Handling the internal structure of the pion in $e^+e^- \to \pi^+\pi^-(\gamma)$

Workshop on Radiative Corrections and Monte Carlo simulations for electron-positron collisions
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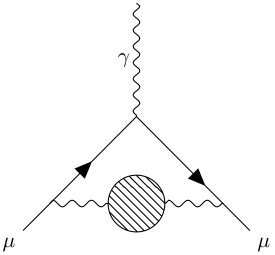
[1]Based on: Budassi et al., "Pion pair production in e^+e^- annihilation at next-to-leading order matched to Parton Shower"

Outline

Introduction

Why do we care about the Pion FF?



The LO Hadronic Vacuum Polarisation contribution to the $(g-2)_\mu$ in the dispersive approach is computed as

$$a_\mu^{\text{HLO}} = \frac{\alpha}{\pi^2} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} K(s) \left(\frac{\alpha(s)}{3} \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \right)$$

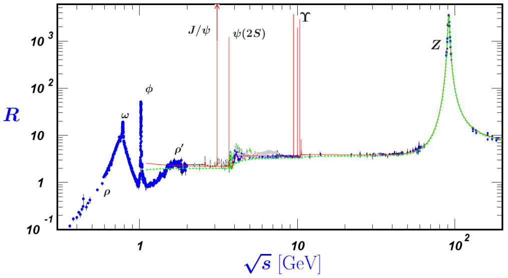
Most of the contribution comes from the $\pi\pi$ channel

$$a_\mu^{\text{HLO}}\Big|_{\pi\pi} \simeq \frac{\alpha}{\pi^2} \int \frac{\mathrm{d}s}{s} K(s) \beta_\pi^2 |F_\pi(s)|^2 f(s)$$

Needed an accurate ($\mathcal{O}(10^{-3})$) description of the process

$$e^+e^- \to \pi^+\pi^-(\gamma)$$

Monte Carlo Generators + Radiative Corrections



Scalar QED

The formal definition of the pion FF accounts for the non-perturbative nature of ud interactions at $q^2 < \Lambda^2_{\text{QCD}}$

$$j^\mu_{\text{em}} = (2\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d - \bar{s}\gamma^\mu s)/3$$

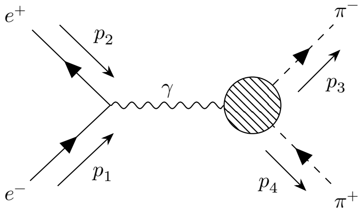
Pion Form factor

$$\langle \pi^\pm(p')|j^\mu_{\text{em}}(0)|\pi^\pm(p)\rangle = \pm(p'+p)^\mu F_\pi\left((p'-p)^2\right)$$

With the condition $F_\pi(0) = 1$

We want to calculate the process

$$e^-(p_1)\, e^+(p_2) \to \gamma^* \to \pi^-(p_3)\, \pi^+(p_4)$$



The cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}c} \propto \frac{1}{(p_1+p_2)^2} \sum_{\text{spins}} |J^\nu_e g_{\mu\nu} J^\mu_\pi|^2$$

- J^ν_e is a conserved spinor-QED current
- J^μ_π can be computed in sQED $\otimes$ form factor

Scalar QED

Pions have $J^{PC}(\pi^{\pm}) = 0^{-}$ and are charged under $U(1)_{\text{em}}$ so they can be described using scalar QED

$$\mathcal{L}_{\text{sQED}} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{Dirac}} + (D_\mu \phi)^\dagger D^\mu \phi$$

sQED $\otimes$ QED Lagrangian

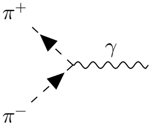
$$\begin{aligned}\mathcal{L}^{\text{int}}_{\text{sQED}} = &- e\bar{\psi}\gamma^\mu \psi A_\mu \\ &- ieA_\mu F_\pi(q^2)\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \\ &+ e^2 A_\mu A^\mu \phi^* \phi F_\pi(q_1^2) F_\pi(q_2^2)\end{aligned}$$

Point-like

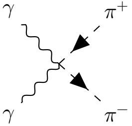
$$D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi$$

Form factor

$$D_\mu \phi = \partial_\mu \phi - ieA_\mu F_\pi(q^2)\phi$$



$$= \quad -ie(p_- - p_+)F_\pi((p_- - p_+)^2)$$



$$= \quad 2ie^2 g^{\mu\nu} F_\pi(q_1^2) F_\pi(q_2^2)$$

- At LO we have **Point-like**$\times F_\pi(q^2)$
- At NLO we have to be careful, since we integrate $\int \mathrm{d}q F_\pi(q^2)\mathcal{M}_{\text{NLO}}(q)$

(The 4-point is not necessarily the most general)

Observables

In **energy scan** experiments, the pion FF is extracted as[2]

$$|F_\pi|^2 = \left(\frac{N_{\pi^+\pi^-}}{N_{e^+e^-}} - \Delta^{\text{bg}} \right) \cdot \frac{\sigma^0_{e^+e^-} \cdot (1+\delta_{e^+e^-}) \cdot \varepsilon_{e^+e^-}}{\sigma^0_{\pi^+\pi^-} \cdot (1+\delta_{\pi^+\pi^-}) \cdot \varepsilon_{\pi^+\pi^-}}$$

We are interested in predictions for the following observables

Cross section

$\sigma^0_{\pi^+\pi^-}$ is the LO cross section

$\delta_{\pi^+\pi^-}$ accounts for the radiative corrections

Charge Asymmetry

$$A_{FB} = \frac{N_{\theta<\pi/2} - N_{\theta>\pi/2}}{N_{\theta<\pi/2} + N_{\theta>\pi/2}}$$

Used to determine the *fiducial volume* of the detector, enters $\varepsilon_{\pi^+\pi^-}$

$$A_{FB}^{\text{NLO}} = A_{FB}^{\text{LO}} + \frac{\alpha}{\pi} A_{FB}^{\alpha} = 0 + \frac{\alpha}{\pi} \left(\frac{\sigma_B^{\text{odd}} - \sigma_F^{\text{odd}}}{\sigma^{\text{NLO}}} \right)$$

$\delta_{\pi^+\pi^-}, A_{FB}$ are sensitive to the insertion of $F_\pi(q^2)$ in loop diagrams

[2] F. V. Ignatov et al., "Measurement of the $e^+e^- \to \pi^+\pi^-$ cross section from threshold to 1.2 GeV with the CMD-3 detector"

Factorised sQED

NLO

The fixed-order NLO cross section can be written as

NLO cross section

$$\sigma_{\text{NLO}} = \sigma_{2\to2} + \sigma_{2\to3} = \sigma_{\text{LO}} + \sigma_{\text{SV}} + \sigma_{\text{H}} \,,$$

where the splitting is given by

$$\sigma_{2\to2} = \frac{1}{\mathcal{F}} \left\{ \int \mathrm{d}\Phi_2 |\mathcal{M}_{\text{LO}}|^2 + \int \mathrm{d}\Phi_2 \, 2\Re \left(\mathcal{M}_{\text{LO}}^\dagger \mathcal{M}_V(\lambda) \right) \right\} \equiv \sigma_{\text{LO}} \left(1 + \delta_V(\lambda)\right)$$

$$\sigma_{2\to3} = \frac{1}{\mathcal{F}} \left\{ \int_{\lambda \le \omega \le \Delta E} \mathrm{d}\Phi_3 \, |\mathcal{M}_{2\to3}|^2 + \int_{\omega > \Delta E} \mathrm{d}\Phi_3 \, |\mathcal{M}_{2\to3}|^2 \right\} \equiv \sigma_S(\lambda, \Delta E) + \sigma_H(\Delta E) \,,$$

- $m_{\text{ph}} = \lambda$ photon mass IR regulator
- On-shell renormalisation of UV divergences
- Phase-space slicing for soft-hard bremsstrahlung

Splitting in gauge-invariant subsets

$$\frac{\mathrm{d}\sigma_{\text{NLO}}}{\mathrm{d}\cos\theta} = \frac{\mathrm{d}\sigma_{\text{LO}}}{\mathrm{d}\cos\theta} \left(1 + \delta_{SV}^{\text{ISR}} + \delta_{SV}^{\text{FSR}} + \delta_{SV}^{\text{IFI}}\right) + \frac{\mathrm{d}\sigma_H}{\mathrm{d}\cos\theta} \,.$$

