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Nonlinear potential theory through the looking-glass

And the Riemannian Penrose inequality we found there

with M. Fogagnolo, L. Mazzieri, A. Pluda and M. Pozzetta

Topics in Geometric Analysis

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Following the White Rabbit down the (black) hole

Introduction

1

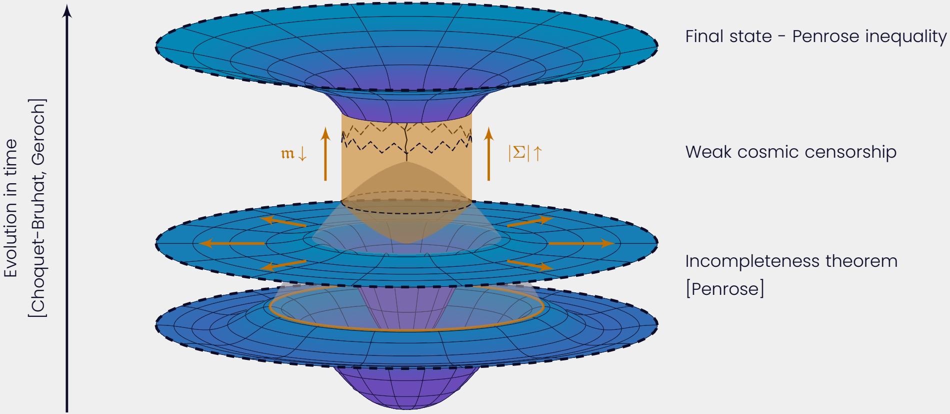
THE PENROSE INEQUALITY

Conjecture - Penrose inequality

Given a time slice of an isolated gravitational system with an energy condition

$$\mathfrak{m} \geq \sqrt{\frac{|\Sigma|}{16\pi}}$$

where $\mathfrak{m}$ is the total mass and Σ is an outermost apparent horizon.



- An **initial data set** (M,g,K) is a triple, where (M,g) is a Riemannian 3-manifold and K is a symmetric $(0,2)$-tensor, satisfying the Einstein constraint equations

$$\mu := \frac{1}{2}\left(\mathrm{R} + \mathrm{tr}K^2 - |K|^2\right), \overset{K=0}{=} \frac{1}{2}\mathrm{R}, \qquad \text{(energy density)}$$

$$J := \mathrm{div}(K - \mathrm{tr}K\, g). \overset{K=0}{=} 0. \qquad \text{(momentum density)}$$

- The **dominant energy condition** is $\mu \geq |J|. \overset{K=0}{\Longleftrightarrow} \mathrm{R} \geq 0.$
- (M,g,K) is **time-symmetric** $\Leftrightarrow K = 0.$
- A surface Σ is an **outermost apparent horizon** $\overset{K=0}{\Longleftrightarrow}$ Σ is a outermost minimal surface.
- (M,g,K) is **isolated** $\overset{K=0}{\Longleftrightarrow}$ (M,g) is asymptotically flat

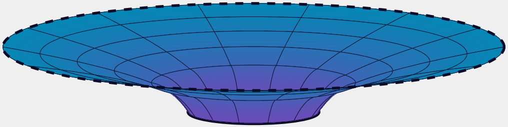
Definition - Asymptotically flat manifold

(M,g) is $\mathscr{C}^k_\tau$-asymptotically flat provided $M \setminus K \cong \mathbb{R}^3 \setminus B_R$ and $|g-\delta| = O_k(|x|^{-\tau})$.

Spatial Schwarzschild manifold

Given $\mathfrak{m} \geq 0$, it is

$$\left(\mathbb{R}^3 \smallsetminus B_{2\mathfrak{m}}, \left(1+\frac{\mathfrak{m}}{2|x|}\right)^4 \delta\right).$$



∂M

- Dominant energy condition: $\mathrm{R}=0$.
- Apparent horizon: ∂M is outermost minimal.
- Isolated: ($\mathscr{C}^\infty_1$-)asymptotically flat.

