Topics in Geometric Analysis 2025
Palazzo del Castelletto, Pisa

Introducing Various Notions of Distances between Space-Times

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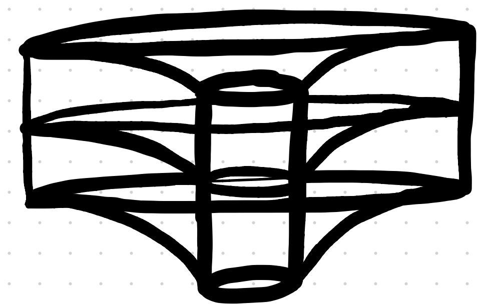
CUNYGC and Lehman College
(with Anna Sakovich)

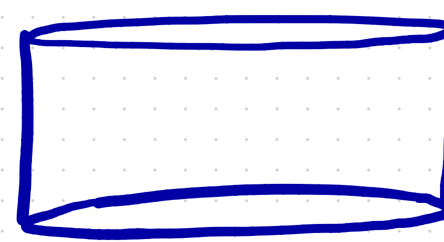
Scroll through, write in black

When we model
the Big Bang using
an FLRW Spacetime,

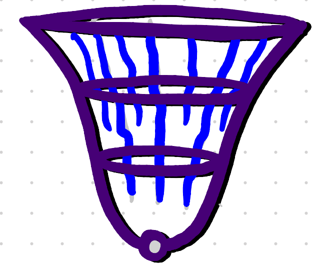
or the neighborhood of
a Black Hole using
Schwarzschild Spacetime



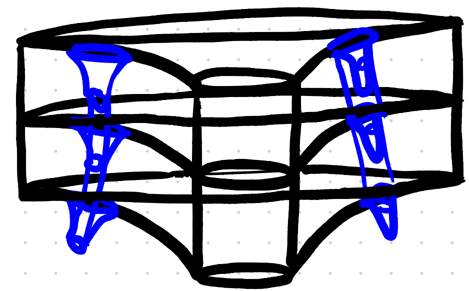
or an isolated empty region
with Minkowski Space



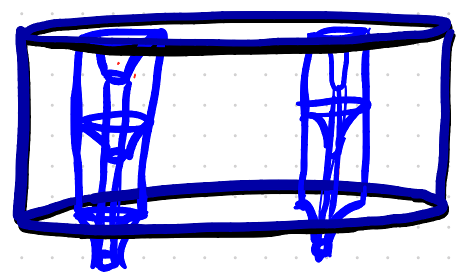
We have ignored all
the black holes, dust,
and gravity wells
within the universe



And ignored the stars
near the black hole

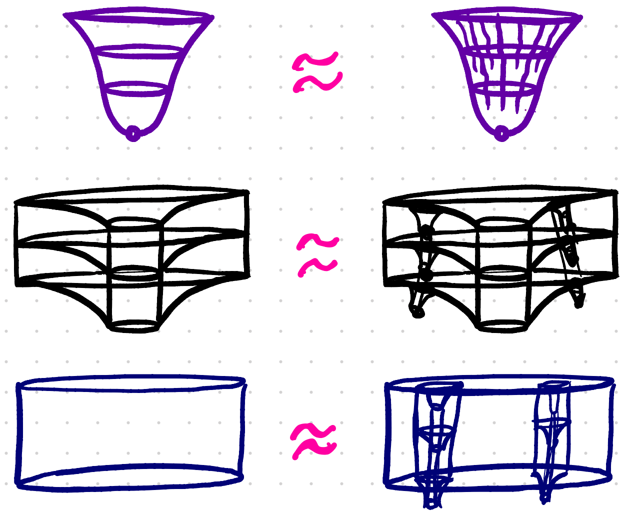


And ignored the possible
black holes and planets of
small mass in the "empty" region



The models "approximate" reality:

but in what sense?



They are not smooth approximations.

With black holes, they are not even diffeomorphic to each other.

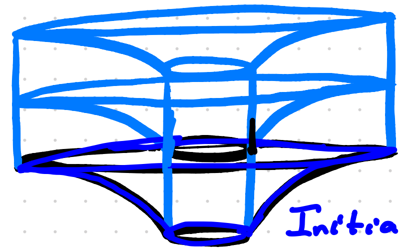
Challenge: define a distance between Space-Times that are not diffeomorphic.

Choquet-Bruhat and Geroch: Solving Einstein's Equations

Given initial data (M, h, K)

$\longrightarrow$ $\exists$ unique maximal

future development (N^+, g)



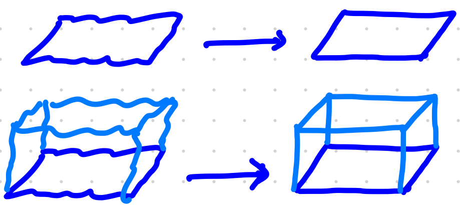
Example: Riemannian Schwarzschild $(M, g, 0)$

$\longrightarrow$ future Schwarzschild spacetime (N^+_{Sch}, g)

Stability Theorems exist using methods from Analysis:

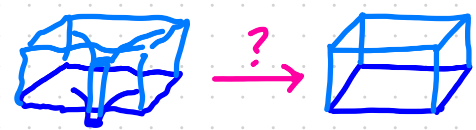
Given initial data $(M, h_j, K_j) \longrightarrow (M, h_0, K_0)$

then future developments $(N^+, g_j) \rightarrow (N, g_0)$



Challenge: Prove Stability Theorems when

the manifolds are <u>not</u> <u>diffeomorphic</u>.