NLO in F×sQED approach

In the F×sQED approach, each diagram is multiplied by $F_\pi(q^2)$ evaluated at the q^2 flowing into the propagator, preserving the soft limit for radiative corrections

NLO differential cross-section

$$\frac{d\sigma_{\text{NLO}}}{d\cos\theta} = \frac{d\sigma_{\text{LO}}}{d\cos\theta}\left(1+\delta_{SV}^{\text{ISR}}+\delta_{SV}^{\text{FSR}}+\delta_{SV}^{\text{IFI}}\right)+\frac{d\sigma_H}{d\cos\theta}\,.$$

The soft-virtual correction has to be IR safe and ΔE-independent

$$\sigma_{SV} = \delta_{SV}\sigma_{\text{LO}} = (\delta_S+\delta_V)\sigma_{\text{LO}}$$

Soft

$$\sigma_S(\lambda,\Delta E) = \delta_S(\lambda,\Delta E)\,\sigma_{\text{LO}}^0 \times |F_\pi(s)|^2$$

Factorises always over the Born, $q^2 = s$

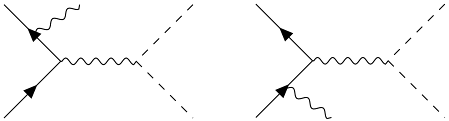
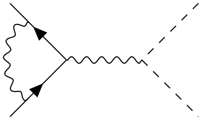
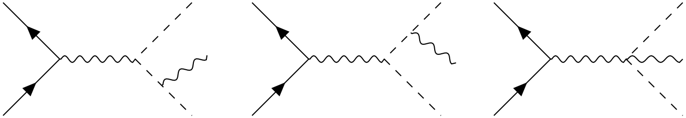
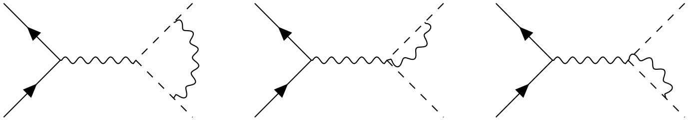
Virtual

$$\sigma_V^i(\lambda) = \int d\Phi_2 2\Re\left(\mathcal{M}_{\text{LO}}^\dagger \mathcal{M}_V^i(\lambda)\right) \times F_\pi^*(s)F_\pi(q_i^2)$$

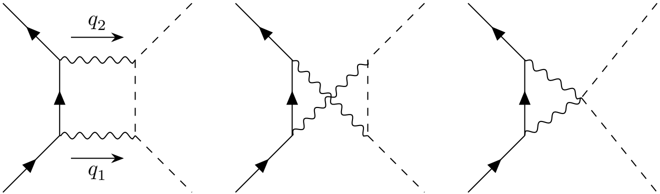
To cancel the $\lambda, \Delta E$ dependence one should have in the soft limit

$$F_\pi^*(s)F_\pi(q_i^2) \to |F_\pi(s)|^2$$

ISR and FSR

Subset		Diagrams	$F_\pi(q^2)$
ISR	real		$F_\pi(m_{\pi\pi}^2)$
	virtual		$F_\pi(s)$
FSR	real		$F_\pi(s)$
	virtual		$F_\pi(s)$

For ISR and FSR the soft limit is clear. In IFI diagrams to which vertex we assign the form factor?



$$\times F_\pi(q_1) F_\pi(q_2) \to F_\pi(s)$$

The $F_\pi \times$ sQED approach is justified because the IR divergence appears when

$$q_2 \to 0 \quad \Rightarrow \quad F(q_1^2) \to F(s),\ F(q_2^2) \to 1$$
$$q_1 \to 0 \quad \Rightarrow \quad F(q_2^2) \to F(s),\ F(q_1^2) \to 1$$

However the factorised prescription is valid only in the **soft limit**

Parton Shower

To take into account additional photon emission, the Higher Order (HO) contribution could be resummed. One way is the Parton Shower

PS master formula

$$\mathrm{d}\sigma_{\text{matched}} = F_{\text{SV}}\, \Pi(\varepsilon, Q^2) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=1}^{n} F_{\text{H},i} \right) |\mathcal{M}_n^{\text{PS}}|^2 \mathrm{d}\Phi_n$$

For the $e^+e^- \to \pi^+\pi^-$ process, the Sudakov form factor $\Pi(\varepsilon, Q^2)$ is a combination of the scalar and spinor one

$$\Pi^{(\text{s})\text{QED}}(\varepsilon, Q^2) = \exp\left\{ -\frac{\alpha}{2\pi} I_+^{(\text{s})\text{QED}} \int \mathrm{d}\Omega_k \, \mathcal{I}(k) \right\} .$$

$$P_f(z) = \frac{1+z^2}{1-z}, \qquad P_s(z) = \frac{2\,z}{1-z} .$$

$$I_+^{\text{QED}}(\varepsilon) = \int_0^{1-\varepsilon} \mathrm{d}z\, P_f(z) = -2\ln\varepsilon - \frac{3}{2} + 2\varepsilon - \frac{1}{2}\varepsilon^2 ,$$

$$I_+^{\text{sQED}}(\varepsilon) = \int_0^{1-\varepsilon} \mathrm{d}z\, P_s(z) = -2\ln\varepsilon - 2 + 2\varepsilon .$$

Internal structure of the pion at NLO

Inserting F_π in loops

The non-trivial task is integrating the pion FF over loop momentum if one wants to write the NLO amplitude in terms of Passarino-Veltman A,B,C,D functions

$$\mathcal{A}^V_{\text{NLO}} \propto 2\text{Re}\,\mathcal{C}_D \int \mathrm{d}^D q\, F^*_\pi(s)\, F_\pi(q^2)\, \mathcal{M}^\dagger_{\text{LO},0} \tilde{\mathcal{M}}_V(q^2,\lambda^2)$$

Numerical Integration

$$\mathcal{A}^V_{\text{NLO}} = \frac{1}{N}\sum_i \mathrm{Re}\left[\tilde{\mathcal{A}}(q_i;s,t)F_\pi(q_i^2)\right]$$

- Need to sample for q divergences
- Very hard numerically
- Unfeasible (?) for MC event generation

Explicit form of F_π

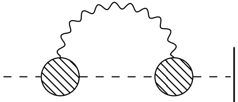
$$F_\pi(q^2) \simeq \text{GVMD}(q^2), \text{FsQED}(q^2)$$

- Do the loop integration 'analytically'
- Evaluate numerically PaVe functions
- Feasible MC event generation
- Relies on a fit function to separate ReF_π and ImF_π

UV Renormalisation and FSR

The insertion of $F_\pi(q^2)$ in the wavefunction renormalisation

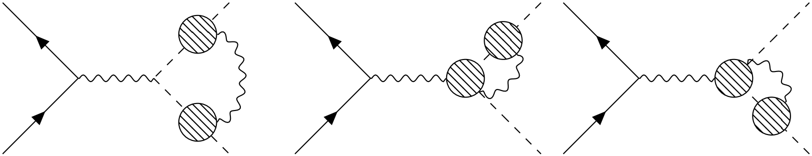
$$\delta Z_{\phi,\text{FF}}(\lambda) = -\frac{\partial}{\partial p^2}$$



$$\Big|_{p^2=m^2} = \delta Z_{\phi,0}(\lambda) + \text{IR-finite terms}\,,$$

The virtual corrections are also modified and can be written as

$$\delta^i_{V,\text{FF}}(\lambda) = \frac{2\Re F^*_\pi(s)\mathcal{M}^\dagger_{\text{LO},0}\,\mathcal{M}^i_{V,\text{FF}}(\lambda)}{|F_\pi(s)|^2\,|\mathcal{M}_{\text{LO},0}|^2}\,, \qquad \begin{array}{c} i = \text{FSR, IFI} \\ \text{FF} = \text{GVMD, FsQED} \end{array}$$

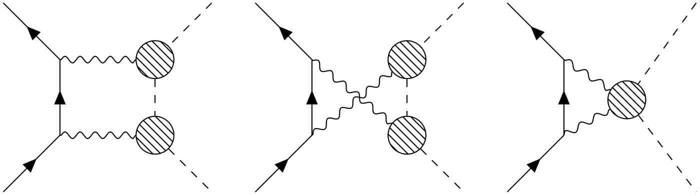


$$\delta^{\text{FSR}}_{V,\text{FF}} = \delta^{\text{FSR}}_{V,0} + \text{IR-finite terms}$$

