifest? Key to quantization?
- Field-theoretical (classical) description of magnetic charges in the same footing as electric ones (local, Lorentz covariant?).
- Non-abelian interactions of (chiral) $p-$forms. In particular, the $6d$ two-forms (related to $M5$ branes and $(2,0)$ theory).
- Electric-magnetic duality in gravity. Key to quantization?

Duality in pure Maxwell theory in $d = 3 + 1$

$SO(2)$ duality symmetry of Maxwell equations in vacuum:

$$\overrightarrow{E} \to \cos\alpha\,\overrightarrow{E} + \sin\alpha\,\overrightarrow{B}\,,$$

$$\overrightarrow{B} \to -\sin\alpha\,\overrightarrow{E} + \cos\alpha\,\overrightarrow{B}\,.$$

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Conventional Lagrangian

The standard form of the Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)\,.$$

transforms under duality:

$$\vec{E}^2 - \vec{B}^2 \to \cos(2\alpha)\,(\vec{E}^2 - \vec{B}^2) + \sin(2\alpha)(\vec{E}\cdot\vec{B})$$

while the Hamiltonian $H = \vec{E}^2 + \vec{B}^2$ is invariant.

Non-linear electrodynamics (NED) in $d = 3+1$

Conventional Lagrangian for general (Plebanski) NED

$$\mathcal{L} = \mathcal{L}(s,p)\,, \quad s = F \wedge \star F\,, \quad p = F \wedge F\,,$$

Equations and duality transformations

$$\mathrm{d}F = 0\,, \qquad \mathrm{d}G = 0\,, \qquad G = \star\frac{\partial \mathcal{L}}{\partial F}\,,$$

Since now G is non-linearly related to F, the duality rotation:

$$F \to \cos\alpha\, F + \sin\alpha\, G\,,$$

$$G \to -\sin\alpha\, F + \cos\alpha\, G\,,$$

is not automatically a symmetry of the theory.

Duality symmetry in NED

Duality-symmetry in conventional NED

$SO(2)$ duality symmetry implies (Schrödinger '35, Gaillard and Zumino '80, Bialynicki-Birula '83, Gibbons and Rasheed '95):

$$F \wedge F = G \wedge G$$

that is satisfied for Lagrangians $\mathcal{L}(s,p)$ solving the equation:

$$\left(\frac{\partial \mathcal{L}}{\partial s}\right)^2 - \frac{2\,s}{p}\left(\frac{\partial \mathcal{L}}{\partial s}\right)\left(\frac{\partial \mathcal{L}}{\partial p}\right) - \left(\frac{\partial \mathcal{L}}{\partial p}\right)^2 = 1\,,$$

Explicit solutions: Maxwell, Born-Infeld, a few more by M. Hatsuda, K. Kamimura and S. Sekiya '99, M. Svazas '21 (master thesis), K.M. and M. Svazas '22, Russo and Townsend '24. General solution via an *implicit* function of one variable (Courant-Hilbert, 1924).

General democracy for free $p-$forms

Equations

Free $p-$form dynamics:

$$\star F = G\,, \quad \mathrm{d}F = 0\;(F = \mathrm{d}A)\,, \quad \mathrm{d}G = 0\;(G = \mathrm{d}B)\,,$$

The eq. $\mathrm{d}G = \mathrm{d}\star F = \mathrm{d}\star dA = 0$ follows from the Lagrangian:

$$\mathcal{L} = F \wedge \star F\,, \qquad F = \mathrm{d}A\,.$$

Does not work for self-dual fields $(G = F)$.

Democratic (twisted selfduality) equation:

$$\star F = G\,, \quad F = \mathrm{d}A\,, \quad G = \mathrm{d}B$$

Self-dual (Chiral) fields

Covariant equations of self-dual fields:

$$\star F = \pm F\,, \quad F = \mathrm{d}A$$

implying regular "Maxwell equations" $d \star F = 0$, but stronger.

Lagrangian?