The quantity $\mathfrak{m}$ represents the mass of the black hole and satisfies the equality in the Penrose inequality

$$\mathfrak{m} = \sqrt{\frac{|\partial M|}{16\pi}}.$$

The natural choice would be to integrate the mass density

$$\frac{1}{8\pi}\int_M \mu \,\mathrm{dVol} \overset{K=0}{=} \frac{1}{16\pi}\int_M \mathrm{R}\,\mathrm{dVol}.$$

but we have no superposition principle, and the mass of Schwarzschild is 0

Linearizing around δ

$$\frac{1}{16\pi}\int_M \nabla \mathrm{R}_{|_\delta}(g-\delta)\mathrm{d}x = \lim_{R\to+\infty}\frac{1}{16\pi}\int_{B_R}\partial_k(\partial_j g_{kj}-\partial_k g_{jj})\mathrm{d}x$$

Definition - ADM mass *[Arnowitt, Deser, Misner '61 · Phys. Rev.]*

$$\mathfrak{m}_{\mathrm{ADM}} := \lim_{R\to+\infty}\frac{1}{16\pi}\int_{\partial B_R}(\partial_j g_{kj}-\partial_k g_{jj})\frac{x^k}{|x|}\mathrm{d}\sigma_\delta$$

Theorem - *[Bartnik '86 · CPAM], [Chruściel '86 · SPRINGER]*

$\mathfrak{m}_{\mathrm{ADM}}$ is a geometric invariant if (M,g) is $\mathscr{C}^1_\tau$-asymptotically flat, $\tau > 1/2$.

In the general case...

- *[Malec, Murchadha '94 · Phys. Rev. D]* under spherical symmetries.
- *[Malec, Mars, Simon '02 · Phys. Rev. Lett.]* under additional constraints.
- *[Mars, Soria '16 · Class. Quantum Gravity]* Penrose inequality along null hypersurfaces.
- *[Allen, Bryden, Kazaras, Khuri '25]* suboptimal Penrose inequality.

... and in the time-symmetric case.

- *[Bray '01 · JDG]* for disconnected horizons (up to dimension 8 *[Bray, Lee '09 · DUKE]*).
- *[Huisken, Ilmanen '01 · JDG]* for connected horizons using inverse mean curvature flow (IMCF).
- *[Agostiniani, Mantegazza, Mazzieri, Oronzio '22]* for connected horizons using nonlinear potential theory (NPT) (cf. *[Xia, Yin, Zhou '24 · Adv. Math.]* for a sharp version on spatial Schwarzschild).

(M,g) is an asymptotically flat Riemannian manifold with $\mathrm{R}\geq 0$ and compact outermost minimal connected boundary ∂M.

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Foreshadowing the fight against the Jabberwocky

Riemannian Penrose inequality

Theorem – *[Huisken, Ilmanen '01 · JDG]*

Let (M,g) be a $\mathscr{C}^1_1$-asymptotically flat 3-Riemannian manifold with $\mathrm{R} \geq 0$ and $\mathrm{Ric} \geq -\mathrm{C}/|x|^2$ and a minimal connected outermost boundary ∂M. Then,

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\mathrm{ADM}}.$$

Moreover, the equality holds <u>if and only if</u> (M,g) is isometric to the Schwarzschild of mass $\mathfrak{m}_{\mathrm{ADM}}$.

Does the Riemannian Penrose inequality holds even for $\mathscr{C}^1_\tau$-asymptotically flat 3-Riemannian manifold, for $\tau > 1/2$?

- Consider the **Hawking mass**

$$\mathfrak{m}_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma \mathrm{H}^2 \, \mathrm{d}\sigma \right).$$

  - ∂M is minimal $\Rightarrow \mathfrak{m}_H(\partial M) = \sqrt{|\partial M|/16\pi}$

- **Evolve ∂M using the IMCF**, namely a family of diffeomorphisms $F_t(\Sigma) = \Sigma_t \subseteq M$ such that

$$\frac{\partial}{\partial t} F_t = \frac{\nu}{\mathrm{H}}.$$

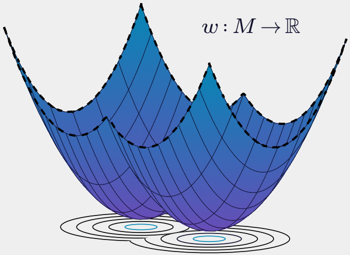
  - By the asymptotic behavior of $g \Rightarrow \Sigma_t$ asymptotically approaches large coordinate spheres as $t \to +\infty$ and

$$\varlimsup_{t \to +\infty} \mathfrak{m}_H(\Sigma_t) \leq \mathfrak{m}_{\mathrm{ADM}}.$$