IFI

The soft virtual correction should be IR-finite

$$\delta^{\mathsf{IFI}}_{SV,\mathsf{FF}} = \delta^{\mathsf{IFI}}_{SV,0}\Big|_{\mathrm{IR}} + \text{IR-finite terms}\,. \tag{1}$$



This is true only if the IR coefficient matches the soft-photon emission

$$\delta^{\mathsf{IFI}}_{S,\mathsf{IR}} = \mathcal{C}^{\mathsf{IFI}}_{\mathsf{IR}} \log \frac{4\Delta E^2}{\lambda^2}$$

Massive Photons

What happens is that photon propagators are substituted as

$$\frac{1}{q^2-\lambda^2\ (+i\varepsilon)} \quad\rightarrow\quad \frac{1}{q^2--m^2-\lambda^2\ (+i\varepsilon)} = \frac{1}{q^2-s'\ (+i\varepsilon)}$$

$F_\pi(q^2)$ in Loops $\iff$ Massive photons $m^2_{\text{ph}} = s'$

We can define *massive photon kernels* to compute the virtual corrections

$$\bar{\delta}_V^{\text{FSR}}(s') = \frac{2\mathcal{M}^\dagger_{\text{LO},0}\,\mathcal{M}^{\text{FSR}}_{V,0}(s')}{|\mathcal{M}_{\text{LO},0}|^2} \qquad \bar{\delta}_V^{\text{IFI}}(s',s'') = \frac{2\mathcal{M}^\dagger_{\text{LO},0}\,\mathcal{M}^{\text{IFI}}_{V,0}(s',s'')}{|\mathcal{M}_{\text{LO},0}|^2}\,,$$

GVMD Approach

GVMD[3]

In the GVMD approach, form factor is written as a sum over additional propagators, written as Breit-Wigners

GVMD Form factor

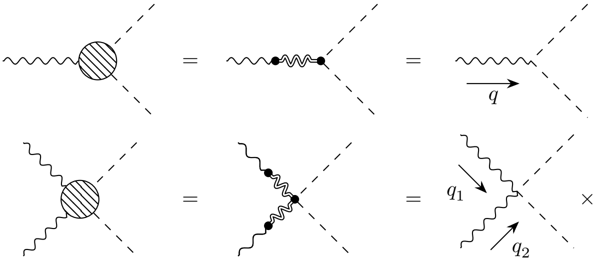
$$F_\pi^{\mathrm{BW}}(q^2) = \sum_{v=1}^{n_r} F_{\pi,v}^{\mathrm{BW}}(q^2) = \frac{1}{c_t}\sum_{v=1}^{n_r} c_v \frac{\Lambda_v^2}{\Lambda_v^2 - q^2}$$

$$\Lambda_v^2 = m_v^2 - i m_v \Gamma_v$$

$$c_v = |c_v| e^{i\phi_v}$$

$$c_t = \sum_v c_v$$

Vertices are modified diagrammatically



$$\times F_\pi^{\mathrm{BW}}(q^2)\,,$$

$$\times F_\pi^{\mathrm{BW}}(q_1^2)\, F_\pi^{\mathrm{BW}}(q_2^2)\,.$$

[3] Fedor Ignatov and Roman N. Lee. "Charge asymmetry in $e^+e^- \to \pi^+\pi^-$ process". In: *Phys. Lett. B* 833 (2022), p. 137283. DOI: 10.1016/j.physletb.2022.137283. arXiv: 2204.12235 [hep-ph]

GVMD: Virtual

The virtual FSR and ISR amplitudes can be written as

$$\mathcal{M}^{\text{FSR}}_{V,\text{GVMD}} = \int \mathrm{d}^D q\, \mathcal{M}^{\text{FSR}}_{V,0}\, F^{\text{BW}}_{\pi}(s) \sum_{v,w=1}^{n_r} F^{\text{BW}}_{\pi,v}(q^2)\, F^{\text{BW}}_{\pi,w}(q^2)$$

$$\mathcal{M}^{\text{IFI}}_{V,\text{GVMD}} = \int \mathrm{d}^D q\, \mathcal{M}^{\text{IFI}}_{V,0} \sum_{v,w=1}^{n_r} F^{\text{BW}}_{\pi,v}(q^2)\, F^{\text{BW}}_{\pi,w}((q-p_3-p_4)^2)$$

- Preserves gauge invariance
- Since $F_\pi(q^2)$ is propagator-like $\Rightarrow$ standard loop techniques

Two Propagator identity

$$\frac{1}{q_i^2-\lambda^2}\frac{1}{q_i^2-\Lambda_i^2} = \frac{1}{\Lambda_i^2-\lambda^2}\left[\frac{1}{q_i^2-\Lambda_i^2} - \frac{1}{q_i^2-\lambda^2}\right]$$

The identity can be iterated to the three propagator case with masses $\lambda^2, \Lambda_i^2, \Lambda_j^2$

Some technical details in the next slides

GVDM: FSR

In the FSR we have three propagators, we sum over all possible resonances

$$\delta^{\text{FSR}}_{V,\text{GVMD}}(\lambda) = \frac{2\Re F_\pi(s)^* \mathcal{M}^\dagger_{\text{LO},0}\, \mathcal{M}^{\text{FSR}}_{V,\text{GVMD}}(\lambda)}{|F_\pi(s)|^2\, |\mathcal{M}_{\text{LO},0}|^2} = \sum_{v=1}^{n_r}\sum_{w=1}^{n_r} \Re \frac{c_v\, c_w}{c_t^2}\, \Delta^{\text{FSR}}_{V,\text{GVMD}}(\Lambda_v^2, \Lambda_w^2)\,.$$

In the case $\Lambda_v \neq \Lambda_w$, we obtain

$$\Delta^{\text{FSR}}_{V,\text{GVMD}}(\Lambda_v^2, \Lambda_w^2) = \bar{\delta}^{\text{FSR}}_V(\lambda^2) + \frac{1}{\Lambda_v^2 - \Lambda_w^2}\left[\Lambda_w^2\, \bar{\delta}^{\text{FSR}}_V(\Lambda_v^2) - \Lambda_v^2\, \bar{\delta}^{\text{FSR}}_V(\Lambda_w^2)\right],$$

while for $\Lambda_v = \Lambda_w$ we have

$$\Delta^{\text{FSR}}_{V,\text{GVMD}}(\Lambda_v^2, \Lambda_v^2) = \bar{\delta}^{\text{FSR}}_V(\lambda^2) - \bar{\delta}^{\text{FSR}}_V(\Lambda_v^2) + \Lambda_v^2 \frac{\partial}{\partial \Lambda_v^2} \bar{\delta}^{\text{FSR}}_V(\Lambda_v^2)\,,$$

The IR singularities have the same structure of the F×sQED case

$$\sum_{v=1}^{n_r}\sum_{w=1}^{n_r} \Re \frac{c_v\, c_w}{c_t^2}\, \bar{\delta}^{\text{FSR}}_V(\lambda^2) = \delta^{\text{FSR}}_{V,0}(\lambda)$$

GVMD: IFI

In the IFI we have only two FFs

$$\delta^{\mathsf{IFI}}_{V,\mathsf{GVMD}}(\lambda) = \frac{2\Re F_\pi(s)^* \mathcal{M}^\dagger_{\mathsf{LO},0}\,\mathcal{M}^{\mathsf{IFI}}_{V,\mathsf{GVMD}}(\lambda)}{|F_\pi(s)|^2\,|\mathcal{M}_{\mathsf{LO},0}|^2} = \sum_{v=1}^{n_r}\sum_{w=1}^{n_r}\Re\frac{c_v\,c_w}{c_t^2 F_\pi(s)}\,\Delta^{\mathsf{IFI}}_{V,\mathrm{GVMD}}(\Lambda_v^2,\Lambda_w^2)$$

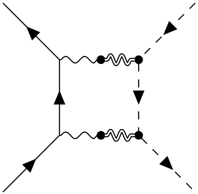
With the propagator identity we obtain the simple relation

$$\Delta^{\mathsf{IFI}}_{V,\mathsf{GVMD}}(\Lambda_v^2,\Lambda_w^2) = \bar\delta^{\mathsf{IFI}}_V(\lambda^2,\lambda^2) - \bar\delta^{\mathsf{IFI}}_V(\Lambda_v^2,\lambda^2) - \bar\delta^{\mathsf{IFI}}_V(\lambda^2,\Lambda_w^2) + \bar\delta^{\mathsf{IFI}}_V(\Lambda_v^2,\Lambda_w^2)$$

All terms are IR divergent apart from the last one. In the soft limit we have

$$\delta^{\mathsf{IFI}}_{V,\mathsf{GVMD}}\Big|_{\mathsf{IR}} = \frac{1}{F_\pi(s)}\left\{\bar\delta^{\mathsf{ISR}}_V(\lambda^2,\lambda^2)\Big|_{\mathsf{IR}}\left(F_\pi(s) + F_\pi(0) - 1\right)\right\}$$

in which the λ^2 dependence exactly cancels with δ^{IFI}_S, using $F_\pi(0) = 1$.