Lagrangian formulation of the (free) chiral fields has a long history. Siegel '84, Kavalov-Mkrtchyan '87, Florianini-Jackiw '87, Henneaux-Teitelboim '88, Harada '90, Tseytlin '90, McClain-Yu-Wu '90, Wotzasek '91,..., Pasti-Sorokin-Tonin '95, Riccioni-Sagnotti '98, Kuzenko-Theisen '00,..., Sen '15,...

Pure Electrodynamics

Duality vs Lorentz symmetry?

"Conventional" Lagrangians manifest only one of the two.

"Democratic" equations for $3+1$ dimensional electrodynamics:

$$\star F^b = \epsilon^{bc} F^c \,, \quad F^b = \mathrm{d}A^b \,, \quad b, c = 1, 2 \,.$$

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Related problem

Lagrangian for self-duality equation?

$$\star F = F \,, \quad F = \mathrm{d}A \,,$$

self-dual $p-$forms in $d = 2p + 2$ dimensions.

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"...As this field is non-Lagrangian..." Witten '96

Conventional wisdom

ChatGPT

Final Conclusion:

A **self-dual rank-3 tensor in 6D has no nonzero independent scalar invariants**.

This is an important result in **6D superconformal field theories** and **string theory** because it means that the **dynamics of self-dual tensors cannot be captured by a standard Lagrangian formulation** (which requires a nonzero invariant action). Instead, they must be treated using **non-Lagrangian or constraint-based formalisms**.

Would you like to explore the consequences of this result, such as how 6D self-dual tensors appear in M-theory?

Main problem

Lagrangian for (twisted) self-duality equation.

A new democratic approach

Spoiler

The problematic case turns into the simplest in a new formulation.

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New action for self-dual $p-$forms $(d = 2p + 2)$

$$\mathcal{L} = \frac{1}{2}(F + a\,Q)^2 \pm a\,F \wedge Q\,,$$

where $F = dA$ and $Q = dR$.

Equations

On-shell a and R are pure-gauge and the e.o.m.'s can be gauged to:

$$\star F \pm F = 0\,.$$

K.M. *JHEP* '19

Democratic Electromagnetism

The Lagrangian for a single massless spin-one field in $d = 3 + 1$

$$\mathcal{L}_{Maxwell} = \frac{1}{2} H^b \wedge \star H^b \pm \epsilon^{bc}\, a\, F^b \wedge Q^c\,,$$

where $H^b \equiv F^b + a\, Q^b$, $b = 1, 2$, and

$$F^b = \mathrm{d}A^b\,, \quad Q^b = \mathrm{d}R^b\,.$$

On-shell, a, R^b are pure gauge, and the equations imply

$$\star F^a \pm \epsilon^{ab}\, F^b = 0\,,$$

with a single propagating Maxwell field.

K.M. *JHEP* '19

The general democratic non-linear electrodynamics

Democratic non-linear electrodynamics

$$\mathcal{L} = \mathcal{L}_{Maxwell} + g(\lambda_1, \lambda_2)\,,$$

where

$$\lambda_1 = G^1 \wedge G^1\,, \quad \lambda_2 = G^1 \wedge \star G^1\,, \quad G^b \equiv \star H^b - \epsilon^{bc}\, H^c$$

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Nonlinear electrodynamics with duality symmetry

Theories with $SO(2)$ duality symmetry will have:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)\,,$$

$$w = \sqrt{-\det \mathcal{H}} = \sqrt{\lambda_1^2 + \lambda_2^2}\,, \quad \mathcal{H}^{ab} \equiv G^a \wedge \star G^b\,.$$

General Lagrangian given via an **explicit** function of one variable.

Z. Avetisyan, O. Evnin, **K.M.**, *PRL* '21.

Example: ModMax

The conformal duality-symmetric electrodynamics

The conformal and duality-symmetric Electrodynamics:

$$\mathcal{L} = -\frac{1}{2}\, H^b \wedge \star H^b + a\, \epsilon_{bc} F^b \wedge Q^c + \delta\, w$$

can be translated to single-potential formulation

$$L(s,p) = -\cosh\gamma\, s + \sinh\gamma \sqrt{s^2 + p^2}$$

with a relation: $\delta = \coth\frac{\gamma}{2}$. This is the so-called ModMax theory (Bandos, Lechner, Sorokin, Townsend '20). For $\delta = 1$, the map breaks down, the single-field formulation diverges.