Intrinsic Distances between Riemannian Manifolds which are not diffeomorphic:

- Gromov-Hausdorff (GH) distance
- Metric Measure (mm) distance
- Intrinsic Flat ($\mathcal{F}$) distance

All convert (M, g) into a metric space (M, d_g) using the Riemannian metric tensor, g.

All are definite: $d_{GH}(M_1, M_2) = 0 \iff M_1$ isometric to M_2

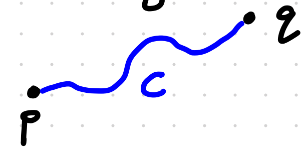
Riemannian Isometry $\iff$ Distance Preserving Bijection

Challenge: Convert Spacetimes into Metric Spaces.

Riemannian Manifolds (M_1, g_1) and (M_2, g_2)

viewed as metric spaces (M_1, d_{g_1}) and (M_2, d_{g_2})

$$d_g(p,q) = \inf \{ L_g(c) \mid c(0)=p \text{ and } c(1)=q \}$$



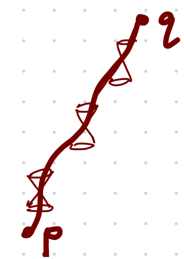
g is positive definite so $L_g(c) > 0$

$F: M_1 \to M_2$ is Distance Preserving if $d_{g_1}(p,q) = d_{g_2}(F(p), F(q))$ $\forall p,q \in M_1$

Dist. Pres. bijections are Riemannian Isometries: $F_* g_1 = g_2$

Space-times have Lorentzian metric tensors:

$g(v,v) \leq 0$ when v is "causal", g $-+++$,



$J_+(p) =$ causal future of $p = \{ q \mid \exists$ future directed causal curve c from $c(0)=p$ to $c(1)=q \}$

Lorentz Distance between points such that $q \in J_+(p)$

$$d_g(p,q) = \sup \{ L_g(c) \mid c(0)=p \text{ and } c(1)=q \text{ and } c \text{ causal} \}$$

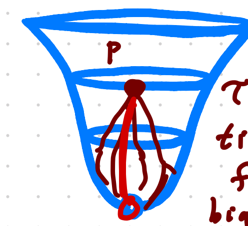
Lorentzian Isometries $F_* g_1 = g_2$ preserve Causal Structure J_+ and Lorentz Distance d_g

How to Convert a Space-time into a metric space?

We will use the Null Distance $\hat{d}_\tau$ of Sormani-Vega

Cosmological time, τ, of Andersson-Galloway-Howard

$$\tau_g(p) = \sup\{ d_g(p,q) \mid q \in J_-(p)\} \in [0,\infty]$$

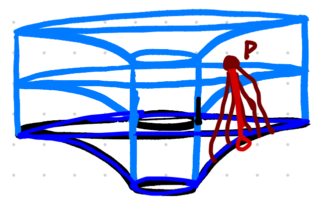


$\tau(p)$ time from big bang is regular

Regular if $\tau_g(p)$ is finite and

$\lim_{s\to 0} \tau_g(c(s)) = 0$ for past inextensible curves.

Future Development Space-time (N,g) has regular τ_g



Regular $\Rightarrow$

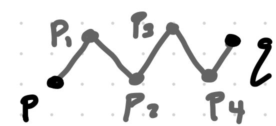
$\forall p \in M \;\; \exists\, \gamma_p : (0, \tau(p)] \to M$ s.t.

$\tau(\gamma_p(t)) = t$ and $\gamma_p(\tau(p)) = p$

$(N, \hat{d}_\tau)$ is a metric space when τ is regular

$$\hat{d}_\tau(p,q) = \inf \sum_{i=1}^{n} |\tau(p_i) - \tau(p_{i-1})|$$

where inf is over $p = p_0,\ q = p_N,\ p_i \in J^+(p_{i-1}) \cap J^+(p_{i+1})$

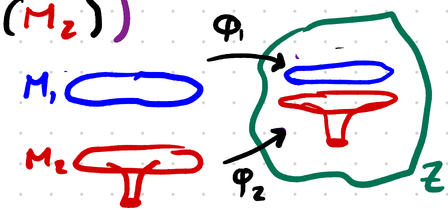


Once we have metric spaces (M_j, d_j), we have intrinsic distances like d_{GH}, d_{mm}, and $d_{\mathcal{F}}$:

$$d_{GH}(M_1, M_2) = \inf d_H^Z(\varphi_1(M_1), \varphi_2(M_2))$$

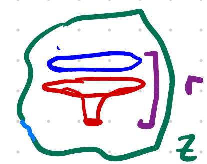
where the inf is over compact Z

and distance preserving $\varphi_i : M_i \to Z$



Recall the Hausdorff Distance is

$$d_H^Z(U_1, U_2) = \inf\{r \mid U_1 \subset T_r(U_2)\ \ U_2 \subset T_r(U_1)\}$$



This depends on the extrinsic Z but d_{GH} is intrinsic!