  - M is 3-dimensional and $\mathrm{R} \geq 0 \Rightarrow \mathfrak{m}_H(\Sigma_t)$ is monotone non decreasing. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{m}_H(\Sigma_t) = \frac{1}{16\pi} \sqrt{\frac{|\Sigma_t|}{16\pi}} \Bigg(\underbrace{8\pi - \int_{\Sigma_t} \mathrm{R}^\top \mathrm{d}\sigma}_{\geq 0 \text{ Gauss-Bonnet}} + \int_{\Sigma_t} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + \frac{|\nabla^\top \mathrm{H}|^2}{\mathrm{H}^2} \mathrm{d}\sigma \Bigg) \geq 0.$$

LEVEL SET FLOW



Replace $F_t(\Sigma)$ with the level sets $\Sigma_t = \partial\{w \leq t\}$ of a function $w : M \to \mathbb{R}$ such that $\partial M = \{w = 0\}$

+ the flow is defined for every t;
- less control on regularity/topology;

Inverse mean curvature flow (IMCF)

[Huisken, Ilmanen '01]

$$\begin{cases} \Delta_1 w_1 = |\nabla w_1| & \text{on } M \setminus \partial M \\ w_1 = 0 & \text{on } \partial M \\ w_1 \to +\infty & \text{on } \mathrm{d}(x, \partial M) \to +\infty. \end{cases}$$

$\Delta_1 w_1 = \mathrm{H}$: level set *evolve* by IMCF.

Nonlinear potential theory (p-IMCF, $p \in (1,2]$)

[Agostiniani, Mantegazza, Mazzieri, Oronzio '22]

$$\begin{cases} \Delta_p w_p = |\nabla w_p|^p & \text{on } M \setminus \partial M \\ w_p = 0 & \text{on } \partial M \\ w_p \to +\infty & \text{on } \mathrm{d}(x, \partial M) \to +\infty. \end{cases}$$

$u_p = \mathrm{e}^{-\frac{w_p}{p-1}}$ is p-harmonic.

NPT & IMCF AT A GLANCE

Inverse mean curvature flow (IMCF)

[Huisken, Ilmanen '01]

Regularity

- is Lipschitz.

Regularity of level sets

- are strictly outward minimizing;
- are $\mathscr{C}^{1,1}$;
- weak H and h.

Energy

- $|\partial K^*| = \inf\{|\partial E| : E \supseteq K\}$
- grows exponentially

$$|\Sigma_t^{(1)}| = \mathrm{e}^t |\partial M|$$

Nonlinear potential theory (p-IMCF, $p \in (1,2]$)

[Agostiniani, Mantegazza, Mazzieri, Oronzio '22]

Regularity

- $w_p \in \mathscr{C}^{1,\beta}$ and smooth on $\{\nabla w_p \neq 0\}$;
- $|\nabla w_p| \in W^{1,2}$.

$(p = 2) \rightsquigarrow$ smooth.

Regularity of level sets

- has a negligible critical part for a.e. t;
- smooth away from the critical set.

$(p = 2) \rightsquigarrow$ smooth for a.e. t.

Energy

- $\mathfrak{c}_p(\partial K) = \inf\{\mathrm{C}_p \int |\nabla \phi|^p : \phi \in \mathscr{C}_c^\infty, \phi \geq 1_K\}$
- grows exponentially

$$\mathfrak{c}_p(\Sigma_t^{(p)}) = \mathrm{e}^t \, \mathfrak{c}_p(\partial M)$$

- The technique based on IMCF requires
  1. the monotonicity of the Hawking mass;
  2. the convergence of evolved surfaces to coordinate spheres.
- **Problem**: the smooth IMCF does not always exist, and the flow could develop singularities.
- **Solution**: pass to a level set/weak formulation.

1. Can we prove the monotonicity despite the regularity of w_p?
2. Do the level sets converge to coordinate spheres for large t?

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Finding allies at the tea party

Monotonicity formulas

We introduce the p**-Hawking mass**, which is

$$\mathfrak{m}_H^{(p)}(\Sigma) = \frac{\mathfrak{c}_p(\Sigma)^{\frac{1}{3-p}}}{8\pi}\left(4\pi - \int_\Sigma \frac{\mathrm{H}^2}{4}\,\mathrm{d}\sigma + \int_\Sigma \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma\right)$$

$\mathfrak{c}_1(\Sigma_t^{(1)}) = |\Sigma_t^{(1)}|/4\pi$ and $\mathrm{H} = |\nabla w_1| \Rightarrow \mathfrak{m}_H^{(1)}(\Sigma_t^{(1)}) = \mathfrak{m}_H(\Sigma_t^{(1)})$.