Extended duality

Free action with $sp(2n,R)$ symmetry

$$\mathcal{L}_{Vec} = \frac{1}{2} G_{MN}\, H^M \wedge \star H^N + \Omega_{MN}\, F^M \wedge Q^N\,,$$

G_{MN} is built from the scalars and Ω_{MN} is the invariant tensor of $sp(2n,R)$ (or $E_{7(7)} \subset Sp(56,R)$), and

$$H^M \equiv F^M + Q^M\,, \qquad F^M \equiv \mathrm{d}A^M\,, \qquad Q^M \equiv \mathrm{d}a \wedge R^M\,.$$

Equations of motion imply:

$$\star H^N - J^N{}_P\, H^P = 0\,, \qquad J^N{}_P = G^{NM}\Omega_{MP}\,.$$

and on-shell we can gauge fix: $H^N = F^N$

Interactions for extended duality case

Interacting Lagrangian with $\mathfrak{sp}(2n, R)$ symmetry

$$\mathcal{L} = \frac{1}{2} G_{MN}\, H^M \wedge \star H^N + \Omega_{MN}\, F^M \wedge Q^N + g(\mathcal{H}^N)\,,$$

where $\mathcal{H}^N = \star H^N + J^N{}_M H^M$. No quadratic invairants of $\mathcal{H}^N$, but there is a quartic one:

$$w = U^{NP}\, G_{PM}\, U^{MQ}\, G_{QN}\,, \quad U^{NP} = \mathcal{H}^N \wedge \star \mathcal{H}^P\,.$$

For the case of $\mathrm{E}_{7(7)}$, there are more invariants.

Can be used in $\mathcal{N} = 8$ SUGRA, $\mathcal{N} = 2$ SYM, etc.

Work in progress...

Lorentz covariant equations for interacting self-dual forms

A comment on nonlinear theory of chiral two-forms in 6d

"It appears that not only is there no manifestly Lorentz invariant action, but *even the field equation lacks manifest Lorentz invariance*."
Perry, Schwarz '96

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Equations of motion (Z. Avetisyan, O. Evnin, K.M. '22)

$$\star F + F = f(\star F - F)\,.$$

General equation for non-linear self-dual $p-$form in $d = 2p+2$ dimensions, with arbitrary $f : \Lambda^- \to \Lambda^+$. In particular,

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,,$$

with arbitrary scalar $g(Y)$ of $Y \in \Lambda^-$.

Abelian interactions for (chiral) p-forms

Free Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q\,, \qquad (F = dA\,,\ Q = dR)\,.$$

Self-interacting Lagrangian: general recipe

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q + g(H^-)\,,$$

$$H^- = \star(F + a\,Q) - (F + aQ)\,.$$

Equations of motion

In the on-shell gauge $Q = 0$, the equations of motion are:

$$\star F + F = f(\star F - F)\,,$$

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,.$$

Z. Avetisyan, O. Evnin, K.M. *JHEP* '22

Comparison to other formulations

Formalism:	PST	Sen's	Our way
Interactions by arbitrary functions	x	✓	✓
Auxiliary fields gauged away on-shell	✓	x	✓
Gauge potential as fundamental field	✓	x	✓

More details in:
Oleg Evnin and K.M., "Three approaches to chiral form interactions", *Differ. Geom. and Appl.* 89 (2023), 102016.

Derivation from Chern-Simons

Chern-Simons with a boundary term

The action:

$$S_{\text{free}} = \int_M H \wedge \mathrm{d}H - \frac{1}{2}\int_{\partial M} H \wedge \star H$$

Full variation:

$$\delta S_{\text{free}} = 2\int_M \delta H \wedge \mathrm{d}H - \frac{1}{2}\int_{\partial M} \delta H^+ \wedge H^- \,.$$

$$H^{\pm} = H \pm \star H$$

Reduction

Main idea

Decompose the field as ($v = \mathrm{d}a$ satisfies $v^2 \neq 0$ on the boundary):

$$H = \hat{H} + v \wedge \check{H}\,, \qquad \mathrm{d}v = 0\,.$$

Then the field $\check{H}$ is a Lagrange multiplier enforcing a constraint

$$v \wedge \mathrm{d}H = 0\,,$$

with a solution

$$H = \mathrm{d}A + v \wedge R$$

Plugging this back into the action gives our chiral Lagrangian.