These intrinsic distances are definite:

$$d_{GH}(M_1, M_2) = 0 \iff \text{dist. pres. bijection } F: M_1 \to M_2$$

when (M_j, d_{g_j}) are Riemannian this holds $\iff F_* g_1 = g_2$

Challenge: dist pres bijections for Space-Times do <u>not</u> give $F_* g_1 = g_2$

Examples of Space-times exist with distance preserving bijections

$F:(N_1,\hat{d}_{g_1}) \to (N_2,\hat{d}_{g_2})$ that are not Lorentzian Isometries: $F_* g_1 \neq g_2$

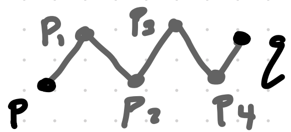
Lorentzian Isometries preserve the Causal Structure:

$$F(p)\in J_+(F(q)) \iff p\in J_+(q)$$

Recall the

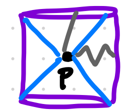
Null Distance $\hat{d}_g(p,q) = \inf \sum_{i=1}^{N} |\tau(p_i)-\tau(p_{i-1})|$

where inf is over $p=p_0,\ q=p_N,\ p_i\in J^+(p_{i-1})\cap J^+(p_{i+1})$



Defn: The null distance encodes causality:

$$p\in J_+(q) \iff \hat{d}_\tau(p,q) = \tau(p)-\tau(q)$$



[Sakovich-Sor]: τ_j is regular and proper on N_j $\Longrightarrow$ $\hat{d}_{\tau_j}$ encodes causality.

[Sak-S]: If $F: N_1\to N_2$ preserves time: $\tau_2(F(p)) = \tau_1(p)\ \forall p\in N_1$

and preserves null distance: $\hat{d}_2(F(p_1),F(p_2)) = \hat{d}_1(p_1,p_2)\ \forall p_1,p_2\in N_1$

then F is a Lorentzian Isometry $F_* g_1 = g_2$

also more general theorems of Burtscher-Garcia Heveling and Galloway.

New paper with Sakovich: We define various notions of

definite intrinsic distances between

"Causally Null Compactifiable Space-Times" (N_j, g_j), which

have associated compact metric spacetimes $(\bar{N}_j, \hat{d}_{g_j}, \tau_j)$

defined using bounded regular cosmological times, τ_j, whose

null distances encode causality: $p \in J_+(q) \iff \hat{d}_\tau(p,q) = \tau(p) - \tau(q)$

"Intrinsic Distance": $d_{S\text{-}dist}((N_1,g_1),(N_2,g_2)) = d_{dist}((\bar{N}_1, \hat{d}_{g_1}, \tau_{g_1}), (\bar{N}_2, \hat{d}_{g_2}, \tau_{g_2}))$

"Definite" means: $d_{S\text{-}dist}((\bar{N}_1,g_1),(\bar{N}_2,g_2)) = 0$ if and only if

$\exists\, F: \bar{N}_1 \to \bar{N}$ which preserves time: $\tau_2(F(p)) = \tau_1(p)\ \forall p \in N_1$

and preserves null distance: $\hat{d}_{g_2}(F(p_1), F(p_2)) = \hat{d}_{g_1}(p_1, p_2)\ \forall p_1, p_2 \in N_1$

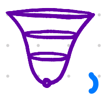
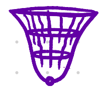
$\iff$ F is a Lorentzian Isometry $F_* g_1 = g_2$ (by encoding causality)

Timeless Intrinsic distances: (Sormani-Vega, Allen-Burtscher, Kunzinger, Saemann)

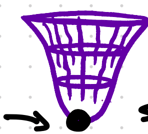
$d^{tls}_{S\text{-}GH}((N_1,g_1),(N_2,g_2)) = d_{GH}((N_1,\hat{d}_{g_1}),(N_2,\hat{d}_{g_2}))$ are not definite!

we must keep track of time and encode causality.