GVMD: Complete NLO

Take the cross section differential in the photon energy

$$\sigma_{\mathrm{NLO}} = \sigma_{\mathrm{LO}}(F_\pi^{\mathrm{LO}}) + \frac{\alpha}{\pi}\left[\int_\lambda^{\Delta E} \mathrm{d}\sigma_{\mathrm{LO}}\left(\delta_{SV}^{\mathrm{ISR}} + \delta_{SV,\mathrm{GVMD}}^{\mathrm{FSR}}(F_\pi^{\mathrm{BW}}) + \delta_{SV,\mathrm{GVMD}}^{\mathrm{IFI}}(F_\pi^{\mathrm{BW}})\right) + \int_{\Delta E}^{\sqrt{s_{\mathrm{max}}}} \mathrm{d}\sigma_H(F_\pi^{?})\right]$$

Born

In principle one could use any form factor in the Born, being consistent at each perturbative order

Soft+Virtual

To cancel λ^2 dependence, we have to use F_π^{BW} **both** in soft and virtual corrections

Hard

To ensure E-independence we use F_π^{BW}, other choices would have

$$\delta_{SV}(\Delta E) \neq \delta_H(\Delta E)$$

- For consistency we use the same F_π **everywhere**
- The BW fit has **limited** accuracy (TBD in next slides)

FsQED Approach

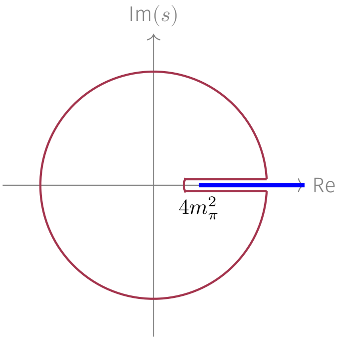
The dispersive Form factor relies on the analiticity of $F_\pi(s)$ on all the complex plane, except for the physical branch cut at $s \geq 4m_\pi^2$

Dispersive approach (FsQED)

$$F_\pi(q^2) = 1 + \frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\frac{\mathrm{Im}F_\pi(s')}{s'-q^2-i\varepsilon'}$$

Comes with the sum rule

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\mathrm{d}s'}{s'}\mathrm{Im}F_\pi(s') = 1$$



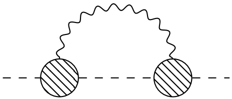
In loop integrals the ratio $F_\pi(q^2)/q^2$ has to be regularised as

$$\frac{F_\pi(q^2)}{q^2} \to \frac{1}{q^2-\lambda^2+i\varepsilon'} - \frac{1}{\pi}\int_{4m_\pi^2-\lambda^2}^{\infty}\frac{\mathrm{d}s'}{s'}\frac{\mathrm{Im}F_\pi(s'+\lambda^2)}{q^2-s'-\lambda^2+i\varepsilon'}$$

[4] Gilberto Colangelo et al. "Radiative corrections to the forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$". In: *JHEP* 08 (2022), p. 295. DOI: 10.1007/JHEP08(2022)295. arXiv: 2207.03495 [hep-ph]

FsQED: Pion Self Energy

$$\Sigma_\pi(p^2) \quad = $$



$$= e^2 \int \frac{\mathrm{d}^D q}{(2\pi)^D} \left\{ - \frac{(2p+q)^2 F_\pi^2(q^2)}{((q+p)^2 - m_\pi^2) q^2} \right\}$$

Since no particular extra divergence arises, the operations can be performed in this order (λ=0)

$\int \mathrm{d}^D q$ $\longrightarrow$ $-\frac{\partial}{\partial p^2}$ $\longrightarrow$ $\int \frac{\mathrm{d}s'}{s'}$

$$\delta Z_\phi^0 \equiv - \frac{\partial \Sigma_\pi(p^2, m_\pi^2, 0)}{\partial p^2}\bigg|_{p^2 = m_\pi^2} \qquad \delta Z_\phi^{\text{FsQED}} \equiv - \frac{\partial \Sigma_\pi(p^2, m_\pi^2, s')}{\partial p^2}\bigg|_{p^2 = m_\pi^2}$$

The counterterm in the dispersive approach reads

$$\begin{aligned} \delta Z_\phi = & \Bigg\{ \delta Z_\phi^0 - \frac{2}{\pi} \int \frac{\mathrm{d}s'}{s'} \text{Im} F_\pi(s') \delta Z_\phi^{\text{FsQED}}(s') \\ & + \frac{1}{\pi^2} \int \mathrm{d}s' \int \frac{\mathrm{d}s''}{s''} \frac{\text{Im} F_\pi(s') \text{Im} F_\pi(s'')}{s'' - s'} \left(\delta Z_\phi^{\text{disp}}(s'') s'' - Z_\phi^{\text{disp}}(s') s' \right) \Bigg\} \end{aligned}$$

FsQED: FSR

The FSR contribution works in the same way

$$\tilde{\delta}^{\text{FSR}}_{V,\text{FsQED}}(\lambda) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} 2\text{Re} \int \mathrm{d}^D q \frac{\mathcal{M}^{\dagger}_{\text{LO},0}\, \overline{\mathcal{M}}^{\text{FSR}}_{V,\text{FsQED}}(q,\lambda)}{|\mathcal{M}_{\text{LO},0}|^2} \frac{F^2_\pi(q^2)}{q^2-\lambda^2+i\varepsilon}$$

with the two FF expressed as

$$\begin{aligned}\tilde{\delta}^{\text{FSR}}_{V,\text{FsQED}} =& 2\mathcal{C}_D \text{Re} \int \mathrm{d}^D q \frac{\mathcal{M}^{\dagger}_{\text{LO},0}\, \overline{\mathcal{M}}^{\text{FSR}}_{V,\text{FsQED}}(q,\lambda)}{|\mathcal{M}_{\text{LO},0}|^2} \Bigg[\frac{1}{q^2-\lambda^2} - \frac{2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'} \frac{\text{Im}F_\pi(s')}{q^2-\lambda^2-s'+i\varepsilon'} \\ &+ \frac{1}{\pi^2}\int_{\Omega_\infty} \frac{\mathrm{d}s'}{s'}\frac{\mathrm{d}s''}{s''} \frac{\text{Im}F_\pi(s')\text{Im}F_\pi(s'')}{s''-s'-i\varepsilon''+i\varepsilon'} \left(\frac{s''}{q^2-\lambda^2-s''+i\varepsilon''} - \frac{s'}{q^2-\lambda^2-s'+i\varepsilon'} \right) \Bigg],\end{aligned}$$

where massless propagators are replaced by massive photons propagators

$$\begin{aligned}\delta^{\text{FSR}}_{V,\text{FsQED}} =& \Bigg\{ \bar{\delta}^{\text{FSR}}_{V}(0) - \frac{2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'} \text{Im}F_\pi(s') \bar{\delta}^{\text{FSR}}_{V}(s') \\ &+ \frac{1}{\pi^2}\int_{\Omega_\infty} \mathrm{d}s' \frac{\mathrm{d}s''}{s''} \frac{\text{Im}F_\pi(s')\text{Im}F_\pi(s'')}{s''-s'-i\varepsilon''+i\varepsilon'} \left(\bar{\delta}^{\text{FSR}}_{V}(s'')s'' - \bar{\delta}^{\text{FSR}}_{V}(s')s' \right) \Bigg\}\end{aligned}$$