Arvanitakis, Cole, Hulik, Sevrin, and Thompson, PRD '23. (From now on ACHST reduction.)

Extensions to interacting chiral fields

General equations

General equations describing self-interactions of a chiral field are given as

$$H^- = f(H^+)\,, \qquad \mathrm{d}H = 0\,,$$

where $f : \Lambda^+ \to \Lambda^-$ is an antiselfdual form valued function of a selfdual variable.

Action

$$S = \int_M H \wedge \mathrm{d}H - \int_{\partial M} \tfrac{1}{2}\, H \wedge \star H + g(H^+)\,,$$

where $f(Y) = \partial g(Y)/\partial Y$.

Democratic $p-$forms

Generalization to democratic case

Action:

$$S = \int_M (-1)^{d-p}\, G \wedge \mathrm{d}F + \mathrm{d}G \wedge F$$
$$- \int_{\partial M} \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) + g(F + \star G)\,,$$

with bulk equations $\mathrm{d}F = 0 = \mathrm{d}G$ and boundary equations:

$$F - \star G = f(F + \star G)\,,$$

where $f(Y) = \partial g(Y)/\partial Y$ for a $(p+1)-$form argument Y. The ACHST reduction leads to democratic Lagrangians for the most general self-interactions of abelian p-forms.

More

More details in: Evnin, Joung, K.M., PRD '24

3-form of 11d SUGRA

Bulk action

$$S = \int_M G \wedge \mathrm{d}F + \mathrm{d}G \wedge F + \frac{2}{3}\,\lambda_3\, F \wedge F \wedge F$$
$$- \int_{\partial M} \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) - g(\star G + F)\,.$$

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Boundary Lagrangian

After reduction we get:

$$\mathcal{L} = v \wedge S \wedge \mathrm{d}A - \mathrm{d}B \wedge v \wedge R - \frac{\lambda_3}{3} A \wedge \mathrm{d}A \wedge \mathrm{d}A$$
$$- \frac{1}{2}\,(F \wedge \star F + G \wedge \star G) - g(\star G + F)\,,$$

where

$$F = \mathrm{d}A + v \wedge R\,, \qquad G = \mathrm{d}B + v \wedge S - \lambda_3\, A \wedge \mathrm{d}A\,.$$

Gravity and higher-spins

Linearized gravity and higher spins

No **manifestly** covariant democratic formulation so far, however, Henneaux, Teitelboim '04, Henneaux, Hörtner, Leonard '16

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Hamiltonian form for gravity and "higher spins" in $d = 1 + 1$

The linearized spin-s action in the Henneaux-Teitelboim form:

$$S_L \sim \int_{\partial M} \mathrm{d}^2 x \, \mathcal{D}_{\mathcal{L}}^{(2s-1)} \phi^{(s)} \, \partial_- \phi^{(s)}$$

where

$$\mathcal{D}_{\mathcal{L}}^{(2s-1)} = \partial_1 \prod_{n=1}^{s-1} (\partial_1^2 - 4 \, \mathcal{L} \, n^2) \, .$$

Includes Florianini-Jackiw ($s = 1$) and Alekseev-Shatashvili ($s = 2$).

C. Y. R. Chen, E. Joung, K. M. and J. Yoon, [arXiv:2501.16463].

Covariant form for Gravity and higher-spins

Covariant action in $d = 1 + 1$

We get a covariant action for chiral gauged WZW theory:

$$S_{\rm L} = \int_{\partial M} {\rm tr}\left[-\tfrac{1}{2}\,(g^{-1}{\rm d}g + v\,\rho) \wedge \star(g^{-1}{\rm d}g + v\,\rho)\right]$$

$$+{\rm tr}\left[v \wedge (\rho + \lambda)\, g^{-1}{\rm d}g\right] + S_{\rm WZW}[g]\,,$$

For $\mathfrak{sl}(N, R)$: the left-moving boundary modes of the (higher-spin) gravity in $3d$, with spins $s = 2, 3, \ldots, N$ (one copy each).
For $\mathfrak{su}(N, N)$: left-moving boundary modes of coloured gravity in $3d$: N^2 copies of spin-two and $N^2 - 1$ copies of spin-one.