Big Bang Space-Time (announced with Vega, Sakovich-Sor) is definite! when τ is proper

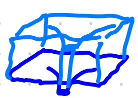
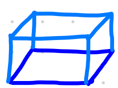
$d^{BB}_{S\text{-}GH}($  ,  $) = d^{pt}_{GH}((\bar{N}_1,\hat{d}_{g_1},p_1^{BB}),(\bar{N}_2,\hat{d}_{g_2},p_2^{BB}))$

↑ pointed GH distance [Grom

where p_j^{BB} is the 

big bang point → such that $\tau_{g_j}(q) = \hat{d}_{g_j}(q,p_j^{BB})$

Future Developed Space-Time is definite when τ is proper

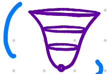
$d^{FD}_{S\text{-}GH}($  ,  $) = d^{FD}_{GH}((\bar{N}_1, M_1, \hat{d}_{g_1}),(\bar{N}_2, M_2, \hat{d}_{g_2}))$

where $M_j \subset \bar{N}_j$  GH of metric pairs [Che, Galaz-Garcia, Guijarro, Membrillo]

is the intial data set → such that $\tau_{g_j}(q) = \inf\{\hat{d}_{g_j}(q,p) \mid p \in M_j\}$

Big Bang and Future Developed Spacetimes with proper, τ, are Causally Null Compactifiable Space-Times

They have cosmic strips $\tau^{-1}(\tau_{min}, \tau_{max})$ which convert into compact metric spaces $(\bar{N}, \hat{d}_\tau)$ that encode causality $p \in J_+(q) \iff \hat{d}_\tau(p,q) = \tau(p) - \tau(q)$

$d^{BB}_{S\text{-}GH}$ Definite Proof: $d^{BB}_{S\text{-}GH}($  ,  $) = d^{pt}_{GH}\left((\bar{N}_1, \hat{d}_{g_1}, p_1^{BB}), (\bar{N}_2, \hat{d}_{g_2}, p_2^{BB})\right) = 0$ iff

$\exists F: \bar{N}_1 \to \bar{N}_2$ that preserves distances and $F(p_1^{BB}) = p_2^{BB}$

so F also preserves time $\tau_{g_j}(q) = \hat{d}_{g_j}(q, p_j^{BB}) \Rightarrow$ Encodes causality $\Rightarrow F_* g_1 = g_2$

$d^{FD}_{S\text{-}GH}$ Definite Proof: $d^{FD}_{S\text{-}GH}($  ,  $) = d^{FD}_{GH}\left((\bar{N}_1, M_1, \hat{d}_{g_1}), (\bar{N}_2, M_2, \hat{d}_{g_2})\right) = 0$ iff

$\exists F: \bar{N}_1 \to \bar{N}_2$ that preserves distances and $F(M_1) = M_2$

so F also pres. time: $\tau_{g_j}(q) = \inf\{\hat{d}_{g_j}(q,p) \mid p \in M_j\} \Rightarrow$ Encodes causality $\Rightarrow F_* g_1 = g_2$

Challenge: What about asymptotically flat and other Space-Times whose τ is not proper?

Exhaust these space-times (N, g) with Sub-Space-Times, (Ω_i, g), such that $N = \bigcup \Omega_i$ such that each (Ω_i, g) are Causally Null Compactifiable

Exhausting an asymptotically flat spacetime by  Ω_i' null compactifiable spacetimes has known methods

To get a Causally Null Compactifiable sub-space time; take $\Omega_i = J^-(\Omega_i')$  and apply a Galloway Thm

Challenge: Define a **definite** intrinsic distance between any pair of Causally Null Compactifiable Space-Times

Defn [Sakovich-S] The intrinsic timed-Hausdorff distance, $d_{S\text{-}t\text{-}H}$, between Causally Null Compactifiable Space-Times, (N_j, g_j), is

$$d_{S\text{-}\tau\text{-}H}((N_1,g_1),(N_2,g_2)) = d_{\tau\text{-}H}((\bar N_1, \hat d_{g_1}, \tau_{g_1}),(\bar N_2, \hat d_{g_2}, \tau_{g_2}))$$ where

the timed Hausdorff distance between compact timed metric spaces

$$d_{\tau\text{-}H}((X_1,d_1,\tau_1),(X_2,d_2,\tau_2)) = \inf d_H^{\ell^\infty}(K_{\tau_1,X_1}(X_1), K_{\tau_2,X_2}(X_2))$$

where the infimum is over all timed Kuratowski maps,

$$K_{\tau,X} : X \to \ell^\infty \quad \text{s.t.} \quad K_{\tau,X}(p) = (\tau(p), d(p,p_1), d(p,p_2), d(p,p_3)\ldots)$$

where $\{p_1, p_2 \ldots\}$ are dense in (X,d)

Thm [Sakovich-S]: The intrinsic timed-Hausdorff distance between Causally Null Compactifiable Space-Times is definite.