$\bar{\delta}^{\text{FSR}}_{V}(s')$ is the massive-photon kernel

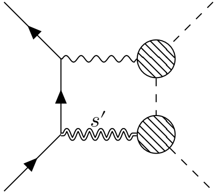
FsQED: IFI

The most difficult contribution is given by the FsQED IFI

$$\delta^{\text{IFI}}_{V,\text{FsQED}} = \frac{2\text{Re}F^*_\pi(s)\mathcal{M}^\dagger_{\text{LO},0}\,\mathcal{M}^{\text{IFI}}_{V,\text{FsQED}}}{|F_\pi(s)|^2\,|\mathcal{M}_{\text{LO},0}|^2} \equiv \frac{\text{Re}F^*_\pi(s)\Delta^{\text{IFI}}_{V,\text{FsQED}}}{|F_\pi(s)|^2}\,.$$

The correction can be written in terms of polar and dispersive contributions

$$\begin{aligned}\Delta^{\text{IFI}}_{V,\text{FsQED}} =& \bar{\delta}^{\text{IFI}}_V(\lambda^2,\lambda^2) && \textbf{pole-pole}\\ &-\frac{1}{\pi}\int_{4m_\pi^2}^\infty \frac{\text{d}s'}{s'}\text{Im}F_\pi(s')\left[\bar{\delta}^{\text{IFI}}_V(s',\lambda^2)+\bar{\delta}^{\text{IFI}}_V(\lambda^2,s')\right] && \textbf{pole-disp.}\\ &+\frac{1}{\pi^2}\int_{\Omega_\infty}\frac{\text{d}s'}{s'}\frac{\text{d}s''}{s''}\text{Im}F(s')\text{Im}F(s'')\bar{\delta}^{\text{IFI}}_V(s',s'') && \textbf{disp.-disp.}\end{aligned}$$



The IR divergence has to be carefully isolated. We write

FsQED Virtual IFI

$$\delta^{\text{IFI}}_{V,\text{FsQED}} = \frac{1}{|F_\pi(s)|^2}\left[\text{Re}F_\pi(s)\text{Re}\Delta^{\text{IFI}}_{V,\text{FsQED}} + \text{Im}F_\pi(s)\text{Im}\Delta^{\text{IFI}}_{V,\text{FsQED}}\right]$$

In the following we shall treat separately Re and Im

Real Part

The divergence in the IFI massive kernel arises in two regions of the loop integration, namely for $q \to 0$ and $q \to p_3 + p_4$:

$$\bar{\delta}^{\text{IFI}}_{V,\text{IR}}(\lambda^2, s') = \frac{s}{2(s - s' + i\varepsilon')} \mathcal{C}_{\text{IR}} \log \frac{\lambda^2}{s}$$

We can add and subtract the IR divergence of the pole-pole part

$$\begin{aligned}
\text{Re}\Delta^{\text{IFI}}_{V,\text{FsQED}} &= \underbrace{\text{Re}\bar{\delta}^{\text{IFI}}_{V}(\lambda^2, \lambda^2) - \text{Re}\bar{\delta}^{\text{IFI}}_{V,\text{IR}}(\lambda^2, \lambda^2)}_{\text{IR-finite}} + \text{Re}\bar{\delta}^{\text{IFI}}_{V,\text{IR}}(\lambda^2, \lambda^2) \\
&- \frac{2}{\pi} \text{Re} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'} \text{Im}F_\pi(s') \left[\underbrace{\bar{\delta}^{\text{IFI}}_{V}(\lambda^2, s') - \bar{\delta}^{\text{IFI}}_{V,\text{IR}}(\lambda^2, s')}_{\text{IR-finite}} + \bar{\delta}^{\text{IFI}}_{V,\text{IR}}(\lambda^2, s') \right] \\
&+ \frac{1}{\pi^2} \int_{\Omega_\infty} \frac{\mathrm{d}s'}{s'} \frac{\mathrm{d}s''}{s''} \text{Im}F(s') \text{Im}F(s'') \text{Re}\bar{\delta}^{\text{IFI}}_{V}(s', s'') \,.
\end{aligned}$$

Real Part: The missing piece

- The pole-pole and disp-disp corrections are fine
- The pole-dispersive correction exhibits a singularity that has to be treated with the **Principal Value**

$$\lim_{\varepsilon'\to 0_+} \mathrm{Re}\int \mathrm{d}s' \frac{f(s')}{s-s'+i\varepsilon'} = \mathrm{P.V.}\int \left(\frac{\mathrm{Re}\, f(s')}{s-s'}\right) + \frac{\pi}{2}\mathrm{Im}\, f(s_+) + \frac{\pi}{2}\mathrm{Im}\, f(s_-)$$

The piece in red was **overlooked** by literature and by a previous version of this work

$$\begin{aligned}
\mathrm{Re}\Delta^{\mathrm{IFI}}_{V,\mathrm{FsQED}} =& \mathrm{Re}F_\pi(s)\mathcal{C}_{\mathrm{IR}} \log\frac{\lambda^2}{s} \\
&+ \left[\mathrm{Re}\bar{\delta}^{\mathrm{IFI}}_V(\lambda^2,\lambda^2) - \mathrm{Re}\bar{\delta}^{\mathrm{IFI}}_{V,\mathrm{IR}}(\lambda^2,\lambda^2)\right]_{\mathrm{fin.}} \\
&- \frac{2}{\pi}\,\mathrm{P.V.}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'} \mathrm{Im}F_\pi(s') \left[\mathrm{Re}\bar{\delta}^{\mathrm{IFI}}_V(\lambda^2,s') - \mathrm{Re}\delta^{\mathrm{IFI}}_{V,\mathrm{IR}}(\lambda^2,s')\right]_{\mathrm{fin.}} \\
&- \frac{\mathrm{Im}F_\pi(s)}{s}\left[\lim_{s'\to s^-} \mathrm{Im}\bar{\delta}^{\mathrm{IFI}}_V(\lambda^2,s')(s-s')\right] \\
&+ \frac{1}{\pi^2}\int_{\Omega_\infty} \frac{\mathrm{d}s'}{s'}\frac{\mathrm{d}s''}{s''} \mathrm{Im}F(s')\mathrm{Im}F(s'')\mathrm{Re}\bar{\delta}^{\mathrm{IFI}}_V(s',s''),
\end{aligned}$$

This missing piece was spotted with the help of **Fedor Ignatov** and completely changes the picture (be patient)

Remark on order of limits

The discussion is valid with this order

$\int \mathrm{d}^D q$ $\longrightarrow$ $\lambda \to 0$ $\longrightarrow$ $\varepsilon \to 0$ $\longrightarrow$ $\int \frac{\mathrm{d}s'}{s'}$

If one wants to reverse the order of limits the following substitution should be applied

$$\bar{\delta}_V^{\text{IFI}}(\lambda^2, s') \to \bar{\delta}_V^{\text{IFI}}(\lambda^2, s' + \lambda^2) \qquad F_\pi(s') \to F_\pi(s' + \lambda^2)$$

$\int \mathrm{d}^D q$ $\longrightarrow$ $\epsilon \to 0$ $\longrightarrow$ $\lambda \to 0$ $\longrightarrow$ $\int \frac{\mathrm{d}s'}{s'}$

All the results are valid also in IR dimensional regularisation

Imaginary Part

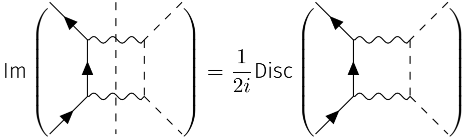
Most of the effort in computing the imaginary part

$$\text{Im}\Delta^{\text{IFI}}_{V,\text{FsQED}} = \text{Im}\bar{\delta}^{\text{IFI}}_V(\lambda^2,\lambda^2) - \frac{2}{\pi}\text{Im}\int_{4m_\pi^2}^{s}\frac{\mathrm{d}s'}{s'}\left(\text{Im}F_\pi(s') - \text{Im}F_\pi(s)\right)\bar{\delta}^{\text{IFI}}_V(\lambda^2,s')$$
$$-\frac{2}{\pi}\text{Im}F_\pi(s)\text{Im}\int_{4m_\pi^2}^{s}\frac{\mathrm{d}s'}{s'}\bar{\delta}^{\text{IFI}}_V(\lambda^2,s')$$
$$+\frac{1}{\pi^2}\int_{\Omega_s}\frac{\mathrm{d}s'}{s'}\frac{\mathrm{d}s''}{s''}\text{Im}F(s')\text{Im}F(s'')\text{Im}\bar{\delta}^{\text{IFI}}_V(s',s'')\,.$$