$\mathfrak{m}_H(\Sigma) \leq \left[|\Sigma|^{\frac{1}{2}}\,\mathfrak{c}_p(\Sigma)^{-\frac{1}{3-p}}\right]\mathfrak{m}_H^{(p)}(\Sigma)$.

Theorem - *[Fogagnolo, Mazzieri '22 · J. Funct. Anal.]*

$\mathfrak{c}_p(\Sigma_t^{(p)}) \xrightarrow{p\to 1^+} \mathfrak{c}_1(\Sigma_t)$ for almost every t.

$$\lim_{p\to 1^+} \mathfrak{m}_H^{(p)}(\partial M) \geq \mathfrak{m}_H(\partial M) = \sqrt{\frac{|\partial M|}{16\pi}}.$$

Theorem - *[B —, Pluda, Pozzetta '24]*

Let w_p be the proper p-IMCF, $p \in [1,2]$. Then, $t \mapsto \mathfrak{m}_H^{(p)}(\Sigma_t^{(p)})$ is monotone nondecreasing and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \geq \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{16\pi(3-p)}\left(8\pi - \int_{\Sigma_t^{(p)}} \mathrm{R}^\top \mathrm{d}\sigma + \int_{\Sigma_t^{(p)}} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + \frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + \frac{5-p}{p-1}\left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma\right).$$

- For $p > 1$, monotonicity was proven in *[Agostiniani, Mantegazza, Mazzieri, Oronzio '22]*.

> - H and h have a geometric interpretation (and agree a.e. with the analytic one).
> - Monotonicity follows by computations and a **Gauss-Bonnet**-type **theorem**.

- For $p = 1$, the theorem was proven in *[Huisken, Ilmanen '01 · JDG]*.

Theorem - *[B —, Pluda, Pozzetta '24]*

$\mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \to \mathfrak{m}_H(\Sigma_t^{(1)})$ in L^1_{loc} and the lower bound for the derivative is lower semicontinuous as $p \to 1^+$.

The difficult part is to send $p \to 1^+$ in

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \geq \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{16\pi(3-p)}\left(8\pi - \int_{\Sigma_t^{(p)}} \mathrm{R}^\top \mathrm{d}\sigma + \int_{\Sigma_t^{(p)}} |\mathring{\mathrm{h}}|^2 + \mathrm{R} + \frac{|\nabla^\top|\nabla w_p||^2}{|\nabla w_p|^2} + \frac{5-p}{p-1}\left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma\right),$$

but the best convergence result we had was

Theorem - *[Mari, Rigoli, Setti '22 · AJM]*†

$w_p \to w_1$ locally uniformly as $p \to 1^+$.

† After *[Moser '07 · JEMS]* and *[Kotschwar, Ni '09 · Ann. Sci. Éc. Norm. Supér.]*.

If $\mathfrak{m}_H^{(p)}(\Sigma_t^{(p)})$ is bounded in L^1, then $\|\mathrm{H} - |\nabla w_p|\|_{L^2(\Sigma_t^{(p)})} \xrightarrow{p \to 1^+} 0$.

Theorem - *[**B** —, Pluda, Pozzetta '24]*

- $w_p \xrightarrow{p \to 1^+} w_1$ in $W^{1,q}_{\mathrm{loc}}$ for every $q < +\infty$.
- $\Sigma_t^{(p)} \xrightarrow{p \to 1^+} \Sigma_t^{(1)}$ (in curvature varifolds sense) for almost every t.

4

Enjoy the world beyond the looking-glass

Asymptotic analysis

Theorem - *[Huisken, Ilmanen '01 · JDG]*

Let (M,g) be $\mathscr{C}^{1,1}$-asymptotically flat and $\mathrm{Ric} \geq -\mathrm{C}|x|^{-2}$. Then

1. $w_1 = 2\log|x| - 2\log(\mathfrak{c}_1(\Sigma_0^{(1)})) + o(1)$,
2. $\Sigma_t^{(1)} \sim \mathbb{S}^2(\mathrm{e}^{t/2})$ in $\mathscr{C}^1$.

In particular,

$$\lim_{t\to+\infty} \mathfrak{m}_H(\Sigma_t^{(1)}) \leq \mathfrak{m}_{\mathrm{ADM}}.$$

Theorem

Let (M,g) be $\mathscr{C}^1_{\tau>1/2}$-asymptotically flat. Then

1. $w_2 = \log|x| - \log\mathfrak{c}_2(\Sigma_0^{(2)}) + \log(1+O(|x|^{-\tau}))$
2. $\nabla_i w_2 = \partial_i \log|x| + O(|x|^{-1-\tau})$,
3. $\left\|\nabla_i\nabla_j w_2 - \partial_i\partial_j \log|x|\right\|_{L^2(\Sigma_t)} = O(\mathrm{e}^{-2\tau t})$.