Gauged version of the covariant non-abelian chiral WZW (ACHST)

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Gauged version of the covariant non-abelian chiral WZW (ACHST)

Duality-symmetric (linearized) gravity and higher spins in $d > 2$

Work in progress. . .

Outlook

A list of related (un)solvable problems

- (1,0) Democratic formulation for non-abelian gauge theory.
- (2,0) Non-abelian interactions for (chiral) $p-$forms.
- (4,0) Democratic GR.
- ... (higher spins on my mind).

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Grazie!

Example: Generalized Born-Infeld theory

Democratic formulation

The democratic Lagrangian is

$$\mathcal{L} = \mathcal{L}_{Maxwell} + h(w)\,,$$

where $h(w)$ for GBI is implicitly given by:

$$h(\lambda) = 4\,T\,\sinh^2\frac{\lambda}{2}\,\cosh(\lambda+\gamma)\,,$$

$$w(\lambda) = -4\,T\,\cosh^2\frac{\lambda}{2}\,\sinh(\lambda+\gamma)\,.$$

Democratic formulation for $p-$form in d dimensions

Lagrangian

$$\mathcal{L} = \frac{1}{2}(F + a\,P)^2 + \frac{1}{2}(G + a\,Q)^2 - \,a\,Q \wedge F + \,a\,G \wedge P$$

where $F = dA\,,\; G = dB\,,\; P = dS\,,\; Q = dR$. The fields A and S are $p-$forms, while B and R are $(d-p-2)-$forms.

This is a democratic formulation for a $p-$form field (together with dual $(d-p-2)-$form field) in d dimensions. The equations imply that S, R are pure gauge (as is the field a), and the only physical d.o.f. are in A, B, satisfying the duality relation:

$$\star\, dA + dB = 0\,.$$

New action for Chiral fields: more details

Lagrangian

$$\mathcal{L} = \frac{1}{2}(F + a\,Q)^2 + \,a\,F \wedge Q\,,$$

where $F = dA$ and $Q = dR$.

Symmetries

$$\begin{aligned} \delta A = dU\,; \qquad\qquad &\delta R = dV\,; \\ \delta A = -\,a\,da \wedge W\,, \quad &\delta R = da \wedge W\,; \\ \delta A = -\,&\frac{a\,\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,, \\ \delta a = \varphi\,, \quad \delta R = &\frac{\varphi}{(\partial a)^2}\,\iota_{da}(Q + \star Q)\,. \end{aligned}$$

Equations and consequences

Equations

$$E_a \equiv \frac{\delta \mathcal{L}}{\delta a} \equiv (F + a\,Q) \wedge \star Q + F \wedge Q = 0\,,$$
$$E_A \equiv \frac{\delta \mathcal{L}}{\delta A} \equiv d\,[\star(F + a\,Q)] + da \wedge Q = 0\,,$$
$$E_R \equiv \frac{\delta \mathcal{L}}{\delta R} \equiv d\,[a \star (F + a\,Q)] - da \wedge F = 0\,.$$

Relations

$$E_R - a\,E_A = da \wedge [F + a\,Q - \star(F + a\,Q)] = 0$$

From here (for $(da)^2 \neq 0$):

$$F + a\,Q - \star(F + a\,Q) = 0$$

and $E_a \equiv [F{+}a\,Q{-}\star(F{+}a\,Q)]{\wedge}Q = 0$ follows from $E_A = 0 = E_R$.

The spectrum

Consequences of e.o.m.

From the equations of motion it follows that:

$$da \wedge dR = 0$$

which can be solved generally as:

$$R = d\lambda + da \wedge \rho$$

This implies that R is pure gauge. In the $R = 0$ gauge, we get:

$$\star F = F$$

Thus the propagating d.o.f. consist of a single self-dual $p-$form.