The IR divergence is contained in

$$\text{Im}\Delta^{\text{IFI}}_{V,\text{FsQED}}\Big|^{\text{pole-disp}}_{\text{IR}} = -\frac{2}{\pi}\text{Im}F_\pi(s)\text{Im}\int_{4m_\pi^2}^{s}\frac{\mathrm{d}s'}{s'}\bar{\delta}^{\text{IFI}}_V(\lambda^2,s')\,,$$

Computed using Cutkosky rules



Imaginary Part

Long story short, after some iteration we got the exact answer

$$\mathrm{Im}\Delta^{\mathrm{IFI}}_{V,\mathrm{FsQED}}\Big|^{\text{pole-disp}}_{\mathrm{IR}} = -\frac{2}{\pi}\mathrm{Im}F_\pi(s)\Big\{C_{1/s'}(t)\,\mathcal{I}_{1/s'}(t) + C_{s'/s'}(t)\,\mathcal{I}_{s'/s'}(t) - (t \to u)\Big\}\,,$$

where the integrals are proportional to the **Eikonal integral**

$$\mathcal{I}_{1/s'}(x) = \mathrm{Im}\int_{4m_\pi^2}^{s} \frac{\mathrm{d}s'}{s'}\mathcal{D}_0^{e,\pi}(x,s',s'') = \frac{1}{4s}\mathcal{L}(x) + \mathcal{I}_2(x)\,,$$

$$\mathcal{I}_{s'/s'}(x) = \mathrm{Im}\int_{4m_\pi^2}^{s} \mathrm{d}s'\mathcal{D}_0^{e,\pi}(x,s',s'') = \frac{1}{4}\mathcal{L}(x)\,,$$

In the end, the IR divergence in IFI diagrams is correctly reconstructed as

Pole-disp IR part

$$\delta^{\mathrm{IFI}}_{V,\mathrm{FsQED}}\Big|_{\mathrm{IR}} = \left(\frac{|\mathrm{Re}F_\pi(s)|^2}{|F_\pi(s)|^2}\,\mathcal{C}_{\mathrm{IR}} + \frac{|\mathrm{Im}F_\pi(s)|^2}{|F_\pi(s)|^2}\,\mathcal{C}_{\mathrm{IR}}\right)\log\frac{\lambda^2}{s} = \mathcal{C}_{\mathrm{IR}}\log\frac{\lambda^2}{s}\,,$$

Numerics

Fitting F_π

We have two ways of fitting the FF, relying on BW functions or GS functions

Breit-Wigner Function

$$\mathrm{BW}_v(q^2) = \frac{m_v^2}{m_v^2 - i m_v \Gamma_v - q^2}$$

- Does not have the right analycal properties
- Has complex poles for $q^2 = m_V^2 - i m_v \Gamma_v$

Gounaris-Sakurai Function

$$\mathrm{BW}_v^{\mathrm{GS}}(q^2) = \frac{m_v^2 + d(m_v)\, m_v\, \Gamma_v}{m_v^2 - q^2 + f(q^2) - i\, m_v\, \Gamma(q^2)}$$

- $\mathrm{Im}(\mathrm{BW}(q^2 < 4m_\pi^2)) = 0$
- Has almost the right analytical properties of F_π (branch cut)

$$F_\pi^{\mathrm{GS}}(q^2) = \frac{1}{1 + c_{\rho'} + c_{\rho''}} \left[\left(1 + \sum_{v=\omega,\phi} c_v \frac{q^2}{m_v^2} \mathrm{BW}_v(q^2) \right) \mathrm{BW}_\rho^{\mathrm{GS}}(q^2) + c_{\rho'}\, \mathrm{BW}_{\rho'}^{\mathrm{GS}}(q^2) + c_{\rho''}\, \mathrm{BW}_{\rho''}^{\mathrm{GS}}(q^2) \right]$$

$$F_\pi^{\mathrm{BW}}(q^2) = \frac{\mathrm{BW}_\rho(q^2) + c_\omega \mathrm{BW}_\omega(q^2) + c_\phi \mathrm{BW}_\phi(q^2) + c_{\rho'} \mathrm{BW}_{\rho'}(q^2) + c_{\rho''} \mathrm{BW}_{\rho''}(q^2)}{1 + c_\omega + c_\phi + c_{\rho'} + c_{\rho''}}$$

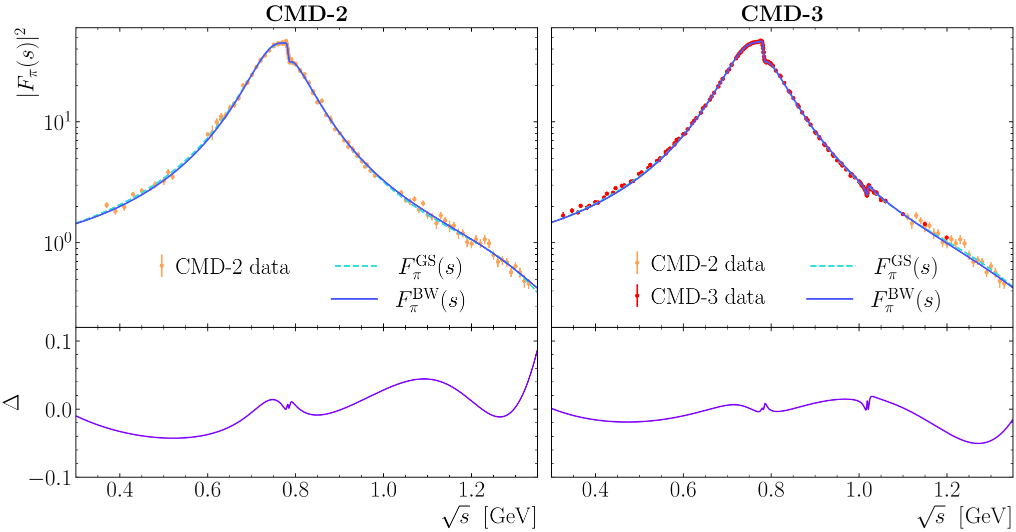
Fitting F_π

The numerical results are produced using a *parameterisation* of F_π *inspired by data*

- **CMD-2 scenario:** we use only CMD-2 data
- **CMD-3 scenario:** we use a combination fo CMD-2+CMD-3 (as done in their works)

		CMD-2			CMD-3				
		ρ	ω	ρ'	ρ	ω	ϕ	ρ'	ρ''
GS	m_v	775.49	782.66	1369.8	773.98	782.22	1019.5	1456.7	1870.74
	Γ_v	145.70	8.560	385.21	147.86	8.174	5.275	524.05	170.49
	$\|c_v\|$	-	0.0016	0.0887	-	0.0016	0.00059	0.097	0.037
	φ_v	-	0.179	3.159	-	0.057	2.836	3.541	2.277
BW	m_v	758.08	782.80	1253.8	755.71	782.07	1019.5	1338.64	1745.02
	Γ_v	136.81	8.004	530.86	142.86	7.997	6.251	982.29	397.85
	$\|c_v\|$	-	0.0079	0.144	-	0.0085	0.00089	0.259	0.098
	φ_v	-	2.014	3.021	-	1.782	5.561	3.340	0.817

Fitting F_π



CMD-2
CMD-3
$|F_\pi(s)|^2$
10^1
10^0
CMD-2 data
CMD-3 data
$F_\pi^{\mathrm{GS}}(s)$
$F_\pi^{\mathrm{BW}}(s)$
Δ
0.1
0.0
−0.1
0.4
0.6
0.8
1.0
1.2
$\sqrt{s}$ [GeV]