Future Work: Relate $d_{S\text{-}\tau\text{-}H}$, $d_{S\text{-}BB\text{-}H}$, and $d_{S\text{-}FD\text{-}H}$ (see conjectures)

Proof $d_{\tau\text{-}H}$ is definite: $d_{\tau\text{-}H}((X_1,d_1,\tau_1),(X_2,d_2,\tau_2))=0$

$\iff \forall \varepsilon>0\ \exists K^{\varepsilon}_{\tau_j X_j}: X_j \to \ell^{\infty}$ s.t. $d_H^{\ell^\infty}\left(K^{\varepsilon}_{\tau_1,X_1}(X_1), K^{\varepsilon}_{\tau_2,X_2}(X_2)\right)<\varepsilon$

$\iff \forall \varepsilon>0\ \exists K^{\varepsilon}_{\tau_j X_j}, \exists F^{\varepsilon}_{1,2}: X_1 \to X_2$ and $\exists F^{\varepsilon}_{2,1}: X_2 \to X_1$ s.t.

$d_{\ell^\infty}\left(K^{\varepsilon}_{\tau_2,X_2}(F^{\varepsilon}_{1,2}(p)), K^{\varepsilon}_{\tau_1,X_1}(p)\right)<\varepsilon\ \ \forall p \in X_1$ and visa-versa.

$|\tau_2(F^{\varepsilon}_{1,2}(p)) - \tau_1(p)|<\varepsilon$ and $d(\cdot,p_i)$ terms of $K^{\varepsilon}_{\tau_i X_i}$ too

$F^{\varepsilon}_{1,2}$ is ε-almost time preserving and ε-almost distance preserving

Same for $F^{\varepsilon}_{2,1}$ and these are also 2ε-almost inverses

Taking $\varepsilon \to 0$, a subseq $\to F: X_1 \to X_2$ time + dist pres bijection $\square$

Proof $d_{S\text{-}\tau\text{-}H}$ is definite: Given (N_j, g_j) Causally Null Compactifiable

$d_{S\text{-}\tau\text{-}H}((N_1,g_1),(N_2,g_2))=0 \iff d_{\tau\text{-}H}((\bar{N}_1,\hat{d}_{g_1},\tau_{g_1}),(\bar{N}_2,\hat{d}_{g_2},\tau_{g_2}))=0$

$\Updownarrow$

$\square\ \ \exists F: N_1 \to N_2$ s.t $F_* g_1 = g_2 \iff \exists F: \bar{N}_1 \to \bar{N}_2$ time + dist pres bijection

Introducing Various Notions of Distances between Space-Times by Sakovich-Sormani (arxiv)

Here we introduce the causally null compactifiable space-times,
prove a number of theorems about how to find such space-times;
and prove definitess for the intrinsic distances described above

We also describe more (possibly definite) intrinsic distances:

- $d^{BB}_{S\text{-}F}$, $d^{FD}_{S\text{-}F}$, $d^{TK}_{S\text{-}F}$ (to appear in future work with Sakovich)
  based on the intrinsic flat distance of Sormani-Wenger using Ambrosio-Kirchheim
  The challenge is setting up an integral current structure
  We are working with Meco on setting up bilipschitz charts
- $d^{BB}_{S\text{-}mm}$, $d^{FD}_{S\text{-}mm}$, $d^{T\text{-}K}_{S\text{-}mm}$ (to appear in future work of Mondino-Perales)
  using metric measure spaces and perhaps Wasserstein distances

We survey other notions of convergence of space-times (that do not involve the null distance) as well.

The paper closes with conjectures and proposed applications.

Thank You!

- Christina Sormani
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Answers to questions on the slides below

↓

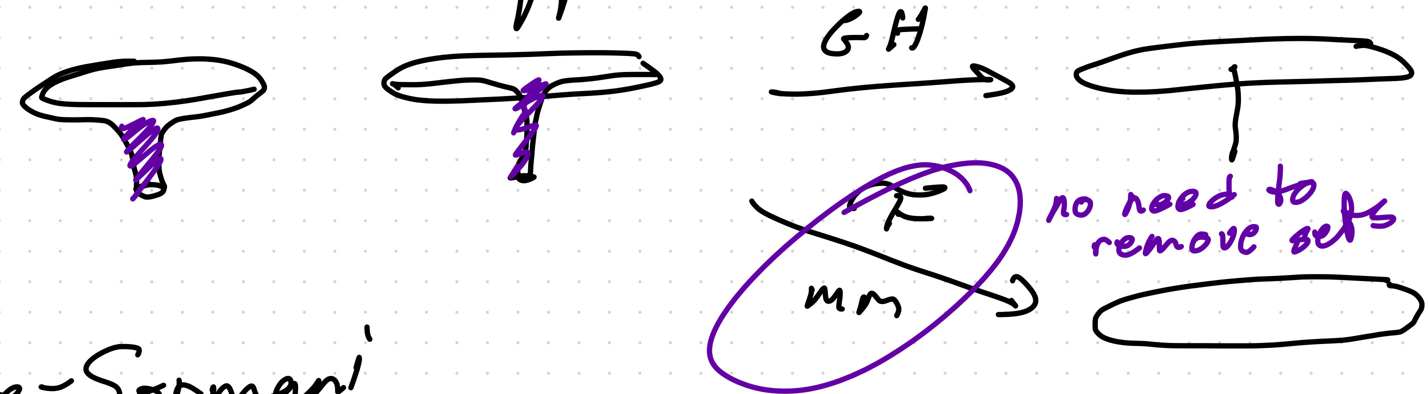
Questions after the talk:

Ariadna Leon-Quiros: is there a Hopf-Rinow Thm in this setting? Anna Sakovich: See work of Burtscher and Garcia-Heveling.