Duality-symmetric Electromagnetism

The Lagrangian for a single massless spin-one field in $d=4$

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} H^b_{\mu\nu} H^{b\mu\nu} + \frac{1}{4}\, \epsilon_{bc}\, \varepsilon^{\mu\nu\lambda\rho}\, a\, F^b_{\mu\nu}\, Q^c_{\lambda\rho}$$

where $H^b_{\mu\nu} \equiv F^b_{\mu\nu} + a\, Q^b_{\mu\nu}$, $b = 1, 2$, and

$$F^b_{\mu\nu} = \partial_\mu\, A^b_\nu - \partial_\nu\, A^b_\mu\,, \quad Q^b_{\mu\nu} = \partial_\mu\, R^b_\nu - \partial_\nu\, R^b_\mu\,.$$

This Lagrangian describes a single Maxwell field, using 4 vectors and 1 scalar. Any solution of the e.o.m. is gauge equivalent to that of

$$R^b_\mu = 0, \qquad \star F^a_{\mu\nu} + \epsilon^{ab}\, F^b_{\mu\nu} = 0\,,$$

with a single propagating Maxwell field.

Non-linear electrodynamics: Ansatz

Ansatz for the consistent non-linear Lagrangian

$$\mathcal{L} = a\,\epsilon_{bc} F^b \wedge Q^c + f(U^{ab}, V^{ab})$$

where

$$U^{ab} \equiv \frac{1}{2}\, H^a_{\mu\nu}\, H^{b\mu\nu}\,, \quad V^{ab} \equiv \frac{1}{2}\, H^a_{\mu\nu} \star H^{b\mu\nu}$$

All symmetries are built in, except for the shift of a. The latter will fix the form of $f(U, V)$.

Equations of motion

$$E_{A^b} \equiv d[(f^U_{bc} + f^U_{cb}) \star H^c - (f^V_{bc} + f^V_{cb})\, H^c + a\,\epsilon_{bc}\, Q^c] = 0\,,$$

$$E_{R^b} \equiv d[a\{(f^U_{bc} + f^U_{cb}) \star H^c - (f^V_{bc} + f^V_{cb})\, H^c - \epsilon_{bc}\, F^c\}] = 0\,.$$

Imposing the missing symmetry

Shift symmetry $\delta a = \varphi$

Equations of motion for a:

$$E_a \equiv Q^b \wedge K_b = 0\,,$$

where

$$K_a \equiv (f^U_{ab} + f^U_{ba}) \star H^b - (f^V_{ab} + f^V_{ba})\, H^b - \epsilon_{ab}\, H^b\,,$$

and $f^U_{ab} \equiv \partial f / \partial U_{ab}$, $f^V_{ab} \equiv \partial f / \partial V_{ab}$ ($f^U_{21} \equiv 0 \equiv f^V_{21}$).

Note, that $E_{R^b} - a\, E_{A^b} = da \wedge K_b = 0\,$, which implies $K_b = 0$ iff

$$K_a \pm \epsilon_{ab} \star K_b \equiv 0$$

Then, the $E_a = 0$ is redundant, implying the shift symmetry for a.

The general democratic non-linear electrodynamics

Solution

The equation $K_a \pm \epsilon_{ab} \star K_b \equiv 0$ implies

$$\pm\, \delta^{ac}\, (f^U_{cb} + f^U_{bc}) - \epsilon^{ac}\, (f^V_{cb} + f^V_{bc}) + \delta^a_b = 0$$

The general solution gives the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{Maxwell} + g(\lambda_1, \lambda_2)\,,$$

where

$$\lambda_1 = \tfrac{1}{2}\, G_{\mu\nu} \star G^{\mu\nu}\,, \quad \lambda_2 = -\tfrac{1}{2}\, G_{\mu\nu}\, G^{\mu\nu}\,, \quad G_{\mu\nu} \equiv \star H^1_{\mu\nu} - H^2_{\mu\nu}$$