Form factors and cuts

The form factor can be chosen from the following list

Data	Λ^2[GeV^2]	F$\times$sQED	GVMD	FsQED
CMD-3	4	✓		✓
CMD-2	2	✓		✓
BW sum 2/3	$2/4$	✓	✓	✓
SND	1	✓		✓
Babar	9	✓		✓
Strong2020	9	✓		✓
BesIII	9	✓		✓
Kloe2	1	✓		✓
Phokhara	16	✓		✓
Bern	4	✓		✓

The cuts we have used are inspired by CMD-3

$$p^{\pm} \equiv |^{\pm}| > 0.45E\,,$$

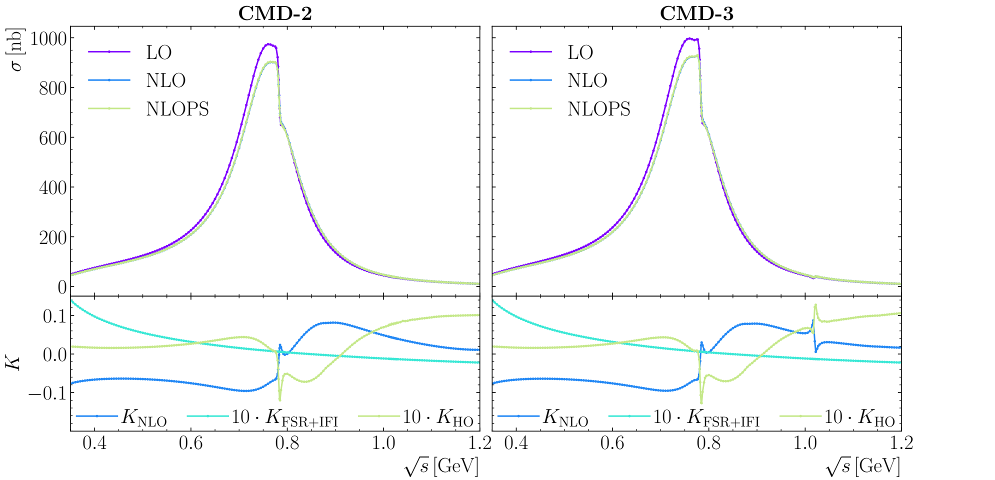
$$\vartheta_{\text{avg}} \equiv \frac{1}{2}(\pi - \vartheta^+ + \vartheta^-) \in [1, \pi - 1]\,,$$

$$\delta\vartheta \equiv |\vartheta^+ + \vartheta^- - \pi| < 0.25\,,$$

$$\delta\phi \equiv ||\phi^+ - \phi^-| - \pi| < 0.15$$

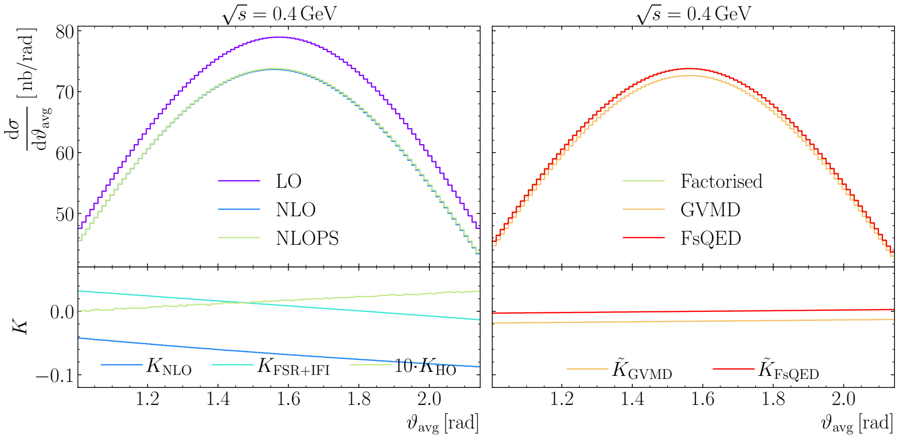
Interplay of Radiative Corrections and F_πs

$$K_{\mathrm{NLO}} = \frac{\sigma_{\mathrm{NLO}} - \sigma_{\mathrm{LO}}}{\sigma_{\mathrm{LO}}}\,, \qquad K_{\mathrm{FSR+IFI}} = \frac{\sigma_{\mathrm{NLO}} - \sigma_{\mathrm{ISR}}}{\sigma_{\mathrm{LO}}}\,, \qquad K_{\mathrm{HO}} = \frac{\sigma_{\mathrm{NLOPS}} - \sigma_{\mathrm{NLO}}}{\sigma_{\mathrm{LO}}}$$

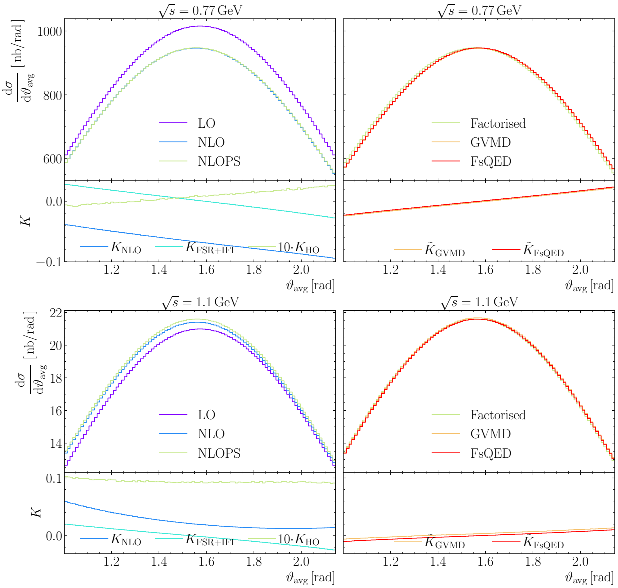


Angular distributions

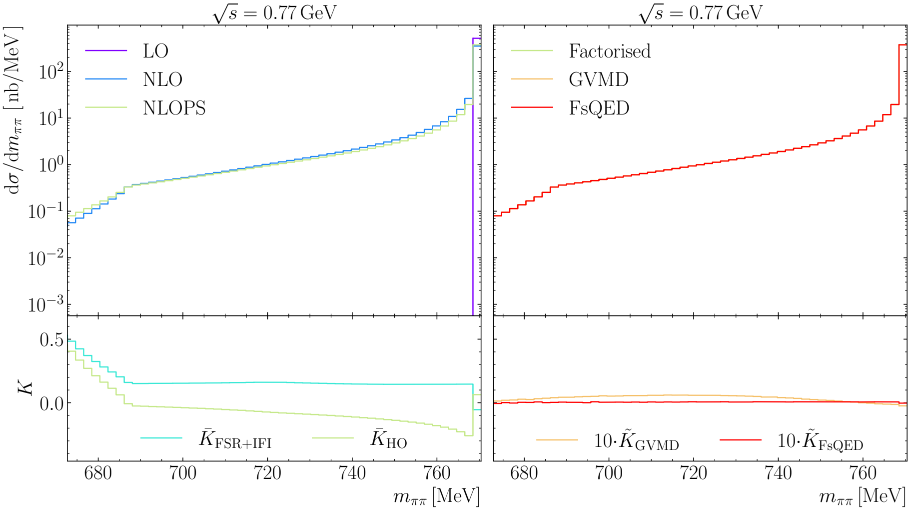
$$\tilde{K}_{\mathrm{FF}} = \left(\frac{\mathrm{d}\sigma_{\mathrm{FF}}}{\mathrm{d}\vartheta_{\mathrm{avg}}}\right)\left(\frac{\mathrm{d}\sigma_{\mathrm{Factorised}}}{\mathrm{d}\vartheta_{\mathrm{avg}}}\right)^{-1} - 1$$