Gerhardt Huisken: can one apply these notions to study the Newtonian Limit, taking the speed of light to infinity as in Cederbaum's Work? Carla Cederbaum: Would be interested in working on this project.

Quick Review of GH vs mm vs $\mathcal{F}$ on the next two slides:

Gromov - Hausdorff (Riemannian)



Lee-Sormani

Intrinsic Flat allows thin gravity wells to disappear

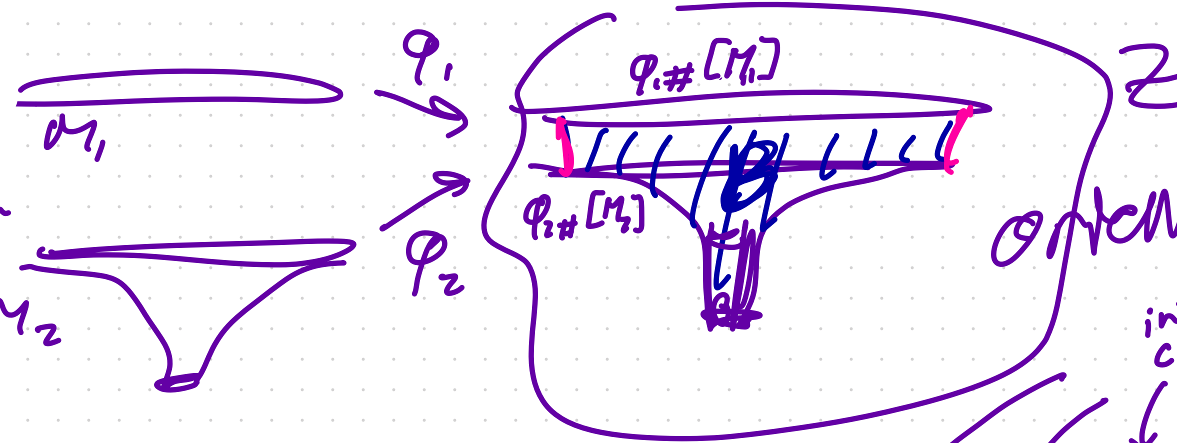
$$\text{ADM Mass}(M_j) \to 0 \stackrel{?}{\Longrightarrow} M_j \setminus K_j \xrightarrow{?} \mathbb{E}^3$$

Conj: Converges in $\mathcal{F}$ sense, GH conv fails

Dong-Song: GH removing sets works

$\mathcal{F}$

Intrinsic Flat
Sormani-Wenger
Ambrosio-Kirchheim



oriented

integral currents

$$d_{\mathcal{F}}(M_1, M_2) = \inf\{ \mathbb{M}(A) + \mathbb{M}(B) \mid A + \partial B = \varphi_{1\#}[M_1] - \varphi_{2\#}[M_2] \}$$

uses Geometric Measure Theory

$$d_{mm}(M_1, M_2) = d_W(\varphi_{1*}\mu_1, \varphi_{2*}\mu_2)$$

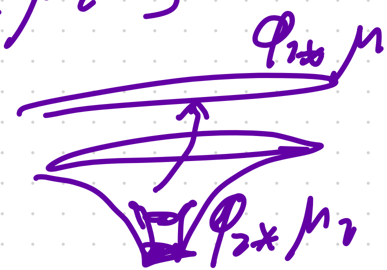
metric measure

Lott
Villani Sturm

Wasserstein distance

Optimal Transport

RCD
CD



Thank You!

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additional slides below
may be of interest

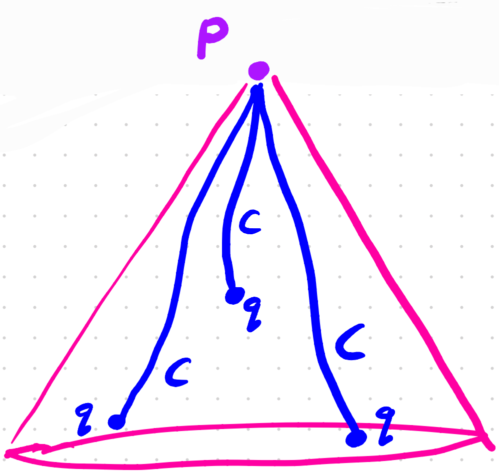
Cosmological Time Function, τ,

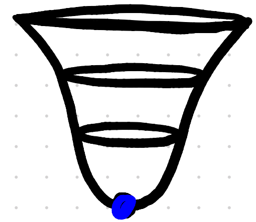
Andersson-Galloway-Howard defined a time function which is independent of a particular gauge on a given spacetime (see also Wald-Yip):

Defn: $\tau_{AGH}(p)$ is the supremum of the Lorentz distance from p over all points q in its past. That is,

$$\tau_{AGH}(p) = \sup_{q<p} \int_{c} \sqrt{-g(c'(s), c'(s))}\, ds$$

where c is a **future causal curve** from q to p. It is said to be "regular" if it is finite on all of M and converges to 0 on all past inextensible curves.