Reminder: non-linear electrodynamics in the conventional language

$$S = \int \mathcal{L}(s,p)\, d^4x, \quad s \equiv \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad p \equiv \frac{1}{2} F_{\mu\nu} \star F^{\mu\nu}$$

Nonlinear theories of chiral $p-$forms

Free Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q\,, \qquad (F = dA\,,\; Q = dR)\,.$$

Self-interacting Lagrangian: general recipe

$$\mathcal{L} = \tfrac{1}{2}(F + a\,Q)^2 + a\,F \wedge Q + g(H^-)\,,$$

$$H^- = F + a\,Q - \star(F + aQ)\,.$$

Equations of motion

In the on-shell gauge $Q = 0$, the equations of motion are:

$$F + \star F = f(F - \star F)\,,$$

$$f(Y) = \frac{\partial g(Y)}{\partial Y}\,.$$

Type II Supergravities: preliminaries

Some notations

Reflection operator: $\star\alpha = (-1)^{\lfloor \frac{\deg \alpha}{2} \rfloor + \deg \alpha} * \alpha \,,$

Mukai pairing: $(\alpha, \beta) := (-1)^{\lfloor \frac{\deg \alpha}{2} \rfloor} (\alpha \wedge \beta)^{top},$

Differential: $D\alpha = d\alpha + H \wedge \alpha \,.$

Properties

$$(\alpha, \star\beta) = (\beta, \star\alpha)\,, \qquad \star^2 = 1\,, \qquad D^2 = 0\,,$$

$$\int_M (\alpha, D\beta) = -\int_M (D\alpha, \beta) \qquad \text{(up to boundary terms)}\,,$$

$$D(f\alpha) = f D\alpha + df \wedge \alpha\,, \qquad \text{(for any function } f\text{)}.$$

K.M., F. Valach '22

Type II Supergravities

Action for Type II SUGRAS

$$S = S_{NS} + S_{RR}$$

where

$$S_{NS} = \frac{1}{2\kappa^2}\int\left[\sqrt{-g}\,e^{-2\varphi}\,\left(\mathcal{R} + 4(d\varphi)^2 - \frac{1}{12}H^2\right)\right]\,,$$

$$S_{RR} = \pm\frac{1}{8\kappa^2}\int\left[\frac{1}{2}(F+aQ,\star(F+aQ)) + (F,aQ)\right]\,,$$

$$F = DA\,,\quad Q = DR\,.$$

Upper/lower sign corresponds to IIA/IIB.

Field content

$$F = F_2 + F_4 + F_6 + F_8 + F_{10}\,, \qquad \text{(IIA case)}$$

$$F = F_1 + F_3 + F_5 + F_7 + F_9\,. \qquad \text{(IIB case)}$$

Democratic type II SUGRA

On-shell reduction

On-shell one can gauge fix $Q = 0$,

$$DF = 0\,, \qquad \star F = F\,.$$

reproducing democratic equations for RR forms.

Manifest $SL(2,R)$-symmetric type IIB SUGRA

Action

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left\{\mathcal{R} - 2[(d\phi)^2 + e^{2\phi}(d\ell)^2] - \frac{1}{3}e^{-\phi}H^2 - \frac{1}{3}e^{\phi}(H' - \ell H)^2\right\} + S_{SD}\,,$$

where

$$S_{SD} = \frac{1}{16\kappa^2}\int \left[(F + aQ)\wedge *(F + aQ) + 2\,F\wedge aQ - 2\,(1 + *)(F + aQ)\wedge X + X\wedge *X\right]\,,$$

and

$$X = \frac{1}{2}(B\wedge H' - B'\wedge H)$$

Dual theories (free p-forms)

A $p-$form and its dual

The Lagrangian is given in the form of ("Maxwell Lagrangian")

$$\mathcal{L} \sim F \wedge \star F\,, \quad F = dA\,.$$

Massless $p-$form and a $(d-2-p)$-form fields describe correspondingly particles of $p-$form and a $(d-2-p)$-form representations of the massless little group $ISO(d-2)$, dual to each other.

Attention!

Dual formulations do not admit the same interacting deformations!