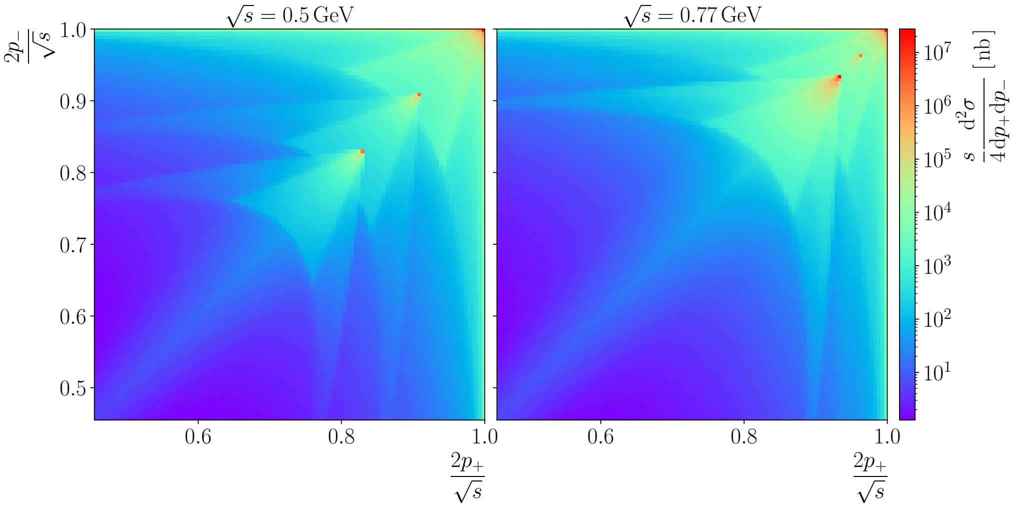
Angular distributions



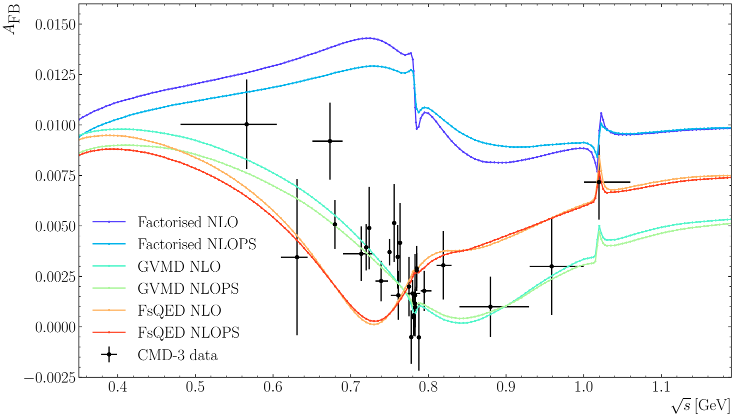
Pion pair invariant mass



2D distributions

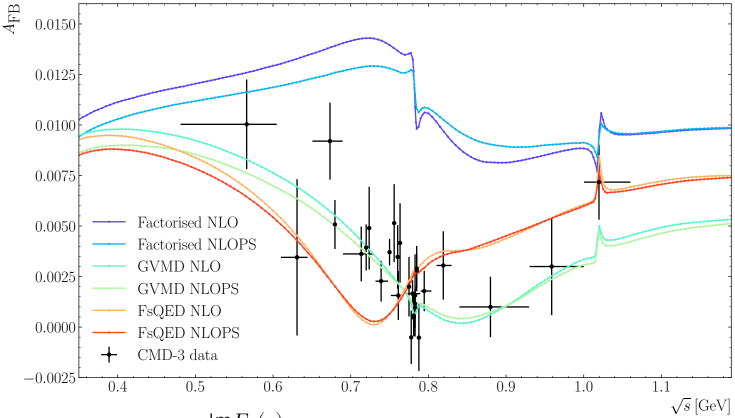


The charge asymmetry: September 2024



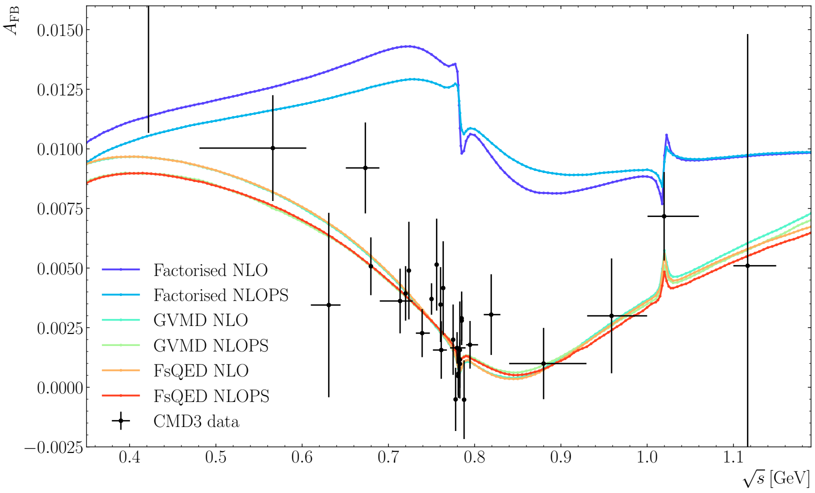
One can conclude, that at some level, GVMD and FsQED disagree, especially in the ρ peak region

The charge asymmetry: September 2024



$$-\frac{\mathrm{Im}F_\pi(s)}{s}\left[\lim_{s'\to s^-}\mathrm{Im}\bar{\delta}_V^{\mathrm{IFI}}(\lambda^2,s')(s-s')\right]$$

The charge asymmetry: accepted version of BabaYaga paper



After adding the Principal Value pole, the discrepancy **is gone**. The two approaches yield to the same results (up to FF limitations)

State of BabaYaga

- As of today, we can generate $e^+e^- \to \pi^+\pi + -$ at NLO and NLO PS with **all masses**

Approach	NLO	NLO PS
$F_\pi \times$ sQED	✓	✓
GVMD	✓	✓
Dispersive	✓	✓

- In the next future, we will calculate $e^+e^- \to \pi^+\pi^-\gamma(+n\gamma)$ Marco's Talk this afternoon
- Investigate if $\pi\pi\gamma\gamma$ vertex needs improvement for radiative return
- Extend to other hadronic channels, $i.e.$ K^+K^-

Further remarks

Fit Procedure

When fitting $F_\pi(s)$ as a sum of BW functions, one should also take into account the sum rule. Take 2 resonances for example

$$F_\pi(s) = e^{i\theta}\left[|c_1|e^{i\varphi_1}\text{BW}(s; m_1, \Gamma_1) + |c_2|e^{i\varphi_2}\text{BW}(s; m_2, \Gamma_2)\right] \tag{5}$$

One could rephase the function

$$\text{Re}F_\pi(s) = \cos\theta\,\text{Re}\left(\sum_i c_i \text{BW}_i(s)\right) - \sin\theta\,\text{Im}\left(\sum_i c_i \text{BW}_i(s)\right)$$

If one imposes the sum rule, the degeneracy is removed

$$\int \frac{\mathrm{d}s'}{s'}\text{Im}F_\pi(s') = \pi = \sin\theta \int \frac{\mathrm{d}s'}{s'}\text{Re}\left(\sum_i c_i \text{BW}_i\right) + \cos\theta \int \frac{\mathrm{d}s'}{s'}\text{Im}\left(\sum_i c_i \text{BW}_i\right) \tag{6}$$

This allows to a correct separation of Re-Im parts of F_π
Another strategy is fitting as a normalisation

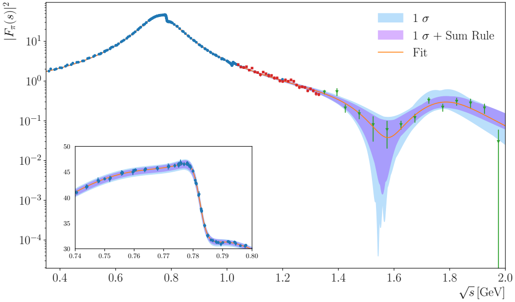
$$F_\pi(s; \vec{c})\text{SR}[\vec{c}]^\beta = \left(\frac{1}{\pi}\int \frac{\mathrm{d}s'}{s'}\text{Im}F_\pi(s')\right)^\beta \sum_i c_i \text{BW}_i(s) \tag{7}$$

This constrained fit improves the agreement with GS functions

Estimation of parametric uncertainty?

We **need** a fit to obtain $\mathrm{Re}F_\pi + i\mathrm{Im}F_\pi$ → Every fit has an associated **uncertainty** → Parametric uncertainty of theory predictions?

What is the role of the sum rule in the fits? (GS sum in the example)



Bootstrap the prediction?

$$A_{FB} \pm \Delta A_{FB}^{\text{pars}}$$

Is it needed by experiments?
WIP in BabaYaga