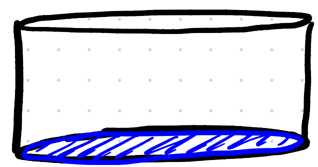




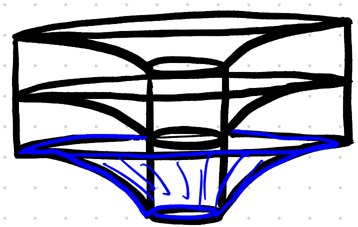
τ is the time since the big bang

AGH: τ is regular $\Rightarrow \forall p \in M$

$\exists\, \gamma_p : [0, \tau(p)] \to M$ s.t. $\tau(\gamma_p(t)) = t$.



τ is not regular on Minkowski but is regular on future Minkowski where $t > 0$.



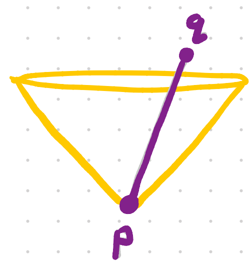
The Null Distance between events in a Spacetime: $\hat{d}_\tau(p,q)$

Joint with Vega: Given a time function, τ, on a spacetime, (M,g),

$$\hat{d}_\tau(p,q) = \inf_\beta \hat{L}_\tau(\beta) = \inf_\beta \sum_{i=1}^{k} |\tau(\beta(t_i)) - \tau(\beta(t_{i+1})|$$

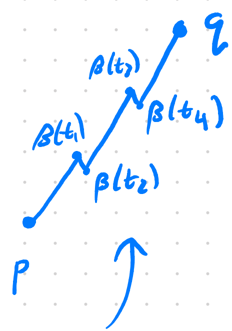
where the inf is over all piecewise causal curves β from p to q, which are causal from $x_i = \beta(t_i)$ to $x_{i+1} = \beta(t_{i+1})$:

If $q > p$ then $\hat{d}_\tau(p,q) = \tau(q) - \tau(p)$ with one step in the sum.



Otherwise we might need many pieces

and the inf is not even piecewise causal



We say $\hat{d}_\tau$ encodes causality:

$$\hat{d}_\tau(p,q) = \tau(q) - \tau(p) \iff q > p.$$

When does $\hat{d}_\tau$ encode causality?

The Null Distance in Minkowski Spacetime

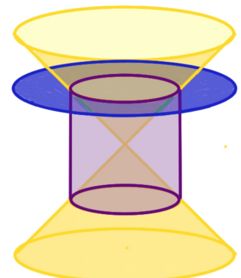
Joint with Vega: Given a time function, τ, on a spacetime, (M, g),

$$\hat{d}_\tau(p,q) = \inf_\beta \hat{L}_\tau(\beta) = \inf_\beta \sum_{i=1}^{k} |\tau(\beta(t_i)) - \tau(\beta(t_{i+1})|$$

where the inf is over all piecewise causal curves β from p to q.

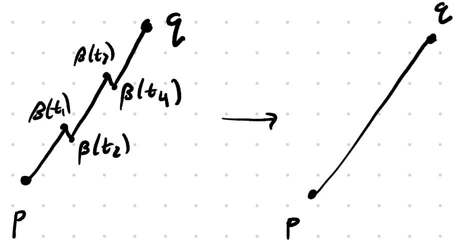
Example: Minkowski Spacetime

The metric tensor is $g = -dt^2 + dx_1^2 + dx_2^2$ So if we take $\tau = t$ then the level sets of $\hat{d}_\tau(p, \cdot)$ are cylinders aligned perfectly with the light cones.



Encodes Causality: $q > p \iff \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$

On Minkowski w/ $\tau = t$:



This captures the infinitesimal behavior of a smooth τ with $g(\nabla\tau, \nabla\tau) = -1$

DIFFICULTY:

But our only canonical τ is Cosmological time and it is not necessarily smooth everywhere.