In particular,

$$\lim_{t\to+\infty} \mathfrak{m}_H^{(2)}(\Sigma_t^{(2)}) \leq \mathfrak{m}_{\mathrm{ADM}}.$$

We obtain a <u>non-sharp upper bound</u> for the Hawking mass, indeed

$$\lim_{t\to+\infty} \mathfrak{m}_H(\Sigma_t) \leq \overline{\lim_{t\to+\infty}} \left[\mathfrak{c}_1(\Sigma_t^{(1)})^{\frac{1}{2}}\, \mathfrak{c}_2(\Sigma_t^{(1)})^{-1}\right] \mathfrak{m}_H^{(2)}(\Sigma_t) \leq \mathrm{C}\,\mathfrak{m}_{ADM}.$$

$\mathfrak{c}_2(\Sigma_t^{(p)}) = \mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}(1+o(1))$ for all $p \in (1,2]$.

Recall the definition of the p-Hawking mass

$$\mathfrak{m}_H^{(p)}(\Sigma) = \frac{\mathfrak{c}_p(\Sigma)^{\frac{1}{3-p}}}{8\pi}\left(4\pi - \int_\Sigma \frac{\mathrm{H}^2}{4}\,\mathrm{d}\sigma + \int_\Sigma \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma\right)$$

I do not know if $\mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \leq \left[\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}\,\mathfrak{c}_2(\Sigma_t^{(p)})^{-1}\right]\mathfrak{m}_H^{(2)}(\Sigma_t^{(p)})$.

- Take any $p \in (1,2]$. For large t, $\Sigma_t^{(p)}$ is almost spherical. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t^{(p)}} \frac{\mathrm{H}^2}{4} - \left(\frac{\mathrm{H}}{2} - \frac{|\nabla w_p|}{3-p}\right)^2 \mathrm{d}\sigma \leq \frac{1}{3-p}\left(\int_{\Sigma_t^{(p)}} \frac{\mathrm{R}^\top}{2} - \frac{\mathrm{H}^2}{4}\,\mathrm{d}\sigma\right) \leq \frac{4\pi}{3-p}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t^{(p)}} \mathrm{H}^2\,\mathrm{d}\sigma\right).$$

- By de l'Hôpital rule we infer

$$\lim_{t\to+\infty} \mathfrak{m}_H^{(p)}(\Sigma_t^{(p)}) \leq \overline{\lim_{t\to+\infty}} \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{\mathfrak{c}_1(\Sigma_t^{(p)})^{\frac{1}{2}}}\,\mathfrak{m}_H(\Sigma_t^{(p)}) \leq \overline{\lim_{t\to+\infty}} \frac{\mathfrak{c}_p(\Sigma_t^{(p)})^{\frac{1}{3-p}}}{\mathfrak{c}_2(\Sigma_t^{(p)})}\,\mathfrak{m}_H^{(2)}(\Sigma_t^{(p)}) \leq \mathfrak{m}_{\mathrm{ADM}}.$$

- We conclude that

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \underline{\lim_{p\to 1^+}}\, \mathfrak{m}_H^{(p)}(\partial M) \leq \mathfrak{m}_{\mathrm{ADM}}.$$

Theorem - *[**B** —, Fogagnolo, Mazzieri '24 · CPAM]*

Let (M, g) be a $\mathscr{C}^1_\tau$-asymptotically flat 3-Riemannian manifold, $\tau > 1/2$, with $\mathrm{R} \geq 0$ and a minimal connected outermost boundary. Then,

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq \mathfrak{m}_{\mathrm{ADM}}.$$

Moreover, the equality holds if and only if (M, g) is isometric to the Schwarzschild of mass $\mathfrak{m}_{\mathrm{ADM}}$.

- We also proved a Penrose inequality for the isoperimetric mass $\mathfrak{m}_{\mathrm{iso}}$ introduced by Huisken *[Huisken '09]*, in $\mathscr{C}^0$-asymptotically flat manifolds.
- Combining the theorem above with *[Jauregui, Lee '19 · CRELLE]*, we show that $\mathfrak{m}_{\mathrm{iso}} = \mathfrak{m}_{\mathrm{ADM}}$ under optimal decay assumptions, which implies *[Bartnik '86 · CPAM]*, *[Chruściel '86 · SPRINGER]*.