[Allen-Burtscher]
[Sakovich-Sormani]
[Burtscher-Garcia-Heveling]
and [Galloway]

Prove τ encodes causality in a variety of settings

Joint with Sakovich:

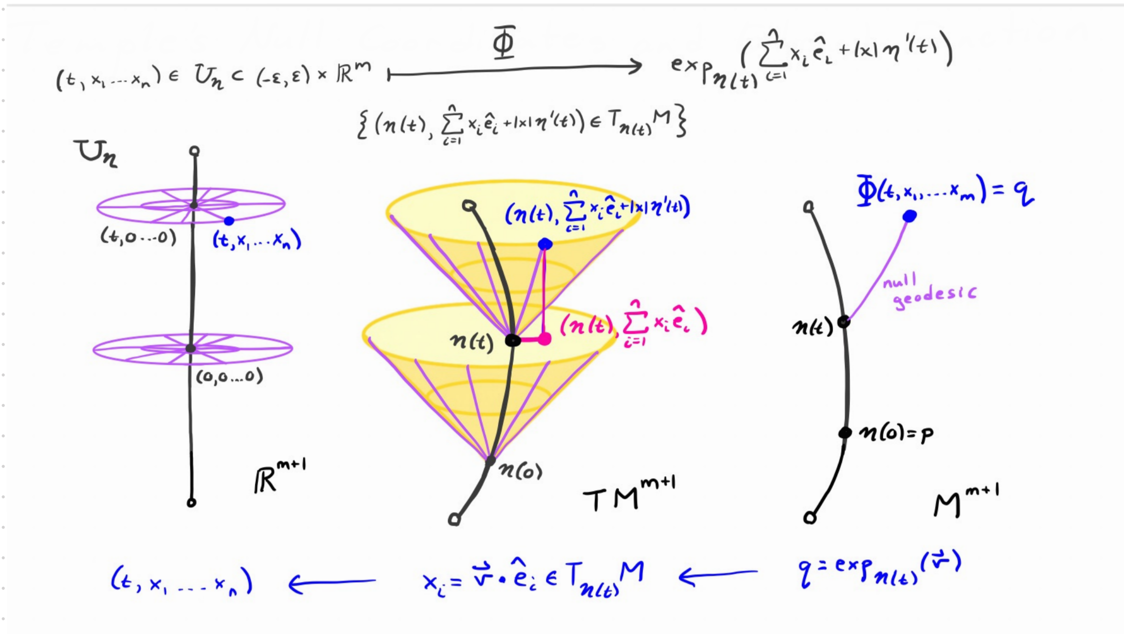
A Regular Cosmological Time Locally Encodes Causality

If (N,g) is a Lorentzian Manifold and $\tau = \tau_{AGH}$ is regular

then $\forall p \in N$ $\exists$ nbd U s.t $\forall q \in U$ $q \succ p \Leftrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$

The proof uses Temple's null coords about a timelike curve $\eta(t)$ with $\eta(0)=p$.

Temple's Eikonal function: is $\omega = t$ of these coords



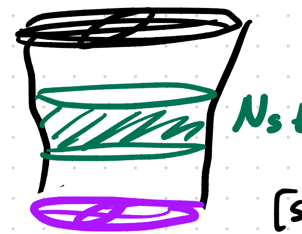
Examples of Causally Null Compactifiable Spacetimes:

Cosmic Strips $N_{st} = \tau^{-1}(s,t)$

in FLRW big bang spacetimes with proper cosmological time functions

[Sakovich-S]

or in future developments of compact initial data sets



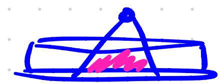
[Sakovich-S]

Don't even need N to have a bounded cosmological time as long as it has a time function which is the shift of a cosmotime on a cosmic strip. [Sako-S]

If N is unbounded in space

you can try to exhaust it, $N = \bigcup_{i=1}^{\infty} W_i$, with

causally null compactifiable subspacetimes, W_i.

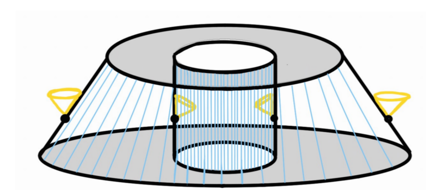
N = Future Minkowski Space 

Pasts of Points $J^{-1}(p) \cap \tau^{-1}(s,t) \subseteq \mathbb{R}^{1,3}$

are causally null compactifiable [Sakovich-S]

N = Future Exterior Schwarzschild

can be exhausted

by these sets:  [Sakovich-S]

Joint with Sakovich: (In dimensions $n \geq 3$)

Given two causally null compactifiable spacetimes, (N_i, g_i, τ_i),

and a bijection $F: N_1 \to N_2$ that is

* distance preserving $\hat{d}_{\tau_2}(F(p), F(q)) = \hat{d}_{\tau_1}(p,q) \quad \forall p,q \in N_1$

* time preserving $\tau_1(p) = \tau_2(F(p)) \quad \forall p \in N_1$

then F is a Lorentzian Isometry $F^* g_2 = g_1$

Proof: by $\tau_j(p) - \tau_j(q) = \hat{d}_{\tau_j}(p,q) \iff p > q$ and * and *

F preserves the Causal Structure, $p > q \iff F(p) > F(q)$

By work of Hawking et al and Levichev which requires $n \geq 3$:

F is C^1 and $F^* g_2 = f^2 g_1$ where f^2 is a conformal factor.

Finally use $g_i(\nabla\tau_i, \nabla\tau_i) = -1$ a.e. for $i = 1, 2$ to see $f^2 \equiv 1$

QED

Thank You!

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